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STRONG TUTTE TYPE CONDITIONS AND FACTORS OF GRAPHS

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Abstract

Let $odd(G)$ denote the number of odd components of a graph G and $k \geq 2$ be an integer. We give sufficient conditions using $odd(G-S)$ for a graph G to have an even factor. Moreover, we show that if a graph G satisfies $odd(G-S) \leq \max\{1, (1/k)|S|\}$ for all $S \subset V(G)$, then G has a $(k-1)$-regular factor for $k \geq 3$ or an $\mathbf{H}$-factor for $k = 2$, where we say that G has an $\mathbf{H}$-factor if for every labeling $h : V(G) \to \{\text{red}, \text{blue}\}$ with $\#\{v \in V(G) : f(v) = \text{red}\}$ even, G has a spanning subgraph F such that $\deg_F(x) = 1$ if $h(x) = \text{red}$ and $\deg_F(x) \in \{0, 2\}$ otherwise.

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1. INTRODUCTION

In this paper we consider finite graphs which have neither multiple edges nor loops. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. The minimum degree of G is denoted by $\delta(G)$. An edge of G joining a vertex u to a vertex v is denoted by uv or vu. For a subset S of $V(G)$, we write $G - S$ for the subgraph of G induced by $V(G) \setminus S$. Let $odd(G)$, $iso(G)$ and $\omega(G)$ denote the number of odd components of G (i.e., components of G of odd order) the number of isolated vertices of G, and the number of components of G, respectively. The complete graph, the path and the cycle of order n are denoted by K_n, P_n and C_n, respectively. The complete bipartite graph with parts of sizes m and n is denoted by $K_{m,n}$. In particular, $K_2 = P_2$ and $K_{1,2} = P_3$. For a set X, the cardinality of X is denoted by $|X|$ or $\#X$. Other notation and definitions not defined in this paper are standard and found in the book [14].

We give some definitions of factors of graphs. Let G be a graph and F be its spanning subgraph. Then for a set $\mathcal{S}$ of connected graphs, F is called an *$\mathcal{S}$-factor* of G if every component of F is isomorphic to an element of $\mathcal{S}$. For a set $\mathbb{S}$ of positive integers, F is called an *$\mathbb{S}$-factor* of G if $\deg_F(x) \in \mathbb{S}$ for all $x \in V(G)$. For an integer $k \geq 1$, F is called a *k-regular factor* or a *k-factor* of G if $\deg_F(x) = k$ for all $x \in V(G)$. For a function $f : V(G) \to \{1, 3, 5, \dots\}$, F is called an *$(1, f)$-odd factor* of G if $\deg_F(x) \in \{1, 3, \dots, f(x)\}$ for all $x \in V(G)$. A $\{2, 4, 6, \dots\}$-factor is called an *even factor*. So every vertex of an even factor F has a positive even degree in F.

We now present some known results on factors of graphs related to our theorems.

Theorem 1. *Let G be a graph. Then the following statements hold.*

1. *G has a 1-factor if and only if $odd(G - S) \leq |S|$ for all $S \subset V(G)$,* (*Tutte* [12]).
2. *Let $n \geq 3$ be an odd integer. Then G has a $\{1, 3, \dots, n\}$-factor if and only if $odd(G - S) \leq n|S|$ for all $S \subset V(G)$,* (*Amahashi* [1]).
3. *Let $f : V(G) \to \{1, 3, 5, \dots\}$. Then G has a $(1, f)$-odd factor if and only if $odd(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$,* (*Cui and Kano* [4]).
4. *Let $m \geq 2$ be an even integer. Then G has a $\{1, 3, \dots, m-1, m\}$-factor if $odd(G - S) \leq m|S|$ for all $S \subset V(G)$,* (*Lu and Wang* [10]).

On the other hand, the following theorem shows some results on factors of graphs G which satisfy certain conditions on $iso(G - S)$ instead of $odd(G - S)$.

Theorem 2. *Let G be a graph. Then the following statements hold.*

1. *G has a $\{K_2, C_n : n \geq 3\}$-factor if and only if $iso(G - S) \leq |S|$ for all $S \subset V(G)$, (Tutte [13]).*
2. *Let $n \geq 2$ be an integer. Then G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$, (Amahashi and Kano [2], Las Vergnas [8]).*
3. *If $iso(G - S) \leq (1/2)|S|$ for all $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, (Kano, Lu and Yu [6]).*
4. *Let $m \geq 2$ be an integer. If $iso(G - S) \leq (1/m)|S|$ for all $S \subset V(G)$, then G has a $\{K_{1,m}, K_{1,m+1}, \ldots, K_{1,2m}\}$-factor, (Kano and Saito [7]).*
5. *Let $m \geq 2$ be an integer. If $iso(G-S) \leq (1/m)|S|$ for all $S \subset V(G)$, then G has a $\{K_{1,m}, K_{1,m+1}, \ldots, K_{1,2m-1}, K_{2m+1}\}$-factor, (Zhang, Yan and Kano [15]).*

Proofs of the most results given in Theorems 1 and 2 and some other results on factors can be found in the book [3] by Akiyama and Kano. In this paper, we consider the following problem.

Problem 3. Suppose that a graph G satisfies $odd(G - S) \leq \max\{1, \alpha|S|\}$ or $odd(G-S) < \max\{2, \alpha|S|\}$ for all $S \subset V(G)$ and for a rational number $0 < \alpha \leq 1$. What kind of a factor does G have?

Notice that if a graph G has even order and $0 < \alpha < 1$, then for a vertex v of G, $odd(G-v) \geq 1$ and $\alpha|\{v\}| < 1$, and so $odd(G-S) \leq \alpha|S|$ does not hold. Hence we need the condition $odd(G - S) \leq \max\{1, \alpha|S|\}$ or $odd(G - S) < \max\{2, \alpha|S|\}$ instead of $odd(G - S) \leq \alpha|S|$.

As shown in Theorem 2, for some rational numbers $0 < \alpha < 1$, there are some results on factors of graphs which satisfy $iso(G-S) \leq \alpha|S|$ for all $S \subset V(G)$. On the other hand, there are no known results concerning Problem 3. We provide some partial solutions of this problem in Theorems 4 and 5.

Theorem 4. *Let G be a graph with $|G| \geq 3$. If G satisfies one of the following conditions* (i) *and* (ii), *then G has an even factor.*

(i) *G is of even order and satisfies*

$$odd(G - S) < |S| \quad \text{for all} \quad S \subset V(G) \text{ with } |S| \geq 2. \tag{1}$$

(ii) *G is of odd order and satisfies*

$$odd(G - S) < \frac{2}{3}|S| \quad \text{for all} \quad \emptyset \neq S \subset V(G). \tag{2}$$

We remark that if G is of even order, then the condition (1) is equivalent to

$$odd(G-S) < \max\{2, |S|\} \quad \text{for all} \quad S \subset V(G). \tag{3}$$

It is obvious that (1) and (3) are equivalent for $S \subset V(G)$ with $|S| \geq 2$. So we consider the case when S consists of one vertex v. It suffices to show that (1) implies $odd(G-v) \leq 1$. Assume that $odd(G-v) > 1$. Then $odd(G-v) \geq 3$ since G is of even order. Take a vertex u from an odd component of $G-v$, then $odd(G-\{v,u\}) \geq 2$. This contradicts (1). Hence (1) implies $odd(G-v) \leq 1$, and thus (1) and (3) are equivalent.

Theorem 5. *Let $k \geq 3$ be an integer, and G be a connected graph with $|G| \geq k$. If $(k-1)|G|$ is even and G satisfies*

$$odd(G-S) \leq \max\left\{1, \frac{1}{k}|S|\right\} \quad \textit{for all} \quad S \subset V(G), \tag{4}$$

then G has a $(k-1)$-regular factor.

Theorem 6. *Let G be a connected graph of order at least 2. If*

$$odd(G-S) \leq \max\left\{1, \frac{1}{2}|S|\right\} \quad \textit{for all} \quad S \subset V(G), \tag{5}$$

then for every vertex-labeling $h : V(G) \to \{red, blue\}$ with $\#\{v \in V(G) : h(v) = red\}$ even, G has a spanning subgraph F such that

$$\deg_F(v) = 1 \quad \textit{if} \quad h(v){=}red, \quad \textit{and} \quad \deg_F(v) \in \{0,2\} \quad \textit{otherwise.} \tag{6}$$

Notice that if for every $h : V(G) \to \{\text{red, blue}\}$ with $\#\{v \in V(G) : h(v) = \text{red}\}$ even, G has a spanning subgraph F satisfying (6), then we say that G has an **H**-*factor* [9]. Moreover, the spanning subgraph F that satisfies (6) and has a minimal edge set consists of vertex disjoint paths which connect two red vertices and whose all inner vertices are blue. We conclude this section by showing that the conditions of Theorem 4 are best possible in some sense.

Example 7. We show that the condition (1) of Theorem 4 cannot be replaced by $odd(G-S) \leq |S|$ for all $S \subset V(G)$ with $|S| \geq 2$. Let $k \geq 2$ be an integer, and let C_{2k} and C_{2k-1} be the cycles of order $2k$ and $2k-1$, respectively. We construct the graph G_1 from C_{2k} and C_{2k-1} by adding a new vertex v together with two edges vu_1, vu_2, where $u_1 \in V(C_{2k})$ and $u_2 \in V(C_{2k-1})$. Then G_1 is of even order and has no even factor. On the other hand, G_1 has a 1-factor, and so by the 1-factor theorem (1. of Theorem 1), G_1 satisfies $odd(G_1 - S) \leq |S|$ for all $S \subset V(G_1)$. Hence the condition (1) is best possible.

In order to show the sharpness of condition (2) of Theorem 4, we need the following theorem. Here for a component C of $G-S$ with $S \subset V(G)$, the number of edges of G joining C to S is denoted by $e_G(C,S)$.

Lemma 8 (Theorem 6.2 of [3]). *A connected graph G has an even factor if and only if*

$$\sum_{x\in S}(\deg_G(x)-2)-q(G,S)\geq 0 \quad \textit{for all} \quad S\subset V(G),$$

where $q(G,S)$ denotes the number of components C of $G-S$ such that $e_G(C,S)\equiv 1 \pmod 2$.

Example 9. We next show that the condition (2) of Theorem 4 cannot be replaced by the condition that $odd(G-S)\leq (2/3)|S|$ for all $\emptyset \neq S \subset V(G)$. Let $n\geq 3$ be an integer, D_1 and D_2 be the complete graphs of order $2n+1$, and $Z=\{z_1,z_2,\ldots,z_5\}$ be a set of 5 vertices. Let us take 10 vertices $u_i\in V(D_1), v_i\in V(D_2)$ for $1\leq i\leq 5$, and define the graph G_2 by

$$\begin{aligned} V(G_2) &= V(D_1)\cup V(D_2)\cup Z \quad \text{(disjoint union)}, \quad \text{and} \\ E(G_2) &= E(D_1)\cup E(D_2)\cup \{z_iu_i, z_iv_i : 1\leq i\leq 5\}. \end{aligned}$$

Then G_2 has odd order. Consider $S_1=\{u_i,v_i,z_5 : 1\leq i\leq 4\}$. Then we have $odd(G_2-S_1)=6=(2/3)|S_1|$ and for other $\emptyset\neq S\subset V(G_2)$, we can easily show that $odd(G_2-S)<(2/3)|S|$. Thus,

$$odd(G_2-S)\leq \frac{2}{3}|S| \quad \text{for all} \quad \emptyset\neq S\subset V(G_2).$$

Since $\sum_{x\in Z}(\deg_G(x)-2)-q(G,Z)=-2$, Lemma 8 implies that G_2 has no even factor. Hence the condition (2) is also best possible.

2. PROOF OF THEOREM 4

In this section we prove Theorem 4. We need the following result.

Lemma 10 (Problem 42 in Section 7 of [11], Theorem 6.3 of [3]). *Every 2-edge connected graph G with $\delta(G)\geq 3$ has an even factor.*

We now prove Theorem 4.

***Proof of Theorem 4 under the condition* (i).** Assume that a graph G satisfies the condition (i). Then $|G|$ is even and $|G|\geq 4$.

Claim 1. *G is connected.*

Proof. Assume that G is not connected. Let $G = H_1 \cup H_2 \cup \cdots \cup H_m$ be the decomposition of G into its components, where $m \geq 2$. We first assume that two components, say H_1 and H_2, are of even order. Take two vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. Then $2 \leq odd(G - \{v_1, v_2\}) < |\{v_1, v_2\}| = 2$, a contradiction. Hence at most one of components H_i has even order.

If a component H_a has even order, then at least two components have odd order and so $m \geq 3$. Take two vertices $v_3, v_4 \in V(H_a)$. Then $2 \leq odd(G - \{v_3, v_4\}) < |\{v_3, v_4\}| = 2$, a contradiction. Hence every component H_i has odd order. If a component H_b has order at least 3, then take two vertices $v_5, v_6 \in V(H_b)$. Then $2 \leq m \leq odd(G - \{v_5, v_6\}) < |\{v_5, v_6\}| = 2$, a contradiction. Hence every component H_i has order 1. It follows from $m = |G| \geq 4$ that $2 \leq m - 2 \leq odd(G - V(H_1) \cup V(H_2)) < |V(H_1) \cup V(H_2)| = 2$, a contradiction. Therefore G is connected. $\square$

Claim 2. *G is 2-edge connected.*

Proof. Assume that G is not 2-edge connected. Then there exists an edge $e \in E(G)$ such that $G - e$ consists of two components D_1 and D_2. Let $e = v_1v_2$ and $v_1 \in V(D_1)$, $v_2 \in V(D_2)$. Since $|G|$ is even, we have $|D_1| \equiv |D_2| \pmod 2$. If $|D_1|$ is even, then $2 \leq odd(G - \{v_1, v_2\}) < |\{v_1, v_2\}| = 2$, a contradiction. If $|D_1|$ is odd, then $|D_2|$ is odd, and so by symmetry and by $|G| \geq 4$, we may assume that $|D_1| \geq 3$. Take one vertex $v_3 \in V(D_1)$ and $v_3 \neq v_1$. Then $2 \leq odd(G - \{v_1, v_3\}) < |\{v_1, v_3\}| = 2$, a contradiction. Hence G is 2-edge connected. $\square$

Claim 3. *$\delta(G) \geq 3$.*

Proof. By Claim 2, $\delta(G) \geq 2$. Assume that $\delta(G) = 2$ and $\deg_G(v) = 2$ for some vertex v. Let x and y be the two vertices adjacent to v in G. Since $|G|$ is even, $G - \{x, y\}$ has at least two odd components including $\{v\}$, which implies $2 \leq odd(G - \{x, y\}) < |\{x, y\}| = 2$, a contradiction. Hence $\delta(G) \geq 3$. $\square$

Consequently, by Lemma 10, G has an even factor. $\blacksquare$

***Proof of Theorem* 4 *under the condition* (ii).** Assume that a graph G satisfies the condition (ii). Then $|G|$ is odd and $|G| \geq 3$.

Claim 1. *G is connected.*

Proof. Assume that G is not connected. Let $G = H_1 \cup H_2 \cup \cdots \cup H_m$ be the decomposition of G into its components, where $m \geq 2$. If a component H_a has even order, then take a vertex $v_1 \in V(H_a)$. Then $1 \leq odd(G - v_1) < (2/3)|\{v_1\}| < 1$, a contradiction. Hence every component H_i has odd order. If a component H_b has order at least 3, we consider two vertices $v_2, v_3 \in V(H_b)$. Then $2 \leq m \leq odd(G - \{v_2, v_3\}) < (2/3)|\{v_2, v_3\}| < 2$, a contradiction. Hence every component

H_i has order 1, and $m = |G| \geq 3$. Then we have $2 \leq m - 1 = odd(G - V(H_1)) < (2/3)|V(H_1)| < 1$, a contradiction. Therefore G is connected. $\square$

Claim 2. *G is 2-edge connected.*

Proof. Assume that G is not 2-edge connected. Then there exists an edge $e = xy \in E(G)$ such that $G - e$ consists of two components D_1 and D_2 with $x \in V(D_1)$ and $y \in V(D_2)$. Since $|G|$ is odd, we may assume that $|D_1|$ is odd and $|D_2|$ is even. Then $2 \leq odd(G - \{y\}) < (2/3)|\{y\}| < 1$, a contradiction. Hence G is 2-edge connected. $\square$

By Claim 2, $\delta(G) \geq 2$. If $|G| = 3$, then G is a cycle of order 3, which is an even factor. Thus we may assume that $|G| \geq 5$.

Claim 3. *G is 3-connected, and so $\delta(G) \geq 3$.*

Proof. Assume that $G - \{x, y\}$ is not connected for some two vertices x and y. Then $1 \leq odd(G - \{x, y\}) < (2/3)|\{x, y\}| = 4/3$. Thus $odd(G - \{x, y\}) = 1$. Let $G - \{x, y\} = C_1 \cup D_1 \cup \cdots \cup D_t$, $t \geq 1$, where C_1 is an odd component and all D_i are even components. Take a vertex $z \in V(D_1)$. Then $2 \leq odd(G - \{x, y, z\}) < (2/3) \cdot 3 = 2$, a contradiction. Hence G is 3-connected. $\square$

Consequently, by Lemma 10, G has an even factor. $\blacksquare$

3. Proofs of Theorems 5 and 6

Recall that $\omega(G)$ denotes the number of components of a graph G. A graph G is said to be *k-tough* if $\omega(G - S) \leq |S|/k$ for all $S \subset V(G)$ with $\omega(G - S) \geq 2$. In order to prove Theorems 5 and 6, we need the following two theorems.

Theorem 11 (Enomoto, Jackson, Katerinis, Saito [5])**.** *Let $k \geq 2$ be an integer and G be a graph. If G is k-tough, $|G| \geq k + 1$ and $k|G|$ is even, then G has a k-regular factor.*

Theorem 12 (Lu and Kano [9])**.** *Let G be a connected graph. If*

$$\omega(G - S) \leq |S| \quad \textit{for all} \quad \emptyset \neq S \subset V(G), \tag{7}$$

then for every vertex-labeling $h : V(G) \to \{red, blue\}$ with $\#\{v \in V(G) : f(v) = red\}$ even, G has a spanning subgraph F such that

$$\deg_F(x) = 1 \quad \textit{if} \quad h(x){=}red, \quad \textit{and} \quad \deg_F(x) \in \{0, 2\} \quad \textit{otherwise.}$$

We simultaneously prove Theorems 5 and 6 since they can be proved in the same way.

***Proofs of Theorems* 5 *and* 6.** Assume that a graph G satisfies the conditions of Theorems 5 or 6. We start with the following claim.

Claim 1. *G is $(k-1)$-tough for all $k \geq 2$.*

Proof. Assume that G is not $(k-1)$-tough. Then there exists a subset $X \subset V(G)$ such that $\omega(G-X) > |X|/(k-1)$ and $\omega(G-X) \geq 2$. Let $G-X = C_1 \cup \cdots \cup C_s \cup D_1 \cup \cdots \cup D_t$, where $C_1, \ldots, C_s$ are the odd components of $G-X$ and $D_1, \ldots, D_t$ are the even components of $G-X$. Thus we have $s+t = \omega(G-X) > |X|/(k-1)$, and so

$$|X| < (k-1)(s+t). \tag{8}$$

Consider a vertex $x_i \in V(D_i)$ for each $1 \leq i \leq t$. Then $odd(D_i - x_i) \geq 1$ since D_i is of even order. Hence, by (4) or (5), we have

$$s + t \leq odd(G - X \cup \{x_1, \ldots, x_t\}) \leq \frac{|X| + t}{k}.$$

Thus we obtain

$$ks + (k-1)t \leq |X|. \tag{9}$$

By (8) and (9), we have

$$ks + (k-1)t < (k-1)(s+t).$$

This implies that $s < 0$, which is a contradiction. Therefore, Claim 1 holds. □

If $k \geq 3$, then G has a $(k-1)$-regular factor by Claim 1 and Theorem 11. Hence Theorem 5 holds. If $k = 2$, then (7) holds by Claim 1. Therefore Theorem 6 follows from Theorem 12. ■

References

[1] A. Amahashi, *On factors with all degrees odd*, Graphs Combin. **1** (1985) 111–114. doi:10.1007/BF02582935

[2] A. Amahashi and M. Kano, *On factors with given components*, Discrete Math. **42** (1982) 1–6. doi:10.1016/0012-365X(82)90048-6

[3] J. Akiyama and M. Kano, Factors and Factorizations of Graphs, in: Lecture Notes in Math. **2031**, (Springer-Verlag, Berlin 2011). doi:10.1007/978-3-642-21919-1

[4] Y. Cui and M. Kano, *Some results on odd factors of graphs*, J. Graph Theory **12** (1988) 327–333.
doi:10.1002/jgt.3190120305

[5] H. Enomoto, B. Jackson, P. Katerinis and A. Saito, *Toughness and the existence of k-factors*, J. Graph Theory **9** (1985) 87–95.
doi:10.1002/jgt.3190090106

[6] M. Kano, H. Lu and Q. Yu, *Components factors with large components in graphs*, Appl. Math. Lett. **23** (2010) 385–389.
doi:10.1016/j.aml.2009.11.003

[7] M. Kano and A. Saito, *Star-factors with large components*, Discrete Math. **312** (2012) 2005–2008.
doi:10.1016/j.disc.2012.03.017

[8] M. Las Vergnas, *An extension of Tutte's* 1*-factor theorem*, Discrete Math. **23** (1978) 241–255.
doi:10.1016/0012-365X(78)90006-7

[9] H. Lu and M. Kano, *Characterization of* 1*-tough graphs using factors* (2017).
arXiv:1702.05873

[10] H. Lu and D. Wang, *On Cui-Kano's characterization problem on graph factors*, J. Graph Theory **74** (2013) 335–343.
doi:10.1002/jgt.21712

[11] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).
doi:10.1016/C2009-0-09109-0

[12] W.T. Tutte, *The factorization of linear graphs*, J. Lond. Math. Soc. (2) **22** (1947) 107–111.
doi:10.1112/jlms/s1-22.2.107

[13] W.T. Tutte, *The* 1*-factors of oriented graphs*, Proc. Amer. Math. Soc. **4** (1953) 922–931.
doi:10.2307/2031831

[14] D. West, Introduction to Graph Theory (Prentice Hall, 1996).

[15] Y. Zhang, G. Yan and M. Kano, *Star-like factors with large components*, J. Oper. Res. Soc. China **3** (2015) 81–88.
doi:10.1007/s40305-014-0066-7

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