ON THE STAR CHROMATIC INDEX OF GENERALIZED PETERSEN GRAPHS

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Abstract

The star $k$-edge-coloring of graph $G$ is a proper edge coloring using $k$ colors such that no path or cycle of length four is bichromatic. The minimum number $k$ for which $G$ admits a star $k$-edge-coloring is called the star chromatic index of $G$, denoted by $\chi'_s(G)$. Let $\text{GCD}(n,k)$ be the greatest common divisor of $n$ and $k$. In this paper, we give a necessary and sufficient condition of $\chi'_s(P(n,k)) = 4$ for a generalized Petersen graph $P(n,k)$ and show that “almost all” generalized Petersen graphs have a star 5-edge-colorings. Furthermore, for any two integers $k$ and $n (\geq 2k + 1)$ such that $\text{GCD}(n,k) \geq 3$, $P(n,k)$ has a star 5-edge-coloring, with the exception of the case that $\text{GCD}(n,k) = 3$, $k \neq \text{GCD}(n,k)$ and $\frac{n}{3} \equiv 1 \pmod{3}$.

\textbf{Keywords:} star edge-coloring, star chromatic index, generalized Petersen graph.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected; for the terminologies and notations not defined here, we follow [3]. For any graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For any vertex $v$ in $G$, a vertex $u \in V(G)$ is said to be a neighbor of $v$ if $uv \in E(G)$. We use $N_G(v)$ to denote the set of neighbors of $v$. For positive integers $n$ and $k$, let $\text{GCD}(n,k)$ be the greatest common divisor of $n$ and $k$.

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A star $k$-edge-coloring of a graph $G$ is a proper edge-coloring using $k$ colors such that at least three distinct colors are assigned to the edges of every path and cycle of length four. The minimum number $k$ for which $G$ admits a star $k$-edge-coloring is called the star chromatic index of $G$ and is denoted by $\chi'_s(G)$.

The star edge-coloring was motivated by the vertex version [1, 4, 5, 7], which was first studied by Liu and Deng [8], who gave an upper bound on the star chromatic index of graph with maximum degree at least 7. Dvořák et al. [6] provided some upper and lower bounds for complete graphs. They also considered cubic graphs and showed that the star chromatic index of such graphs lies between 4 and 7.

Since there exist many cubic graphs with a star chromatic index equal to 6, e.g., $K_{3,3}$ or the Heawood graph, and no example of a subcubic graph with star chromatic index 7 is known, Dvořák et al. proposed the following conjecture.

Conjecture 1.1 [6]. If $G$ is a subcubic graph, then $\chi'_s(G) \leq 6$.

Recently, Bezegová et al. [2] established tight upper bounds for trees and subcubic outerplanar graphs; they derived upper bounds for outerplanar graphs. In this paper, we obtain a necessary and sufficient condition of $\chi'_s(P(n,k)) = 4$, and present a construction of a star 5-edge-colorings of $P(n,k)$ for “almost all” values of $n$ and $k$. Furthermore, we find that the generalized Petersen graph $P(n,k)$ with $n = 3, k = 1$ is the only graph with a star chromatic index of 6 among the investigated graphs. Based on these results, we conjecture that $P(3,1)$ is the unique generalized Petersen graph that admits no star 5-edge-coloring.

2. A Necessary and Sufficient Condition of $\chi'_s(P(n,k)) = 4$

Let $n$ and $k$ be positive integers, $n \geq 2k+1$ and $n \geq 3$. The generalized Petersen graph $P(n,k)$, which was introduced in [9], is a cubic graph with $2n$ vertices, denoted by $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$, and all edges are of the form $u_iu_{i+1}$, $u_iv_i$, $v_iv_{i+k}$ for $0 \leq i \leq n-1$. In the absence of a special claim, all subscripts of vertices of $P(n,k)$ are taken modulo $n$ in the following.

Lemma 1 [6]. If $G$ is a simple cubic graph, then $\chi'_s(G) = 4$ if and only if $G$ covers the graph of the 3-cube $Q_3$ (as shown in Figure 1), where a graph $H$ is said to be covered by $G$ if there is a locally bijective graph homomorphism from $G$ to $H$.

Theorem 2. $\chi'_s(P(n,k)) = 4$ if and only if $n$ is a multiple of 4 and $k$ is an odd number.

Proof. Consider an arbitrary generalized Petersen graph $P(n,k)$ with $n \equiv 0 \pmod{4}$ and $k \equiv 1 \pmod{2}$. We then prove that $P(n,k)$ covers $Q_3$. Define a
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By the structure of \( V(y) \) of \( \phi \), the three neighbors of \( \phi(\mathbf{u}) \) in \( \mathbf{Q}_3 \) are \( x_i \mod 4, x_{i-1} \mod 4 \) and \( y_i \mod 4 \). Therefore, \( N_{\mathbf{Q}_3}(\phi(\mathbf{u})) = \{ \phi(\mathbf{u}_{i+1}), \phi(\mathbf{u}_{i-1}), \phi(\mathbf{v}_i) \} \). Now, we consider a vertex \( \mathbf{v}_i \) in \( \mathbf{P}(n,k) \). The three neighbors of \( \mathbf{v}_i \) in \( \mathbf{P}(n,k) \) are \( \mathbf{u}_i, \mathbf{v}_{i+k}, \mathbf{v}_{i-k} \), and the three neighbors of \( \phi(\mathbf{v}_i) = y_i \mod 4 \) in \( \mathbf{Q}_3 \) are \( x_i \mod 4, y_{i+1} \mod 4, y_{i-1} \mod 4 \). Observe that \( k \) is an odd number, which implies that \( i + k \mod 4 \neq i - k \mod 4 \), and \( i + k \mod 2 = i - k \mod 2 \neq i \mod 2 \). Therefore, \( \{ \phi(\mathbf{v}_{i+k}), \phi(\mathbf{v}_{i-k}) \} = \{ y_{i+1} \mod 4, y_{i-1} \mod 4 \} \), that is, \( N_{\mathbf{Q}_3}(\phi(\mathbf{v}_i)) = \{ \phi(\mathbf{u}_i), \phi(\mathbf{v}_{i+k}), \phi(\mathbf{v}_{i-k}) \} \). Hence, \( \mathbf{P}(n,k) \) covers \( \mathbf{Q}_3 \), and \( \chi_s(\mathbf{P}(n,k)) = 4 \) by Lemma 1.

For the inverse implication, suppose that \( \mathbf{P}(n,k) \) has a star 4-edge-coloring \( f \). For any vertex \( w \in V(\mathbf{P}(n,k)) \), define a (vertex) 4-coloring \( f' \) of \( \mathbf{P}(n,k) \) by letting \( f'(w) \) be the unique color that is missing on edges incident with \( w \) under \( f \). Then, the three neighbors of any vertex are assigned to different colors under \( f' \). Otherwise, assume that there exist some vertex \( w \) and its two neighbors \( w_1, w_2 \) in \( \mathbf{P}(n,k) \) satisfying \( f'(w) = c_1, f'(w_1) = f'(w_2) = c_2, f(ww_1) = c_3 \) and \( f(ww_2) = c_4 \), where \( \{ c_1, c_2, c_3, c_4 \} = \{ 1, 2, 3, 4 \} \). Then color \( c_4 \) appears on an edge incident with \( w_1 \), and \( c_3 \) appears on an edge incident with \( w_2 \). This creates a bichromatic path or cycle of length 4. Thus, if \( f'(w) = c_1 \), the incident edges and adjacent vertices of \( w \) are \( c_2, c_3, c_4 \) under \( f \) and \( f' \), respectively. There are exactly two possibilities as follows: either the edges incident with \( w \) colored \( c_2, c_3, c_4 \) lead to corresponding vertices \( (w's \) neighbors) colored \( c_3, c_4, c_2 \), respectively, or to corresponding vertices colored \( c_4, c_2, c_3 \). These two possibilities are called the local color pattern at \( w \). Then, \( f \) and \( f' \) induce a covering map \( \Phi: \mathbf{P}(n,k) \rightarrow \mathbf{Q}_3 \) such that for each \( w \in V(\mathbf{P}(n,k)) \), \( f'(w) = f'((\Phi(w)) \) (we use \( f' \) also for the vertex coloring of \( \mathbf{Q}_3 \) shown in Figure 1), and \( w \) and \( \Phi(w) \) have the same local color pattern.
Let $X_i$ and $Y_i$ denote the set of vertices of $P(n, k)$ that are mapped to $x_i$ and $y_i$, respectively, under $\Phi$, $i = 0, 1, 2, 3$. Thus, under $f'$ vertices in $X_0$ and $Y_2$ are colored with 1, in $X_1$ and $Y_3$ vertices are colored with 2, in $X_2$ and $Y_0$ vertices are colored with 3, and in $X_3$ and $Y_1$ vertices are colored with 4.

**Claim.** $|X_i| = |Y_j| = \frac{n}{4}$ for $i, j \in \{0, 1, 2, 3\}$.

**Proof.** Observe that by the definition of $\Phi$, for every vertex $w \in X_0$, there is exactly one neighbor of $w$ that belongs to $Y_0$; for every vertex $w' \in Y_0$, there is exactly one neighbor of $w'$ that belongs to $X_0$. This implies that there is a bijection between $X_0$ and $Y_0$. Therefore, $|X_0| = |Y_0|$. Analogously, we have $|X_0| = |X_1| = |X_3|$, $|X_1| = |X_2| = |Y_1|$, $|X_2| = |X_3| = |Y_2|$, and $|X_3| = |Y_3|$. Therefore, $|X_i| = |Y_j|$, and $|V(P(n,k))| = 2n = 8|X_i|$, which indicates that $n = 4|X_i|$, and the claim holds.

Clearly, $n$ is a multiple of 4 by the above claim. In what follows, we show $k$ is an odd number.

From the definition of covering projections, we see that every cycle of length $\ell$ in $P(n, k)$ is mapped to a cycle of length $\ell'$ in $Q_3$ such that $\ell = m\ell'$ for some nonnegative integer $m$. Therefore, the cycle $C = u_0u_1 \cdots u_nu_0$ is mapped to a cycle $C'$ of length 4 or 8. Note that $Q_3$ is a bipartite graph that does not contain any cycle with odd number of vertices. In addition, if $C'$ is a 6-cycle, then with a similar analysis as below, the subgraph of $Q_3$ induced by vertices corresponding to $v_0, v_1, \ldots, v_{n-1}$ consists of two paths with length 1 and a contraction.

If $C'$ is a cycle of length 4, without loss of generality, it is assumed that $C' = x_0x_1y_1y_0x_0$, and then any 4 consecutive vertices on $C$ are mapped to $x_0, x_1, y_1, y_0$ in one order of $(x_0, x_1, y_1, y_0), (y_1, y_0, x_0, x_1)$ or $(y_0, x_0, x_1, y_1)$. In this way, we can assume the following without the loss of generality

$$
\Phi(u_i) = \begin{cases} 
  x_0, i \equiv 0 \pmod{4}, \\
  x_1, i \equiv 1 \pmod{4}, \\
  y_1, i \equiv 2 \pmod{4}, \\
  y_0, i \equiv 3 \pmod{4}.
\end{cases}
$$

Then,

$$
\Phi(v_i) = \begin{cases} 
  x_3, i \equiv 0 \pmod{4}, \\
  x_2, i \equiv 1 \pmod{4}, \\
  y_2, i \equiv 2 \pmod{4}, \\
  y_3, i \equiv 3 \pmod{4},
\end{cases}
$$

$x_3y_2 \notin E(Q_3)$ and $x_2y_3 \notin E(Q_3)$, so the vertex mapped to $x_3$ (or $x_2$) is not adjacent to the vertex mapped to $y_2$ or $x_3$ (or $y_3$ or $x_2$) in $P(n, k)$. Therefore, $k$ is an odd number in this case.
If \( C' \) is a cycle of length 8, then \( n \) is a multiple of 8, and \( C' \) is a Hamilton cycle such as \( C' = x_0x_1x_2x_3y_3y_2y_1y_0x_0 \). Clearly, any 8 consecutive vertices on \( C \) are mapped to \( x_0, x_1, x_2, x_3, y_3, y_2, y_1, y_0 \), preserving the adjacent relation in \( C' \).

Without loss of generality, we assume

\[
\Phi(u_i) = \begin{cases} 
  x_0, i \equiv 0 \pmod{8}, \\
  x_1, i \equiv 1 \pmod{8}, \\
  x_2, i \equiv 2 \pmod{8}, \\
  x_3, i \equiv 3 \pmod{8}, \\
  y_3, i \equiv 4 \pmod{8}, \\
  y_2, i \equiv 5 \pmod{8}, \\
  y_1, i \equiv 6 \pmod{8}, \\
  y_0, i \equiv 7 \pmod{8}.
\end{cases}
\]

Then, it follows that

\[
\Phi(v_i) = \begin{cases} 
  x_3, i \equiv 0 \pmod{8}, \\
  y_1, i \equiv 1 \pmod{8}, \\
  y_2, i \equiv 2 \pmod{8}, \\
  x_0, i \equiv 3 \pmod{8}, \\
  y_0, i \equiv 4 \pmod{8}, \\
  x_2, i \equiv 5 \pmod{8}, \\
  x_1, i \equiv 6 \pmod{8}, \\
  y_3, i \equiv 7 \pmod{8}.
\end{cases}
\]

Since in \( Q_3 \), \( x_3 \) is not adjacent to \( y_2, y_0, x_1 \) or \( x_3 \) itself, it follows that the vertex mapped to \( x_3 \) is not adjacent to the vertex mapped to \( y_2, y_0, x_1 \) or \( x_3 \), in \( P(n,k) \). Therefore, \( k \) is an odd number, which completes the proof.

3. Construction of Star 5-Edge-Colorings for \( P(n,k) \)

A list \( L \) of a graph \( G \) is a mapping from a finite set of colors (positive integers) to each vertex of \( G \). For any \( V' \subseteq V(G) \), \( L(V') \) denotes the set of colors that are assigned to the vertices of \( V' \), i.e., \( L(V') = \{ L(v) \mid v \in V' \} \). A proper edge-coloring \( f \) of \( G \) is called an irlist-edge-coloring if \( f(e) \notin L(u) \cup L(v) \) for any edge \( e(=uv) \in E(G) \). An edge-coloring of \( G \) is strong if any two edges within distance two apart receive different colors.

Let \( C = v_1v_2 \ldots v_nv_1 \) be a cycle of length \( m, m \geq 3 \). We call \( C \) a listed-cycle if \( C \) has a list \( L \) and refer to the colors in \( L(V(C)) \) as listed-colors of \( C \). In particular, if there are exactly two consecutive vertices \( v_i, v_{i+1} \) satisfying \( L(v_i) \) (respectively, \( L(v_{i+1}) \)) \( \neq L(v_j) \) and \( L(v_j) = L(v_{j'}) \) for all \( j, j' \in \{1, 2, \ldots, m\} \setminus \{i, i + 1\} \), then we say \( C \) is quaint and \( v_i \) and \( v_{i+1} \) are the quaint vertices of \( C \), where \( v_{m+1} = v_m \).
Lemma 3. Let $C = v_1v_2 \cdots v_mv_1$ be a cycle, $m \geq 3$ and $m \neq 5$. Then, $C$ has a star $3$-edge-coloring. Particularly, when $m \equiv 0 \pmod{3}$, $C$ has a strong edge-coloring using three colors.

Proof. We construct our desired colorings as follows. When $m \equiv 0 \pmod{3}$, we color edges $v_1v_2, v_2v_3, \ldots, v_mv_1$ with three colors 1, 2, 3, repeatedly. When $m \equiv 1 \pmod{3}$, we color edges $v_1v_2, v_2v_3, \ldots, v_{m-1}v_m$ with three colors 1, 2, 3, repeatedly, and $v_mv_1$ with color 2. When $m \equiv 2 \pmod{3}$, it follows that $m \geq 8$. We color edges $v_1v_2, v_2v_3, \ldots, v_{m-5}v_{m-4}$ with three colors 1, 2, 3, repeatedly, and color $v_{m-4}v_{m-3}, v_{m-3}v_{m-2}, v_{m-2}v_{m-1}, v_{m-1}v_m$ and $v_mv_1$ with 1, 2, 1, 3 and 2, respectively.

Lemma 4. Let $C = v_1v_2 \cdots v_mv_1$, $m \geq 3$, be a quiant listed-cycle with list $L$ such that $|L(v)| = 2$ for every $v \in V(C)$. Suppose that $v_{m-1}$ and $v_m$ are the two quiant vertices of $C$. If $L(v_i) \subseteq (L(v_{m-1}) \cup L(v_m))$ for $i \in \{1, 2, \ldots, m-2\}$, then

(1) when $m \equiv 1 \pmod{3}$, $C$ has a strong irlist-edge-coloring using at most two non-listed-colors;

(2) when $m \equiv 2 \pmod{3}$, $C$ has an irlist-edge-coloring using at most two non-listed-colors, for which any three consecutive edges receive different colors except $v_{m-2}v_{m-1}, v_{m-1}v_m$ and $v_mv_1$.

Proof. Let $L(v_i) = \{c_1, c'_1\}, i \in \{1, 2, \ldots, m-2\}$, and $L(v_{m-1}) = \{c_2, c'_2\}$, $L(v_m) = \{c_3, c'_3\}$. Since $L(v_i) \not\subseteq (L(v_{m-1}) \cup L(v_m))$, there exist three colors, say $c_1, c_2$ and $c_3$, such that $c_1 \in L(v_i)$ and $c_2 \not\in L(v_{m-1}) \cup L(v_m)$, $c_2 \in L(v_{m-1})$ and $c_2 \not\in \{c_1, c'_1\}$, and $c_3 \in L(v_m)$ and $c_3 \not\in \{c_1, c'_1\}$. Let $c_4, c'_4$ be two distinct non-listed-colors. We construct the desired irlist-edge-colorings $f$ of $C$ by the following four rules.

For (1), $m-1 \equiv 0 \pmod{3}$. If $c_2 \in \{c_3, c'_3\}$ and $c_3 \in \{c_2, c'_2\}$, let $f$ be the following: $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_4$, and for $i = 1, 2, \ldots, m-2$, $f(v_{i}v_{i+1}) = c_2$ when $i \equiv 1 \pmod{3}$, $f(v_{i}v_{i+1}) = c'_2$ when $i \equiv 2 \pmod{3}$ and $f(v_{i}v_{i+1}) = c_4$ when $i \equiv 0 \pmod{3}$ (Rule (*1)). Clearly, under $f$, any two edges within distance two receive distinct colors. Note that $c_1 \not\in L(v_{m-1}) \cup L(v_m)$ and $\{c_2, c_4, c'_4\} \cap \{c_1, c'_1\} = \emptyset$. Therefore, $f$ is a strong irlist-edge-coloring of $C$ using two non-listed-colors $c_4, c'_4$. If $c_2 \not\in \{c_3, c'_3\}$ or $c_3 \not\in \{c_2, c'_2\}$, then $c_2 \neq c_3$. Let $f$ be the following: $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_2$ (or $c_4$), and for $i = 1, 2, \ldots, m-2$, $f(v_{i}v_{i+1}) = c_3$ (or $c_2$) when $i \equiv 1 \pmod{3}$, $f(v_{i}v_{i+1}) = c_4$ (or $c_3$) when $i \equiv 2 \pmod{3}$ and $f(v_{i}v_{i+1}) = c_2$ (or $c_4$) when $i \equiv 0 \pmod{3}$ (Rule (*2)). Additionally, under $f$, any two edges within distance two receive distinct colors. Since $\{c_2, c_3\} \cap \{c_1, c'_1\} = \emptyset$ and $c_1 \not\in L(v_{m-1}) \cup L(v_m)$, it holds that $f$ is a strong irlist-edge-coloring of $C$ using one non-listed-color $c_4$.

For (2), $m-2 \equiv 0 \pmod{3}$. If $c_2 = c_3$, let $f$ be $f(v_{m-1}v_m) = c_1$, $f(v_mv_1) = c_4$, and for $i = 1, 2, \ldots, m-2$, $f(v_{i}v_{i+1}) = c_2$ when $i \equiv 1 \pmod{3}$, $f(v_{i}v_{i+1}) = c'_4$
when $i \equiv 2 \pmod{3}$ and $f(u_ivi_{i+1}) = c_4$ when $i \equiv 0 \pmod{3}$ (Rule (⋆3)). By the definition of $f$, it has that $f(e) \neq f(e')$ for any $e, e' \in (E(C) \setminus \{v_{m_2}v_{m_1}, v_1v_{i+1}\})$ such that the distance between them is at most two. Additionally, $c_1 \notin L(v_{m_1}) \cup L(v_{m_2})$ and $\{c_2, c_4, c'_4\} \cap \{c_1, c'_1\} = \emptyset$. Therefore, $f$ is the desired irlist-edge-coloring of $C$ using two non-listed-colors $c_4, c'_4$.

If $c_2 \neq c_3$, let $f$ be the following: $f(v_{m_2}v_{m_1}) = c_1, f(v_1v_{i+1}) = c_4$, and for $i = 1, 2, \ldots, m - 2$, $f(v_1v_{i+1}) = c_2$ when $i \equiv 1 \pmod{3}$, $f(v_1v_{i+1}) = c_3$ when $i \equiv 2 \pmod{3}$ and $f(v_1v_{i+1}) = c_4$ when $i \equiv 0 \pmod{3}$ (Rule (⋆4)). Analogously, $f$ is the desired irlist-edge-coloring of $C$ using one non-listed-colors $c_4$.

**Theorem 5.** Let $\ell$ be the greatest common divisor of $n$ and $k$. When $\ell \geq 3$, with the exception of $\ell = 3$, $k \neq \ell$, and $\frac{n}{\ell} \equiv 1 \pmod{3}$, $P(n,k)$ has a star 5-edge-coloring.

**Proof.** Let $i_j = i + (j - 1)k$ for $j = 1, 2, \ldots, p = \frac{n}{\ell}$. Then, by the definition, the subgraph of $P(n,k)$ induced by $\{v_0, v_1, \ldots, v_{n-1}\}$ is the union of $\ell$ vertex-disjoint $p$-cycles, denoted by $C_i = v_{i_1}v_{i_2} \cdots v_{i_p}v_{i_1}$, $i = 0, 1, \ldots, \ell - 1$. Let $C = u_0u_1 \cdots u_{n-1}u_0$.

We first partition $C$ into five edge-disjoint paths as follows.

**Path-A.** $u_0u_1u_2, \ldots, u_{n-2k-1}u_{n-2k}$.

**Path-B.** $u_{n-2k}u_{n-2k+1}u_{n-2k+2} \cdots u_{n-2k+\ell-1}u_{n-2k+\ell}$.

**Path-C.** $u_{n-2k+\ell}u_{n-2k+\ell+1}u_{n-2k+\ell+2} \cdots u_{n-1}u_{n-k}$.

**Path-D.** $u_{n-k}u_{n-k+1}u_{n-k+2} \cdots u_{n-k+\ell-1}u_{n-k+\ell}$.

**Path-E.** $u_{n-k+\ell}u_{n-k+\ell+1}u_{n-k+\ell+2} \cdots u_{n-1}u_0$.

Note that the length of each path defined above is a multiple of $\ell$. Both Path-B and Path-D contain exactly $\ell$ edges, and when $k = \ell$, Path-C and Path-E are empty.

We now color edges of $C$ by coloring edges of Paths-A, C, E, B and D, respectively, according to the coloring rules indicated in Table 1. We distinguish 11 cases (each row denotes one case) based on values of $p$ and $\ell$. Each column contains 11 coloring rules of the corresponding paths (for example, the second column corresponds to Path-A, Path-C and Path-E). Each rule is a cyclic coloring of $\ell$ colors. When we use the rule to color the edges of the corresponding path, say $P = u_xu_{x+1} \cdots u_{x+m}$, we first partition the path into $q$ small paths of length $\ell(\geq 3)$, $P_1, P_2, \ldots, P_q$, where $P_1 = u_xu_{x+1} \cdots u_{x+\ell}, P_2 = u_{x+\ell+1}u_{x+\ell+2} \cdots u_{x+2\ell+1}, \ldots, P_q = u_{x+m-\ell}u_{x+m-\ell+1} \cdots u_{x+m}$; then, for each $P_i$, we color it from the first edge to the last edge one by one consecutively, according to the rule. For example, in the case of $p \equiv 1 \pmod{3}$ and $\ell \equiv 1 \pmod{3}$, if $P \in \{\text{Path-A, Path-C, Path-E}\}$, then we color $P_i$ ($P_i$ is a subgraph of $P$) with 1, 2, 3, 1, 2, 3, and 4 when $|E(P_i)| = 7$ and with 1, 2, 3, and 4 when $|E(P_i)| = 4$;
Table 1. Coloring rules of edges of $C$.

<table>
<thead>
<tr>
<th>values of $p$ and $\ell$</th>
<th>Path-A, Path-C, Path-E</th>
<th>Path-B</th>
<th>Path-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \equiv 0 \pmod{3}$ and $\ell \equiv 0 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3$, repeatedly</td>
<td>$\ell$ elements</td>
<td>$\ell$ elements</td>
</tr>
<tr>
<td>$p \equiv 0 \pmod{3}$ and $\ell \equiv 1 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4$, repeatedly</td>
<td>$\ell$ elements</td>
<td>$\ell$ elements</td>
</tr>
<tr>
<td>$p \equiv 0 \pmod{3}$ and $\ell \equiv 2 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4, 5$, repeatedly</td>
<td>$\ell$ elements</td>
<td>$\ell$ elements</td>
</tr>
<tr>
<td>$p \equiv 1 \pmod{3}$ and $\ell \equiv 0 \pmod{3}$</td>
<td>$\ell = k$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3$, repeatedly</td>
<td>by $4, 1, 3, \ldots, 4, 1, 3$</td>
</tr>
<tr>
<td>$p \equiv 1 \pmod{3}$ and $\ell \equiv 0 \pmod{3}$</td>
<td>$\ell \neq k$ and $\ell \geq 6$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3$, repeatedly</td>
<td>by $4, 1, 3, \ldots, 4, 1, 3, 2, 4, 3$</td>
</tr>
<tr>
<td>$p \equiv 1 \pmod{3}$ and $\ell \equiv 1 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4, 5$, repeatedly</td>
<td>$\ell$ elements</td>
<td>by $2, 3, 1, \ldots, 2, 3, 1, 2, 3, 5, 4$</td>
</tr>
<tr>
<td>$p \equiv 1 \pmod{3}$, $\ell \equiv 2 \pmod{3}$</td>
<td>$\ell \geq 8$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4, 5$, repeatedly</td>
<td>by $3, 2, 4, \ldots, 3, 2, 4, 1, 3, 4, 2, 5$</td>
</tr>
<tr>
<td>$p \equiv 2 \pmod{3}$ and $\ell \equiv 0 \pmod{3}$</td>
<td>$\ell = 5$</td>
<td>by $1, 2, 3, 4, 5$, repeatedly</td>
<td>by $1, 3, 4, 5, 3$</td>
</tr>
<tr>
<td>$p \equiv 2 \pmod{3}$ and $\ell \equiv 1 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4$, repeatedly</td>
<td>$\ell$ elements</td>
<td>by $2, 3, 5, \ldots, 2, 3, 5, 2, 3, 5, 4$</td>
</tr>
<tr>
<td>$p \equiv 2 \pmod{3}$ and $\ell \equiv 2 \pmod{3}$</td>
<td>by $1, 2, 3, \ldots, 1, 2, 3, 1, 2, 3, 4, 5$, repeatedly</td>
<td>$\ell$ elements</td>
<td>by $2, 4, 1, \ldots, 2, 4, 1, 2, 4, 1, 3, 5$</td>
</tr>
</tbody>
</table>
if $P \in \{\text{Path-B}, \text{Path-D}\}$, then we color $P_i$ with $2, 3, 1, 2, 3, 5$ and $4$ when $|E(P_i)| = 7$ and with $2, 3, 5$, and $4$ when $|E(P_i)| = 4$.

The resulting coloring of $C$ is denoted by $f$. One can readily check that $f$ is a strong edge-coloring. We now assign list $L$ to $C_i$ for $i = 0, 1, \ldots, \ell - 1$. Let

$$L(v_i) = \{f(u_iu_{i+1}), f(u_iu_{i-1})\}, \quad i = 0, 1, \ldots, n - 1.$$ 

Then, we obtain $\ell$ listed-cycles $C_i$ of length $p = \frac{n}{\ell}$, $i = 0, 1, \ldots, \ell - 1$.

**Case 1.** When $p \equiv 0 \pmod{3}$, then $|L(V(C_i))| = 2$ (since $k$ is a multiple of $\ell$) for each $i \in \{0, 1, \ldots, \ell - 1\}$. Observe that $|V(C_i)| = p \equiv 0 \pmod{3}$. Hence, by Lemma 3, $C_i$ has a strong list-edge-coloring with $\{1, 2, 3, 4, 5\} \setminus \{x, y\}$, where $x, y$ are the two listed-colors of $C_i$.

**Case 2.** When $p \equiv 1 \pmod{3}$, we further consider the following three subcases.

**Case 2.1.** $\ell \equiv 0 \pmod{3}$. First, $\ell = k$. Then, $C_i$ is a listed-cycle such that (1) $L(v_{ij}) = \{1, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 4\}$; or (2) $L(v_{ij}) = \{1, 2\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{4, 5\}$; or (3) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 3\}$; or (4) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 3\}$; (5) $L(v_{ij}) = \{1, 2\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 4\}$; (6) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 4\}$.

**Case 2.2.** $\ell \equiv 1 \pmod{3}$. Then, for $j \in \{1, 2, \ldots, p - 2\}$, it follows that (1) $L(v_{ij}) = \{1, 4\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 4\}$; or (2) $L(v_{ij}) = \{1, 2\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 3\}$; or (3) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 3\}$; or (4) $L(v_{ij}) = \{1, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 2\}$; or (5) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 5\}$; or (6) $L(v_{ij}) = \{3, 4\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{4, 5\}$.

**Case 2.3.** $\ell \equiv 2 \pmod{3}$. First, when $\ell \geq 8$, it has that for $j \in \{1, 2, \ldots, p - 2\}$, (1) $L(v_{ij}) = \{1, 5\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 5\}$; or (2) $L(v_{ij}) = \{1, 2\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 3\}$; or (3) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 3\}$; or (4) $L(v_{ij}) = \{1, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 4\}$; or (5) $L(v_{ij}) = \{1, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 4\}$; or (6) $L(v_{ij}) = \{1, 2\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{1, 3\}$; or (7) $L(v_{ij}) = \{2, 3\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{3, 4\}$; or (8) $L(v_{ij}) = \{3, 4\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 4\}$; or (9) $L(v_{ij}) = \{4, 5\}$ and $L(v_{ip-1}) = L(v_{ip}) = \{2, 5\}$.

Second, when $\ell = 5$, $C_i$ is a listed-cycle such that (1) $L(v_{ij}) = \{1, 5\}$ for $j \in \{1, 2, \ldots, p\} \setminus \{j', j'+1\}$, and $L(v_{ij'}) = L(v_{ij'+1}) = \{1, 3\}$, where $j', j'+1$ are
read model $p$; or (2) $L(v_i) = \{1, 2\}$ and $L(v_{p-1}) = \{1, 3\}, L(v_p) = \{1, 4\}$; or (3) $L(v_i) = \{2, 3\}$ and $L(v_{p-1}) = \{3, 4\}, L(v_p) = \{4, 5\}$; or (4) $L(v_i) = \{3, 4\}$ and $L(v_{p-1}) = \{4, 5\}$; or (5) $L(v_i) = \{3, 5\}$ and $L(v_{p-1}) = \{3, 5\}$, where $j \in \{1, 2, \ldots, p-2\}$ in (2)–(5).

Obviously, in each of the above subcases, $C_i$ is a quaint listed-cycle satisfying the condition of Lemma 4(1). Therefore, $C_i$ has a strong irlist-edge-coloring using some colors in $\{1, 2, 3, 4, 5\}$ by Rules ($\star$1) and ($\star$2).

**Case 3.** When $p \equiv 2 \pmod{3}$, there are also three subcases that need to deal with.

**Case 3.1.** $\ell \equiv 0 \pmod{3}$. Then, one of the following holds. For $j \in \{1, 2, \ldots, p-2\}$, (1) $L(v_i) = \{1, 3\}$ and $L(v_{p-1}) = \{3, 4\};$ (2) $L(v_i) = \{1, 2\}$ and $L(v_{p-1}) = L(v_p) = \{4, 5\};$ (3) $L(v_i) = \{2, 3\}$ and $L(v_{p-1}) = L(v_p) = \{3, 5\}.$

**Case 3.2.** $\ell \equiv 1 \pmod{3}$. Then, for $j \in \{1, 2, \ldots, p-2\}$, it has that (1) $L(v_i) = \{1, 4\}$ and $L(v_{p-1}) = L(v_p) = \{2, 4\};$ or (2) $L(v_i) = \{1, 2\}$ and $L(v_{p-1}) = L(v_p) = \{2, 3\};$ or (3) $L(v_i) = \{2, 3\}$ and $L(v_{p-1}) = L(v_p) = \{1, 4\};$ or (4) $L(v_i) = \{1, 3\}$ and $L(v_{p-1}) = L(v_p) = \{2, 5\};$ or (5) $L(v_i) = \{3, 4\}$ and $L(v_{p-1}) = L(v_p) = \{4, 5\}.$

**Case 3.3.** $\ell \equiv 2 \pmod{3}$. Then, for $j \in \{1, 2, \ldots, p-2\}$, one of the following situations holds. (1) $L(v_i) = \{1, 5\}$ and $L(v_{p-1}) = L(v_p) = \{2, 5\};$ (2) $L(v_i) = \{1, 2\}$ and $L(v_{p-1}) = L(v_p) = \{2, 4\};$ (3) $L(v_i) = \{2, 3\}$ and $L(v_{p-1}) = L(v_p) = \{1, 4\};$ (4) $L(v_i) = \{1, 3\}$ and $L(v_{p-1}) = L(v_p) = \{1, 2\};$ (5) $L(v_i) = \{3, 4\}$ and $L(v_{p-1}) = L(v_p) = \{1, 3\};$ (6) $L(v_i) = \{4, 5\}$ and $L(v_{p-1}) = L(v_p) = \{3, 5\}.$

One can readily check that in Cases 3.1–3.3, $C_i$ is also a quaint listed-cycle. Therefore, $C_i$ has a strong irlist-edge-coloring using colors 1, 2, 3, 4, and 5 by Rules ($\star$3) and ($\star$4) in Lemma 4(2).

Until now, we have colored edges of $C_i$, $i = 0, 1, \ldots, \ell - 1$. We denote the resulting coloring of $C_i$ by $f'$. Obviously, for each $i \in \{0, 1, \ldots, n-1\}$, it has that $|\{f(u_i u_{i+1}), f(u_i u_{i-1}), f'(v_i v_{i+1}), f'(v_i v_{i-1})\}| = 4$. We then color each $u_i v_i$ with the unique color $\{1, 2, 3, 4, 5\} \setminus \{f(u_i u_{i+1}), f(u_i u_{i-1}), f'(v_i v_{i+1}), f'(v_i v_{i-1})\}$. This completes the edge-coloring of $P(n, k)$. We now show that such the coloring is a star 5-edge-coloring.

If not, let $P$ be a bichromatic 4-path. Since $f$ is a strong edge-coloring of $C_i$, and $\{f(u_i u_{i+1}), f(u_i u_{i-1})\} \cap \{f'(v_i v_{i+1}), f'(v_i v_{i-1})\} = \emptyset$ for any $i \in \{0, 1, \ldots, n-1\}$, $P$ does not contain any edges of $C$. In addition, by Lemma 4, any three edges of $C_i$ receive different colors under $f'$, except $v_{ip-2}v_{ip-1}, v_{ip-1}v_p, v_pv_{ip}$. Therefore, $P = v_{p-2}v_{p-1}v_pv_iu_i$ or $P = u_{ip-2}v_{ip-2}v_{ip-1}v_pv_i$. However, by Lemma 4 Rule ($\star$3) and ($\star$4), $f'(v_{ip-1}v_p)$ is a listed-color not in $L(v_{ip})$. Then, by the coloring rule of $u_iv_i$, $i = 0, 1, \ldots, n - 1$, it has that $f'(v_{ip-1}v_p) \neq f(v_i u_i)$.
and \( f'(v_{ip-1}v_{ip}) \neq f(u_{ip-2}v_{ip-2}) \). Therefore, \( P \) is not bichromatic, and it is a contradiction.

**Lemma 6.** Let \( P(n,k) \) be a generalized Petersen graph such that \( \text{GCD}(n,k) = 1 \), \( n \equiv 0 \pmod{2} \) and \( k \equiv 1 \pmod{2} \). Then, \( P(n,k) \) has a star 5-edge-coloring.

**Proof.** It is sufficient to construct a star 5-edge-coloring for \( P(n,k) \) in this case. Let \( C = u_0u_1 \cdots u_{n-1}u_0 \) be the cycle induced by \( \{u_0,u_1,\ldots,u_{n-1}\} \). Since \( \text{GCD}(n,k) = 1 \), i.e., \( n, k \) are coprime, the subgraph induced by \( \{v_0,v_1,\ldots,v_{n-1}\} \) is also a cycle, denoted by \( C' \). Since \( n \equiv 0 \pmod{2} \), it follows that \( n \not\equiv 5 \) and by Lemma 3 both \( C \) and \( C' \) have a star 3-edge-coloring. Let \( f_1 \) and \( f_2 \) be the two star edge-colorings of \( C \) and \( C' \), respectively, using colors 1, 2, and 3. Then, we color \( u_iv_i \) with 4 when \( i \equiv 0 \pmod{2} \) and with 5 when \( i \equiv 1 \pmod{2} \), for \( i = 0,1,\ldots,n-1 \). Denote by \( f_3 \) the resulting coloring, and let \( f = f_1 \cup f_2 \cup f_3 \). We now show that \( f \) is a star edge-coloring. On the contrary, we assume there is a bichromatic 4-path \( P \). Let \( c_1 \) and \( c_2 \) be the two colors appearing on the edges of \( P \). By \( f_1 \) and \( f_2 \), it is by no means that \( \{c_1,c_2\} \subset \{1,2,3\} \). In addition, since \( (n,k) = 1 \), \( n \equiv 0 \pmod{2} \) and \( k \equiv 1 \pmod{2} \), \( f_3(u_iv_i) \neq f_3(u_{i+1}v_{i+1}) \) and \( f_3(u_{n-k}v_{n-k}) \neq f_3(u_{n-k+1}v_{n-k+1}) \). Therefore, together with \( f_3 \), \( \{c_1,c_2\} \cap \{4,5\} = \emptyset \). Hence, \( P \) is not bichromatic and is a contradiction.

**Theorem 7.** \( P(n,1), n \geq 5 \), has a star 5-edge-coloring.

**Proof.** By Lemma 6, we only need to consider the case \( n \equiv 1 \pmod{2} \). In this case, we can also obtain a star 5-edge-coloring by a slight change of the coloring in Lemma 6. Let \( P_1 = u_{n-1}u_0u_1 \cdots u_{n-2} \) and \( P_2 = v_0v_1 \cdots v_{n-1} \) be two paths. We now define a star 3-edge-coloring \( f_1 \) of \( P_1 \) as follows. First, let \( f_1(u_{n-1}u_0) = 2, f_1(u_0u_1) = f_1(u_{n-3}u_{n-2}) = 3 \). Then, color edges of sub-path \( u_1u_2 \cdots u_{n-3} \) as follows: when \( n = 5 \), let \( f_1(u_1u_2) = 1 \); when \( n \geq 7 \), color edges \( u_1u_2u_3, \ldots, u_{n-4}u_{n-3} \) by 1, 3, and 2, repeatedly, if \( n - 4 \equiv 0 \pmod{3} \); by 1, 3, 2, \ldots, 1, 3, 2, 1 and 2 if \( n - 4 \equiv 2 \pmod{3} \). Obviously, \( P_2 \) also has a star 3-edge-coloring, say \( f_2 \). By color permutation, we can assume \( f_2(v_{n-3}v_{n-2}) = 3 \), and \( f_2(v_{n-2}v_{n-1}) = 2 \). Based on \( f_1 \) and \( f_2 \), we color \( u_{n-2}u_{n-1} \) with 4 and \( v_{n-1}v_0 \) with 5. And for any \( i \in \{0,1,\ldots,n-2\} \), color \( u_iv_i \) with 4 for \( i \equiv 0 \pmod{2} \) and with 5 for \( i \equiv 1 \pmod{2} \), and finally, color \( u_{n-1}v_{n-1} \) with 1. Until now, we typically obtain an edge-coloring of \( P(n,1) \). One can easily see that such the coloring is a star 5-edge-coloring.

Note that when \( n = 3 \), Theorem 7 by no means hold. However, by a coloring \( P(n,3) \) with an exhausting search, we can see that \( P(n,3) \) does not contain any star 5-edge-coloring.
Lemma 8. Let $P(n, k)$ be a generalized Petersen graph such that $(n, k) = 2$. Let $C_0 = v_0v_k\cdots v_{(n-1)k}v_0$. If $C_0$ has a star 3-edge-coloring $f$ such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \in \{0, 1, \ldots, n-1\}$ and $i \equiv 0 \pmod{2}$, then $P(n, k)$ has a star 5-edge-coloring, where $C_f(v_i) = \{f(v_{i+k}), f(v_{i-k})\}$.

Proof. Since GCD$(n, k) = 2$, it has that $n$ is an even number. Let $f$ be a star 3-edge-coloring of $C_0$, such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \equiv 0 \pmod{2}$ and $i \in \{0, 1, \ldots, n-1\}$. We now color $C_1 = v_1v_{1+k}\cdots v_{1+(\frac{n}{2}-1)k}v_1$ with the same pattern as $C_0$, that is, color each edge $v_jv_{j+k}$ with the color $f(v_{j-1}v_{j+k-1})$, for $j \equiv 1 \pmod{2}$ and $j \in \{0, 1, \ldots, n-1\}$. Denote the resulting coloring of $C_0$ and $C_1$ also by $f$. Then, $C_f(v_i) = C_f(v_{i+1})$ for any $i = 0, 2, 4, \ldots, n-2$. Based on $f$, for any $i \in \{0, 1, \ldots, n-1\}$, we color $u_iu_{i+1}$ with the color in $\{1, 2, 3\} \setminus C_f(v_i)$ when $i \equiv 0 \pmod{2}$, and with 4 when $i \equiv 1 \pmod{2}$. Finally, color $u_iv_i$ with 5, $i = 0, 1, \ldots, n-1$. Obviously, such the coloring is a star 5-edge-coloring.

Theorem 9. $P(6m, 2)$ has a star 5-edge-coloring, where $m \geq 1$ is a positive number.

Proof. Let $n = 6m$, and $C_0 = v_0v_k\cdots v_{(n-1)k}v_0$. Obviously, $C_0$ has a star 3-edge-coloring $f$ such that $C_f(v_i) \neq C_f(v_{i+1})$ for any $i \in \{0, 1, \ldots, n-1\}$ and $i \equiv 0 \pmod{2}$ (since $\frac{n}{2} = 3m \equiv 0 \pmod{3}$, we can color edges of $C_0$ with 1, 2, 3, repeated). Therefore, by Lemma 8, $P(6m, 2)$ has a star 5-edge-coloring.

In this article, we determine the star chromatic index of generalized Petersen graphs $P(n, k)$ for “almost all” values of $n$ and $k$. By using more involved analysis, we can also prove $P(n, k)$ has a star 5-edge-coloring for some remaining values of $n$ and $k$, particularly, for the case $\ell = 3$, $k \neq \ell$, and $\frac{n}{2} \equiv 1 \pmod{3}$. However, we prefer to present a short or uniform proof. In addition, we would like to stress that only one generalized Petersen graph, i.e., $P(3, 1)$, is found to have the star chromatic index 6. Therefore, we conjecture that $P(n, k)$ has a star 5-edge-coloring for any $n \geq 4$.

References


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