LONGER CYCLES IN ESSENTIALLY 4-CONNECTED PLANAR GRAPHS

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Abstract

A planar 3-connected graph $G$ is called essentially 4-connected if, for every 3-separator $S$, at least one of the two components of $G - S$ is an isolated vertex. Jackson and Wormald proved that the length $\text{circ}(G)$ of a longest cycle of any essentially 4-connected planar graph $G$ on $n$ vertices is at least $\frac{2n+4}{3}$ and Fabrici, Harant and Jendrol improved this result to $\text{circ}(G) \geq \frac{1}{2}(n + 4)$. In the present paper, we prove that an essentially 4-connected planar graph on $n$ vertices contains a cycle of length at least $\frac{3}{5}(n + 2)$ and that such a cycle can be found in time $O(n^2)$.

Keywords: essentially 4-connected planar graph, longest cycle, circumference, shortness coefficient.

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For a finite and simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$, let $N(x)$ and $d(x) = |N(x)|$ denote the neighborhood and the degree of any $x \in V(G)$ in $G$, respectively. The circumference $\text{circ}(G)$ of a graph $G$ is the length of a longest cycle of $G$. A subset $S \subseteq V(G)$ is an $s$-separator of $G$ if $|S| = s$ and $G - S$ is disconnected. From now on, let $G$ be a 3-connected planar graph. For every 3-separator $S$ of $G$, it is well-known that $G - S$ has exactly two components. We call $S$ trivial if at least one component of $G - S$ is a single vertex. If every 3-separator $S$ of $G$ is trivial, we call the 3-connected planar graph $G$ essentially 4-connected. In the present paper, we are interested in lower bounds on the circumference of essentially 4-connected planar graphs.

Jackson and Wormald [4] proved that $\text{circ}(G) \geq 2n + 4$ for every essentially 4-connected planar graph on $n$ vertices and presented an infinite family of essentially 4-connected planar graphs $G$ such that $\text{circ}(G) \leq c \cdot n$ for each real constant $c > \frac{2}{3}$. Moreover, there is a construction of infinitely many essentially 4-connected planar graphs with $\text{circ}(G) = \frac{2}{5}(n + 4)$ (for example see [2]). It is open whether there exists an essentially 4-connected planar graph $G$ on $n$ vertices with $\text{circ}(G) < \frac{2}{5}(n + 4)$. Further results on the length of longest cycles in essentially 4-connected planar graphs can be found in [2, 3, 7].

Fabrici, Harant and Jendrol’ [2] extended the result of Jackson and Wormald by proving that $\text{circ}(G) \geq \frac{1}{2}(n + 4)$ for every essentially 4-connected planar graph $G$ on $n$ vertices.

Our result is presented in the following theorem.

**Theorem 1.** For any essentially 4-connected planar graph $G$ on $n$ vertices, $$\text{circ}(G) \geq \frac{3}{5}(n + 2).$$

We remark that the assertion of the theorem can be improved to $\text{circ}(G) \geq \frac{3}{5}(n + 4)$ if $n \geq 16$. This follows from using Lemma 5 in [2] and a more special version of the forthcoming inequality (i). We will also show how cycles of $G$ of length at least $\frac{3}{5}(n + 2)$ can be found in quadratic time.

Let $C$ be a plane cycle and let $B$ be a set disjoint from $V(C)$. A plane graph $H$ is called a $(B, C)$-graph if $B \cup V(C)$ is the vertex set of $H$, the cycle $C$ is an induced subgraph of $H$, the subgraph of $H$ induced by $B$ is edgeless, and each vertex of $B$ has degree 3 in $H$. The vertices in $B$ are called outer vertices of $C$.

A face $f$ of $H$ is called minor (major) if it is incident with at most one (at least two) outer vertices. Note that $f$ is incident with no outer vertex if and only if $C$ is the facial cycle of $f$.

For every $(B, C)$-graph $H$, let $\mu(H)$ denote the number of minor faces of $H$. Then

(i) $$\mu(H) \geq |V(H)| - |V(C)| + 2.$$
**Proof of (i).** Let $H$ be a smallest counterexample. Since $B = \emptyset$ implies $|V(H)| = |V(C)|$ and $\mu(H) = 2$, which satisfies the inequality (i), we may assume that $B$ is non-empty. For each vertex $y \in B$, the three neighbors of $y$ divide $C$ into three internally disjoint paths $P_1(y)$, $P_2(y)$, and $P_3(y)$ with endvertices in $N(y)$. We may assume that $|V(P_1(y))| \leq |V(P_2(y))| \leq |V(P_3(y))|$ and define $\phi(y) = |V(P_1(y))| + |V(P_2(y))| - 1$ in this case.

Let $x \in B$ be chosen such that $\phi(x) = \min\{\phi(y) \mid y \in B\}$. Consider the two cycles $A_1$ and $A_2$ induced by $V(P_1(x)) \cup \{x\}$ and $V(P_2(x)) \cup \{x\}$, respectively. We claim that the interior of $A_1$ as well as the interior of $A_2$ is a face of $H$ and hence, both are minor faces. Suppose that there is a vertex $z$ in the interior of $A_i$ for $i \in \{1, 2\}$. Then $\phi(z) = |V(P_1(z))| + |V(P_2(z))| - 1 \leq \max\{|V(P_1(x))|, |V(P_2(x))|\} < |V(P_1(x))| + |V(P_2(x))| - 1 = \phi(x)$, which contradicts the choice of $x$.

Let $H' = H - x$. Note that $H'$ is a $((B \setminus \{x\}), C)$-graph and has fewer vertices than $H$. Then $|V(H')| = |V(H)| - 1$, $\mu(H') \leq \mu(H) - 1$, and $\mu(H') \geq |V(H')| - |V(C)| + 2$, hence $\mu(H) \geq 1 + \mu(H') \geq 1 + |V(H')| - |V(C)| + 2 = |V(H)| - |V(C)| + 2$. 

**Proof of Theorem 1.** Let $G$ be an essentially 4-connected plane graph on $n$ vertices. If $G$ has at most 10 vertices, then it is well known that $G$ is Hamiltonian [1]. In this case, we are done, since $n \geq \frac{4}{3}(n + 2)$ for $n \geq 3$. Thus, we assume $n \geq 11$. A cycle $C$ of $G$ is called an outer-independent-3-cycle (O13-cycle) if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for every $x \in V(G) \setminus V(C)$. An edge $e = xy \in E(C)$ of a cycle $C$ is called an extendable edge of $C$ if $x$ and $y$ have a common neighbor in $V(G) \setminus V(C)$.

In [2], it is shown that every essentially 4-connected planar graph $G$ on $n \geq 11$ vertices contains an O13-cycle. In this proof, let $C$ be a longest O13-cycle of $G$, let $c = |V(C)|$, and let $H$ be the graph obtained from $G$ by removing all chords of $C$, i.e. by removing all edges in $E(G) \setminus E(C)$ that connect vertices of $C$. Clearly, $C$ does not contain an extendable edge. Obviously, $H$ is a $(B, C)$-graph, with $B = V(H) \setminus V(C)$.

For the number $\mu$ of minor faces of $H$, we have by (i) $\mu \geq n - c + 2$.

Moreover, we will show

$$6 \mu \leq 4c$$

and then, the theorem follows immediately.

**Proof of (ii).** An edge $e$ of $C$ is incident with exactly two faces $f_1$ and $f_2$ of $H$. In this case, we say $f_1$ is opposite to $f_2$ with respect to $e$. A face $f$ of $H$ is called $j$-face if it is incident with exactly $j$ edges of $C$ and the edges of $C$ incident with $f$ are called $C$-edges of $f$. Because $C$ does not contain an extendable edge, we have $j \geq 2$ for every minor $j$-face of $H$. 
We define a weight function $w_0$ on the set $F(H)$ of faces of $H$, by setting weight $w_0(f) = 6$ for every minor face $f$ of $H$ and weight $w_0(f) = 0$ for every major face $f$ of $H$. Then $\sum_{f \in F(H)} w_0(f) = 6 \mu$. Next, we redistribute the weights of faces of $H$ by the rules R1 and R2.

**Rule R1.** A minor 2-face $f$ of $H$ sends weight 1 through both $C$-edges to the opposite (possibly identical) faces.

**Rule R2.** A minor 3-face $f$ of $H$ with $C$-edges $ux$, $xy$, and $yz$ sends weight 1 through its middle $C$-edge $xy$ to the opposite face.

Let $w_1$ denote the new weight function; clearly, $\sum_{f \in F(H)} w_1(f) = 6 \mu$ still holds.

For the proof of (ii), we will show $w_1(f) \leq 2j$ for each $j$-face $f$ of $H$.

To see that (ii) is a consequence of (iii), let each $j$-face $f$ of $H$ satisfying $j \geq 1$ send the weight $\frac{w_1(f)}{j}$ to each of its $C$-edges. Note that each 0-face $f$ is major, thus $w_1(f) = 0$. Hence, the total weight of all minor and major faces is moved to the edges of $C$. Since every edge of $C$ gets weight at most 4, we obtain $6 \mu = \sum_{f \in F(H)} w_1(f) \leq 4c$, and (ii) follows.

**Proof of (iii).** Next we distinguish several cases. In most of them, we construct a cycle $\tilde{C}$ that is obtained from $C$ by replacing a subpath of $C$ with another path. In every case, $\tilde{C}$ will be an OI3-cycle of $G$ that is longer than $C$. This contradicts the choice of $C$ and therefore shows that the considered case cannot occur. Note that all vertices of $C$ in the following figures are different, because the length of the longest OI3-cycle $C$ in a planar graph on $n \geq 11$ vertices is at least 8 [2, Lemma 4(ii)].

**Case 1.** $f$ is a major $j$-face. Because $w_0(f) = 0$ and $f$ gets weight $\leq 1$ through each of its $C$-edges, we have $w_1(f) \leq j$.

**Case 2.** $f$ is a minor 2-face (see Figure 1). We will show that $f$ does not get any new weight by R1 or by R2; this implies $w_1(f) = w_0(f) - (1 + 1) = 4$. Let $xy$ and $yz$ be the $C$-edges of $f$ and $a$ be the outer vertex incident with $f$ (see Figure 1).

If $f$ gets new weight by R1 or by R2 from a face $f'$ opposite to $f$ with respect to a $C$-edge of $f$, then $f'$ is a minor 2-face or a minor 3-face of $H$. Without loss of generality, we may assume that $f'$ is opposite to $f$ with respect to the edge $yz$. Then $yz$ is a common $C$-edge of $f$ and $f'$ and we distinguish the following subcases.

**Case 2a.** $f'$ is a 2-face and $xy$ is a $C$-edge of $f'$. Then $\{x, z\}$ is the neighborhood of $y$ in $G$, which contradicts the 3-connectedness of $G$. 


Case 2b. \( f' \) is a 2-face and \( xy \) is not a \( C \)-edge of \( f' \) (see Figure 2). Then a longer OI3-cycle \( \tilde{C} \) is obtained from \( C \) by replacing the path \((x, y, z, u)\) with the path \((x, a, z, y, u, b)\), which gives a contradiction.

Case 2c. \( f' \) is a 3-face. Since \( f' \) sends weight to \( f \), then, by rule R2, a \( C \)-edge of \( f \) is the middle \( C \)-edge of \( f' \). It follows that both \( C \)-edges of \( f \) are also \( C \)-edges of \( f' \) and the situation as shown in Figure 3 occurs. The edge \( yu \) exists in \( G \), because otherwise \( d(y) = 2 \) and \( G \) would not be 3-connected. Then \( \tilde{C} \) is obtained from \( C \) by replacing the path \((x, y, z, u)\) with the path \((x, a, z, y, u)\).

Case 3. \( f \) is a minor 3-face (see Figure 4). Since \( f \) loses weight 1 by rule R2 and possibly gets weight \( w \) by R1 or by R2, we have
\[ w_1(f') = 5 + w. \]

If \( w \leq 1 \), then we are done.

If \( w \geq 2 \), then \( f \) does not get any weight through the edge \( xy \) from the opposite face \( f' \). Otherwise, if \( f' \) is a 2-face, then we have the situation as in Case 2c and if \( f' \) is a 3-face, then \( w = 1 \), with contradiction in both cases. Hence, \( f \) gets weight 1 through \( vx \) from the opposite face \( f_1 \) and weight 1 through \( yz \) from the opposite face \( f_2 \). Clearly, \( f_1 \neq f_2 \) and they are not simultaneously 3-faces.

Case 3a. Both \( f_1 \) and \( f_2 \) are 2-faces. The situation is as illustrated in Figure 5 and \( \tilde{C} \) is obtained from \( C \) by replacing the path \((w, v, x, y, z, u)\) with the path \((w, b, x, v, a, z, y, c, u)\). Note that \( b \neq c \), because \( d(b) = d(c) = 3 \).
Case 3b. \( f_1 \) is a 2-face and \( f_2 \) is a 3-face. Then \( e_2 = yz \) is the middle \( C \)-edge of \( f_2 \), as shown in Figure 6, and \( \tilde{C} \) is obtained from \( C \) by replacing the path \((w, v, x, y, z, u)\) with the path \((w, v, a, z, y, x, c, u)\).

Case 4. \( f \) is a minor 4-face (see Figure 7).

If \( w_1(f) = w_0(f) + w = 6 + w \) and \( w \leq 2 \), then we are done.

If otherwise \( w \geq 3 \), there are at least three edges \( e_1, e_2, \) and \( e_3 \) among the four \( C \)-edges \( vw, wx, xy, \) and \( yz \) of \( f \) such that \( f \) gets weight from minor faces which are opposite to \( f \) with respect to \( e_1, e_2, \) and \( e_3 \), respectively.

Case 4a. \( w = 3 \) and \( \{e_1, e_2, e_3\} = \{vw, wx, xy\} \). Then no edge of \( \{e_1, e_2, e_3\} \) is the middle \( C \)-edge of a minor 3-face and \( yz \) is not a \( C \)-edge of a minor 2-face. We have the situation of Figure 8 and one of the edges \( vx \) or \( xz \) exists in \( G \), because otherwise \( x \) would have degree 2 in \( G \).
Then $\tilde{C}$ is obtained again from $C$ by replacing the path $(v, w, x, y, z)$ with the path $(v, x, w, c, y, z)$ or with the path $(v, w, c, y, x, z)$, respectively.

**Case 4b.** $w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ and $wx$ is not a $C$-edge of a minor 3-face. Then $vw$ is not the middle $C$-edge of a minor 3-face opposite to $f$. We have the situation of Figure 9 and one of the edges $vy$ or $wy$ exists in $G$, because otherwise $y$ would have degree 2 in $G$.

Note that $b \neq c$, because $d(b) = d(c) = 3$. Then $\tilde{C}$ is obtained from $C$ by replacing the path $(t, v, w, x, y, z)$ with the path $(t, b, v, w, y, x, c, z)$ or with the path $(t, v, w, y, x, c, z)$.

**Case 4c.** $w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ and $wx$ is a $C$-edge of a minor 3-face. Then $vw$ is the middle $C$-edge of a minor 3-face opposite to $f$ (see Figure 10).

Then at least one of the edges $vy$ or $wy$ exists, because otherwise $y$ would have degree 2 in $G$, and $\tilde{C}$ is obtained from $C$ by replacing the path $(t, v, w, x, y, z)$ with the path $(t, b, x, w, v, y, z)$ or with the path $(t, v, w, y, x, c, z)$.

**Case 4d.** $w = 4$. Then the edges $vw$, $wx$, $xy$, and $yz$ are $C$-edges of minor 2-faces of $H$. Either a situation similar to Case 4a occurs, a contradiction, or the situation of Figure 11 follows.
Then the edge $wy$ exists in $G$, because otherwise $d(w) = 2$ or $d(y) = 2$ in $G$, and $\tilde{C}$ is obtained from $C$ by replacing the path $(v, w, x, y, z)$ with the path $(v, w, y, x, c, z)$.

**Case 5.** $f$ is a minor 5-face. Let $w_1(f) = w_0(f) + w = 6 + w$. If $w \leq 4$, then $w_1(f) \leq 10$ and we are done. If $w = 5$, then all five $C$-edges of $f$ are also $C$-edges of minor 2-faces and we have the situation of Figure 12.

If the edge $vx$ exists, then $\tilde{C}$ is obtained from $C$ by replacing the path $(s, v, w, x)$ with the path $(s, b, w, v, x)$.

If $vx$ does not exist, then, because $d(v) \geq 3$, $y$ or $z$ is a neighbor of $v$. If the edge $vy$ exists, we get $d(x) = 2$, a contradiction. Hence, $vz$ exists and, since $d(x) \geq 3$, $xz$ exists as well. In this case, $\tilde{C}$ is obtained from $C$ by replacing the path $(w, x, y, z)$ with the path $(w, c, y, x, z)$.

The remaining case completes the proof of (iii) and therefore the proof of (ii).

**Case 6.** $f$ is a minor $j$-face with $j \geq 6$. Then $w_1(f) = w_0(f) + w = 6 + w \leq 6 + j \leq 2j$.

**Algorithm.** We now show that a cycle of length at least $\frac{3}{2}(n + 2)$ in any essentially 4-connected planar graph $G$ on $n$ vertices can be computed in time $O(n^2)$. For $n \leq 10$, we may compute even a longest cycle in constant time, so assume $n \geq 11$. The existential proof of the theorem proceeds by using a longest not extendable OI3-cycle of $G$. However, it is straightforward to observe that the proof is still valid when we replace this cycle by an OI3-cycle $C$ that is not extendable and for which none of the local replacements described in the Cases 1–6 can be applied to increase its length (as argued, all these replacements preserve an OI3-cycle).

It suffices to describe how such a cycle $C$ can be computed efficiently; the desired length of $C$ is then implied by the theorem. In [2, Lemma 3], an OI3-cycle
of $G$ is obtained by constructing a special Tutte cycle with the aid of Sander’s result on Tutte paths [5]. Using the recent result in [6], we can compute such Tutte paths and, by prescribing its end vertices accordingly, also the desired Tutte cycle in time $O(n^2)$. This gives an OI3-cycle $C_i$ of $G$.

If $C_i$ is extendable, we compute an extendable edge of $C_i$ and extend $C_i$ to a longer cycle $C_{i+1}$; this takes time $O(n)$ and preserves that $C_{i+1}$ is an OI3-cycle. Otherwise, if there is no extendable edge of $C_i$ (in this case, the length of $C_i$ is at least 8 due to $n \geq 11$ and [2, Lemma 4(ii)]), we decide in time $O(n)$ whether one of the local replacements of the Cases 1–6 can be applied to $C_i$. If so, we apply any such case and obtain the longer OI3-cycle $C_{i+1}$ (which however may be extendable); since all replacements modify only subgraphs of constant size, this can be done in constant time. Iterating this implies a total running time of $O(n^2)$, as the length of the cycle is increased at most $O(n)$ times.

References


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