INCIDENCE COLORING—COLD CASES

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Abstract

An incidence in a graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ is an edge of $G$ incident to $v$. Two incidences $(v, e)$ and $(u, f)$ are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge $vu$ is from the set \{e, f\}. An incidence coloring of $G$ is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the incidence chromatic number.

It was proved that at most $\Delta(G) + 5$ colors are enough for an incidence coloring of any planar graph $G$ except for $\Delta(G) = 6$, in which case at most 12 colors are needed. It is also known that every planar graph $G$ with girth at least 6 and $\Delta(G) \geq 5$ has incidence chromatic number at most $\Delta(G) + 2$.

In this paper we present some results on graphs regarding their maximum degree and maximum average degree. We improve the bound for planar graphs with $\Delta(G) = 6$. We show that the incidence chromatic number is at
most $\Delta(G) + 2$ for any graph $G$ with $\text{mad}(G) < 3$ and $\Delta(G) = 4$, and for any graph with $\text{mad}(G) < \frac{\Delta(G)}{2}$ and $\Delta(G) \geq 8$.

**Keywords:** incidence coloring, incidence chromatic number, planar graph, maximum average degree.

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1. **Introduction**

Incidence coloring was defined by Brualdi and Massey [2] as a tool to study strong edge colorings of bipartite graphs. However, soon after its definition, the coloring itself attracted the attention of several researchers from different points of view.

An incidence in a graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ is an edge of $G$ incident to $v$. Two incidences $(v, e)$ and $(u, f)$ are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge $vu$ is from the set $\{e, f\}$. An incidence coloring of $G$ is a coloring of its incidences assigning distinct colors to adjacent incidences. The minimum number of colors needed for incidence coloring of a graph is called the incidence chromatic number of $G$, denoted by $\chi_i(G)$.

Brualdi and Massey [2] conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for any graph $G$, where $\Delta(G)$ denotes the maximum degree of $G$. The conjecture was disproved by Guiduli [3], who showed that Paley graphs with maximum degree $\Delta$ have incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. However, for many of the commonly considered graph classes the incidence chromatic number is bounded by $\Delta + c$ for some constant $c$, and several papers are devoted to the proof of this type of result, including the following one.

**Theorem 1** (Maydanskiy, 2005). Five colors suffice for an incidence coloring of any subcubic graph.

In order to obtain upper bounds on the incidence chromatic number, in many cases, stronger statements concerning incidence colorings with further local constraints are proved, allowing to apply induction in a more efficient way.

An incidence coloring of a graph $G$ using $k$ colors is an incidence $(k, p)$-coloring of $G$ if for every vertex $v$ of $G$, the number of colors used for coloring the incidences of the form $(u, uv)$ is at most $p$.

Hosseini Dolama, Sopena and Zhu [5] proved that every planar graph with maximum degree $\Delta$ admits an incidence $(\Delta + 7, 7)$-coloring and, thus, has incidence chromatic number at most $\Delta + 7$. This bound was further improved to $\Delta + 4$ for triangle-free planar graphs [6], to $\Delta + 3$ (respectively, $\Delta + 2$, $\Delta + 1$) for planar graphs of girth at least 6 (respectively, 11, 16) [6]. The last result was further improved to girth 14 [1].
Some of these results were proved for more general graph classes, namely graphs with bounded maximum average degree. The average degree of a graph $G$ is the mean value of the degrees of its vertices. The maximum average degree mad($G$) of a graph $G$ is then defined as the maximum value of the average degrees of its subgraphs. When $G$ is a planar graph with girth $g$, it is folklore to establish the inequality $\text{mad}(G) < \frac{2g}{g-2}$.

In [6] the authors proved the following result.

**Theorem 2** (Hosseini Dolama, Sopena, 2005). Let $G$ be a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 5$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

In Section 2 we extend this result to $\text{mad}(G) < 3$ and $\Delta(G) \geq 4$ (Theorem 4). Moreover, we present another result for graphs with larger maximum average degree (Theorem 5).

Recall that the star arboricity of an undirected graph $G$ is the smallest number of star forests needed to cover $G$. Yang [8] observed the following: let $G$ be an undirected graph with star arboricity $\text{st}(G)$, let $s : E(G) \rightarrow \{1, \ldots, \text{st}(G)\}$ be a mapping such that $s^{-1}(i)$ is a forest of stars for every $i$, $1 \leq i \leq \text{st}(G)$, and let $\lambda$ be a proper edge coloring of $G$. Now define the mapping $f$ by $f(u, uv) = s(uv)$ if $v$ is the center of a star in some forest $s^{-1}(i)$ (if some star is reduced to one edge, we arbitrarily choose one of its end vertices as the center) and $f(u, uv) = \lambda(uv)$ otherwise. It is not difficult to check that $f$ is indeed an incidence coloring of $G$. Therefore, thanks to the classical result of Vizing, the relation $\chi_i(G) \leq \Delta(G) + \text{st}(G)$ (respectively, $\chi_i(G) \leq \Delta(G) + \text{st}(G) + 1$) holds for every graph of class 1 (respectively, of class 2). (Recall that the chromatic index $\chi'(G)$ of any graph $G$ is either $\Delta(G)$—such graphs are said to be of class 1—or $\Delta(G) + 1$—such graphs are said to be of class 2.) The facts that planar graphs with $\Delta \geq 7$ are class 1 [7] and that the star arboricity of any planar graph is at most 5 [4] led to the following result.

**Theorem 3** (Yang, 2007). If $G$ is a planar graph with $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$. If $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$.

Yang [8] proposed the following question: Are $\Delta(G) + 5$ colors enough for graphs with maximum degree 6? We give a positive answer to this question (in a stronger form) in Section 3.
Theorem 4. Let $G$ be a graph with $\text{mad}(G) < 3$ and $\Delta(G) \geq 4$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

Theorem 5. Let $G$ be a graph with $\text{mad}(G) < \frac{10}{3}$ and $\Delta(G) \geq 8$. Then $G$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.

2.1. Reducible configurations

We first introduce some additional notation used in the proofs of both results. We denote by $\text{deg}_G(v)$ the degree of a vertex $v$ in a graph $G$. By a $k$-vertex, a $k^+$-vertex and a $k^-$-vertex, we mean a vertex of degree $k$, at least $k$ and at most $k$, respectively. A $(k_1, k_2)$-edge is an edge $v_1v_2$ such that for every $i \in \{1, 2\}$, $v_i$ is a $k_i$-vertex. More generally, a $(k_1, k_2, \ldots, k_{\ell})$-path (respectively, a $(k_1, k_2, \ldots, k_{\ell})$-cycle), $\ell \geq 3$, is a path (respectively, a cycle) $v_1v_2\cdots v_{\ell}$ such that for every $i$, $1 \leq i \leq \ell$, $v_i$ is a $k_i$-vertex.

Let $c$ be a partial incidence coloring of a graph $G$. We say that a color $a$ is admissible for an (uncolored) incidence $(v, e)$ in $G$ if there is no incidence colored by $a$ adjacent to $(v, e)$; otherwise the color $a$ is forbidden. We denote $F_c(v, e)$ the set of forbidden colors for the incidence $(v, e)$.

Let $v$ be a vertex of $G$. We set $I_v := \{(v, uv) : uv \in E(G)\}$ and $A_v := \{(u, uv) : uv \in E(G)\}$. If $c$ is a partial incidence coloring of $G$, we necessarily have $c(I_v) \cap c(A_v) = \emptyset$ for each vertex $v$ of $G$. Moreover, if $c$ is a partial $(k, 2)$-incidence coloring of $G$, then $|c(A_v)| \leq 2$. By $A^c(v)$ we will denote a set of exactly two colors such that $A^c(v) \supseteq c(A_v)$ and $A^c(v) \cap c(I_v) = \emptyset$.

We now prove a series of lemmas.

Lemma 6. Let $G$ be a graph, $v$ be a 1-vertex in $G$ and $k \geq \Delta(G) + 2$ be an integer. If $G - v$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $c$ be a $(k, 2)$-incidence coloring of $G - v$, and $w$ denote the unique neighbor of $v$ in $G$. We will extend $c$ to a $(k, 2)$-incidence coloring of $G$. Since $|F^c(w, vw)| = |c(I_w) \cup c(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$, there is an admissible color $a$ for $(w, vw)$. We then set $c(w, vw) = a$ and $c(v, vw) = b$ for any color $b$ in $A^c(w)$. Clearly, $c$ is a $(k, 2)$-incidence coloring of $G$.

Lemma 7. Let $G$ be a graph, $k \geq \Delta(G) + 2$ be an integer, and $uv$ be a $(2, (k - 3)^-)$-edge in $G$. If $G - uv$ admits a $(k, 2)$-incidence coloring, then $G$ also admits a $(k, 2)$-incidence coloring.

Proof. Let $w$ be the other neighbor of $u$ in $G$ and $c$ be a $(k, 2)$-incidence coloring of $G - e; e = uv$. We extend $c$ to a $(k, 2)$-incidence coloring of $G$ in the following way. We first uncolor $(u, uw)$. We then set $c(u, e) = a$, for some color $a \in A^c(u) - c(u, uw)$, and $c(u, uw) = b$ for some color $b \in A^c(u) - c(u, e)$. Finally,
since $|F^c(v,e)| = |c(I_v) \cup c(A_v) \cup \{c(u,uw)\}| \leq (k-4) + 2 + 1 = k-1 < k,$ there is an admissible color for $(v,e),$ so that we can complete the coloring. 

Lemma 8. Let $G$ be a graph with no 1-vertices and $k \geq \Delta(G) + 2$ be an integer. Let $v$ be an $s$-vertex in $G$, $s \geq 3$, adjacent to at most one $3^+$-vertex, and let $u_i$, $1 \leq i \leq s-1$, denote the 2-neighbors of $v$. If the graph $G - \{vu_i, 1 \leq i \leq s-1\}$ admits a $(k,2)$-incidence coloring, then $G$ also admits a $(k,2)$-incidence coloring.

Proof. Let $e_i = vu_i$, $f_i = u_iw_i$ be the other edge incident to $u_i$ for every $i$, $1 \leq i \leq s-1$, and $u_s$ be the last neighbor of $v$ and $e_s = vu_s$. Let $c$ be a $(k,2)$-incidence coloring of $G - \{e_i, 1 \leq i \leq s-1\}$. We extend $c$ to a $(k,2)$-incidence coloring of $G$ as follows.

We first uncolor $(v,e_s)$ and all incidences $(u_i, f_i)$, $1 \leq i \leq s-1$. Let $a_i = c(w_i, f_i)$, $1 \leq i \leq s-1$. Since we have $k$ colors and $k \geq \Delta(G) + 2$, there is a color $t$ not in $\{a_i, 1 \leq i \leq s-1\}$; moreover, we can choose $t$ such that $t \not\in A^c(w_1)$. We then set $c_i(u_i, u_s) = t$, $1 \leq i \leq s-1$.

Next, for every $i$, $2 \leq i \leq s-1$, we set $c(u_i, f_i) = t_i$ with $t_i \in A^c(w_i) - \{t\}$, $c(v, e_s) = t_s$ with $t_s \in A^c(w_s) - \{t\}$, and $c(u_s, f_1) = t_1$ with $t_1 \in A^c(w_1) - \{t_2\}$.

Now $F^c(v, e_i) = \{t, c(u_i, f_i), c(u_s, e_s), c(v, e_s)\}$. Therefore we have at least $k - 4 \geq s - 2$ admissible colors for every uncolored incidence. As $c(u_i, f_1) \neq c(u_2, f_2)$, we can choose at least $s - 1$ distinct colors $b_i$ such that $b_i \not\in F^c(v, e_i)$, and we set $c(v, e_i) = b_i$ for every $i$, $1 \leq i \leq s-1$. 

Lemma 9. Let $G$ be a graph with $\Delta(G) \geq 7$, $k \geq \Delta(G) + 2$ be an integer, and $C = v_1v_2v_3$ be a $(3,3,3)$-cycle in $G$. If the graph $G - \{v_1v_2, v_2v_3, v_3v_1\}$ admits a $(k,2)$-incidence coloring, then $G$ also admits a $(k,2)$-incidence coloring.

Proof. Let $c$ be a $(k,2)$-incidence coloring of $G - \{v_1v_2, v_2v_3, v_3v_1\}$. Let $u_i$ be the neighbor of $v_i$ not included in $C$, $1 \leq i \leq 3$. We extend $c$ to a $(k,2)$-incidence coloring of $G$ as follows. Let $a_i = c(u_i, u_{i+1})$, $b_i = c(v_i, v_{i+1})$, $1 \leq i \leq 3$. Since $k \geq 9$, there are three colors $c_1, c_2, c_3 \not\in \{a_i, 1 \leq i \leq 3\} \cup \{b_i, 1 \leq i \leq 3\}$. We then color the six incidences of $C$, cyclically, with colors $c_1, c_2, c_3, c_1, c_2, c_3$. 

Lemma 10. Let $G$ be a graph with $\Delta(G) \geq 8$, $k \geq \Delta(G) + 2$ be an integer, and $P = u_1v_1u_2v_2$ be a $(4^-, 3, 3, 4^-)$-path in $G$. If the graph $G - \{u_1v_1, v_1v_2, v_2u_2\}$ admits a $(k,2)$-incidence coloring, then $G$ also admits a $(k,2)$-incidence coloring.

Proof. Let $c$ be a $(k,2)$-incidence coloring of $G - \{u_1v_1, v_1v_2, v_2u_2\}$ and $w_i$ be the third neighbor of $v_i$, $i = 1, 2$. We will extend $c$ to a $(k,2)$-incidence coloring of $G$.

We can assume that $\{c(w_i, w_iv_i), c(v_i, v_iw_i)\} \neq A^c(u_i), i = 1, 2$ (otherwise we recolor $(v_i, v_iw_i)$ using the other color from $A^c(w_i)$). Thus we can set $c(v_i, v_iu_i) = t_i$ with $t_i \in A^c(u_i) - \{c(w_i, w_iv_i), c(v_i, v_iw_i)\}, i = 1, 2$. 


We now consider three cases:

Case 1. \( c(w_2, w_2v_2) \notin c(I_v) \cup c(A_v) \). We first set \( c(v_1, v_1v_2) = c(w_2, w_2v_2) \). Since \( k \geq 10 \), there exists a color \( c_1 \notin c(I_v) \cup c(A_v) \cup \{ c(v_1, v_1v_1), c(v_2, v_2v_2), c(w_2, w_2v_2), c(v_2, v_2u_2) \} \). We then set \( c(u_1, u_1v_1) = c(v_2, v_2v_1) = c_1 \). Since the incidence \( (w_2, w_2v_2) \) is adjacent to at most nine other incidences, it can be colored.

Case 2. \( c(w_1, w_1v_1) \notin c(I_v) \cup c(A_v) \). We proceed similarly as in the previous case.

Case 3. \( c(w_1, w_1v_1) \in c(I_v) \cup c(A_v) \) and \( c(w_2, w_2v_2) \in c(I_v) \cup c(A_v) \). We will color the incidences \((u_1, u_1v_1)\) and \((v_2, v_2v_1)\) with a common color \( c_1 \), and the incidences \((u_2, u_2v_2)\) and \((v_1, v_1v_2)\) with a common color \( c_2 \). Note that we have at most nine forbidden colors for each of \( c_1 \) and \( c_2 \). If we can choose \( c_1 \neq c_2 \), we are done. If not, we necessarily have \( k = 10 \), the sets of forbidden colors for \( c_1 \) and \( c_2 \) are the same, and both contain nine distinct colors. Since in this case we have \( c(w_1, w_1v_1) \in c(I_v) \cup c(A_v) \) and \( c(w_1, w_1v_1), c(v_2, v_2v_2), c(v_2, v_2v_2) \) are different (they are different forbidden colors for \( c_2 \)), we get \( c(w_1, w_1v_1) = c(w_2, w_2v_2) \). Without loss of generality, we may assume that \( c(w_1, w_1v_1) = c(w_2, w_2v_2) = 9 \), \( c(v_1, v_1v_1) = 8, c(v_1, v_1u_1) = 7, c(v_2, v_2v_2) = 6, \) and \( c(v_2, v_2v_2) = 5 \) (see Figure 1). Then \( c(I_v) \cup c(A_v) = \{ 1, 2, 3, 4, 5 \} \) and \( c(I_u) \cup c(A_u) = \{ 1, 2, 3, 4, 7 \} \).

We can replace \( c(v_1, v_1u_1) \) with the other color from \( c(A_u) \). Now, 7 is no more forbidden for \( c_2 \), so we have only eight forbidden colors for \( c_2 \). Therefore, we can now choose \( c_1 \neq c_2 \) to obtain the desired coloring.

![Figure 1. A partial incidence coloring of a \((4^-, 3, 3, 4^-)\)-path.](image)

2.2. Discharging rules

2.2.1. Proof of Theorem 4

We prove Theorem 4 by contradiction. Let \( \Delta_0 \geq 4 \) and \( G \) be a minimal counterexample (with respect to the number of vertices) with \( \text{mad}(G) < 3 \), \( \Delta(G) \leq \Delta_0 \) and
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with no \((\Delta_0 + 2, 2)\)-incidence coloring. From Theorem 1 and Lemmas 6, 7 and 8 it follows that \(\delta(G) \geq 2\), every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices and every \(3^+\)-vertex is adjacent to at least two \(3^+\)-vertices. Moreover, \(\Delta_0 = \Delta(G)\). We will reach a contradiction by using the discharging method.

We assign an initial charge \(\omega(v) = \deg_G(v)\) to each vertex \(v\) of \(G\), and we use the following discharging rule: each \(4^+\)-vertex gives \(\frac{1}{2}\) to each of its 2-neighbors.

We shall prove that the new charge \(\omega'(v)\) of each vertex \(v\) of \(G\) is at least 3, which contradicts our assumption \(\mad(G) < \frac{10}{3}\) (since \(\sum_{v \in G} \omega'(v) = \sum_{v \in G} \omega(v)\)).

Let \(v\) be a vertex of \(G\). We consider three cases, according to \(\deg_G(v)\).

Case 1. \(\deg_G(v) = 2\). Every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices. Therefore, \(\omega'(v) = 2\) by \(\Delta(G) \geq 4\).

Case 2. \(\deg_G(v) = 3\). The discharging rule does not involve 3-vertices, thus \(\omega'(v) = \omega(v) = 3\).

Case 3. \(\deg_G(v) = d \geq 4\). Since every \(d\)-vertex is adjacent to at most \((d - 2)\) 2-vertices, \(\omega'(v) \geq d - \frac{1}{2}(d - 2) = \frac{4d + 2}{2} \geq 3\).

2.2.2. Proof of Theorem 5

We prove Theorem 5 by contradiction. Let \(\Delta_0 \geq 8\) and \(G\) be a minimal counterexample (with respect to the number of vertices) with \(\mad(G) < \frac{10}{3}\), \(\Delta(G) \leq \Delta_0\) and no \((\Delta_0 + 2, 2)\)-incidence coloring. From Lemmas 6, 7, 8, 9 and 10 it follows that \(\delta(G) \geq 2\), every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices, every \(3^+\)-vertex is adjacent to at least two \(3^+\)-vertices, \(G\) does not contain any 3-cycle only on 3-vertices as a subgraph and \(G\) contains no \((4^-, 3, 3, 4^-)\)-path as a subgraph.

Let us define a cluster as a maximal connected subgraph of \(G\) induced on 3-vertices.

We will reach a contradiction by using the discharging method.

We assign an initial charge \(\omega(v) = \deg_G(v)\) to each vertex \(v\) of \(G\), and we use the following discharging rules:

(R1) Each \(\Delta_0\)-vertex gives \(\frac{2}{3}\) to each of its 2-neighbors.

(R2) Each 4-vertex gives \(\frac{1}{3}\) to each of its 3-neighbors.

(R3) Each \(5^+\)-vertex gives \(\frac{1}{3}\) to each of its 3-neighbors.

We shall prove that the new charge \(\omega'(v)\) of each \(k\)-vertex \(v\) of \(G\), \(k = 2\) or \(k \geq 4\), is at least \(\frac{10}{3}\) and that each cluster has average charge at least \(\frac{10}{3}\) too, which contradicts our assumption \(\mad(G) < \frac{10}{3}\).

Let \(v\) be a vertex of \(G\). We consider four cases, according to \(\deg_G(v)\).

Case 1. \(\deg_G(v) = 2\). Every 2-vertex in \(G\) is adjacent to two \(\Delta_0\)-vertices. Therefore, \(\omega'(v) = 2 + 2 \times \frac{2}{3} = \frac{10}{3}\) by R1.
Case 2. \( \deg_G(v) = 4 \). Due to R2, we have \( \omega'(v) \geq 4 - 4 \times \frac{1}{9} = \frac{32}{9} > \frac{10}{3} \).

Case 3. \( \deg_G(v) = d \), with \( 5 \leq d < \Delta_0 \). According to R3, vertex \( v \) sends a charge at most \( \frac{2}{9} \) to each of its neighbors. Hence, \( \omega'(v) \geq d - \frac{2}{9} d = \frac{7}{9} d \geq \frac{35}{9} > \frac{10}{3} \).

Case 4. \( \deg_G(v) = \Delta_0 \). Each \( \Delta_0 \)-vertex sends \( 2 \) to each of its 2-neighbors and at most \( \frac{2}{9} \) to its other neighbors. Moreover \( v \) is adjacent to at most \( \left( \Delta_0 - 2 \right) \) 2-vertices and, therefore, we have \( \omega'(v) \geq \Delta_0 - \frac{2}{9} (\Delta_0 - 2) - 2 \times \frac{2}{9} = \frac{10}{3} + \frac{3\Delta_0 - 22}{9} > \frac{10}{3} \).

Finally, we consider a cluster \( K \). The initial charge of \( K \) is \( 3|K| \). We will show that the final charge \( \omega'(K) = \sum_{v \in K} \omega'(v) \) is at least \( \frac{10}{3}|K| \). As \( G \) contains no \((3,3,3)\)-cycle and no \((4^-,3,3,4^-)\)-path, we have only four possibilities for \( K \).

- **K** is a single 3-vertex \( v \). In this case \( \omega'(K) = \omega'(v) \geq 3 + 3 \times \frac{1}{9} = \frac{10}{3} \).

- **K** is a \((3,3)\)-edge. By Lemma 10, \( K \) is adjacent to at least two \( 5^+ \)-vertices and we have \( \omega'(K) \geq 2 \times 3 + 2 \times \frac{1}{9} + 2 \times \frac{2}{9} = 2 \times \frac{10}{9} \).

- **K** is a \((3,3,3)\)-path. Again by Lemma 10, \( K \) has at least four \( 5^+ \)-vertices in its neighborhood and \( \omega'(K) \geq 3 \times 3 + 1 \times \frac{1}{9} + 4 \times \frac{2}{9} = 3 \times \frac{10}{9} \).

- **K** is a star on four 3-vertices. In this case each neighbor of \( K \) is a \( 5^+ \)-vertex and \( \omega'(K) = 4 \times 3 + 6 \times \frac{2}{9} = 4 \times \frac{10}{3} \).

3. **Graphs with Maximum Degree 6**

Yang [8] proved that \( \chi_i(G) \leq \Delta(G) + 5 \) for every planar graph \( G \) with \( \Delta(G) \neq 6 \), using the relation between the incidence chromatic number, the star arboricity and the chromatic index of a graph. For planar graphs with \( \Delta(G) = 6 \) he only proved \( \chi_i(G) \leq 12 \). We improve this bound and get the following result for a more general class of graphs.

![Figure 2. An Eulerian (multi)graph G' with an additional (multi)edge.](image)

**Theorem 11.** If \( G \) is a graph with \( \Delta(G) \leq 6 \) and with no 6-regular component on an odd number of edges, then \( \chi_i(G) \leq 10 \).
Proof. Let $G$ be a graph with $\Delta(G) \leq 6$ which has no 6-regular component on an odd number of edges. Without loss of generality we may assume that $G$ is connected, otherwise we consider each of its components separately. If $G$ is an Eulerian graph, then we color the edges of an Eulerian trail $T$ alternately with red and blue, starting at a vertex of degree less than 6 (if there exists one; otherwise we start at an arbitrary vertex). The subgraphs $R$ and $B$ of $G$ induced by the sets of red and blue edges, respectively, are subcubic. Hence, by Theorem 1, $\chi_i(R) \leq 5$ and $\chi_i(B) \leq 5$. Using two disjoint sets of colors for incidence coloring of the subgraphs $R$ and $B$, we obtain an incidence coloring of $G$ with (at most) 10 colors.

If $G$ is connected but not Eulerian, then we add edges joining pairs of vertices of odd degree in $G$ to obtain an Eulerian (multi)graph $G'$. Clearly, $\Delta(G') \leq 6$. We then assign colors red and blue alternately to edges of an Eulerian trail $T$ in $G'$. It is easily seen that the subgraphs $R$ and $B$ of $G$ obtained as before are subcubic, unless $G'$ is 6-regular and has an odd number of edges. We can avoid this by starting a trail $T$ at a vertex of degree less than 6 (if such a vertex exists) or by some added (multi)edge (see Figure 2). Therefore, we can ensure that $R$ and $B$ are subcubic. Again, using two disjoint sets of colors for incidence coloring the subgraphs $R$ and $B$, we obtain an incidence coloring of $G'$ (and of $G$) with (at most) 10 colors. Therefore, $\chi_i(G) \leq 10$.

As a consequence of the previous theorem, we positively answer Yang’s question about planar graphs with maximum degree 6, even improving the suggested bound.

**Corollary 12.** Every planar graph $G$ with $\Delta(G) = 6$ satisfies $\chi_i(G) \leq 10$.

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