AN IMPROVED UPPER BOUND ON NEIGHBOR EXPANDED SUM DISTINGUISHING INDEX

BOJAN VUČKOVIĆ

Mathematical Institute, Serbian Academy of Science and Arts,
Kneza Mihaila 36 (P.O. Box 367), 11001 Belgrade, Serbia

e-mail: b.vuckovic@turing.mi.sanu.ac.rs

Abstract

A total $k$-weighting $f$ of a graph $G$ is an assignment of integers from the set $\{1, \ldots, k\}$ to the vertices and edges of $G$. We say that $f$ is neighbor expanded sum distinguishing, or NESD for short, if $\sum_{w \in N(v)} (f(vw) + f(w))$ differs from $\sum_{w \in N(u)} (f(uw) + f(w))$ for every two adjacent vertices $v$ and $u$ of $G$. The neighbor expanded sum distinguishing index of $G$, denoted by $\text{egndi}_G^\Sigma$, is the minimum positive integer $k$ for which there exists an NESD weighting of $G$. An NESD weighting was introduced and investigated by Flandrin et al. (2017), where they conjectured that $\text{egndi}_G^\Sigma \leq 2$ for any graph $G$. They examined some special classes of graphs, while proving that $\text{egndi}_G^\Sigma(G) \leq \chi(G) + 1$. We improve this bound and show that $\text{egndi}_G^\Sigma(G) \leq 3$ for any graph $G$. We also show that the conjecture holds for all bipartite, 3-regular and 4-regular graphs.

Keywords: general edge coloring, total coloring, neighbor sum distinguishing index.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ and $\chi(G)$ denote the vertex set, the edge set, the maximum degree, the minimum degree, and the chromatic number of a graph $G$, respectively. Let $N_G(v)$ and $\deg_G(v)$ denote the set of neighbors and the degree of a vertex $v$ in $G$, respectively. For all other terminology used in this paper, we refer the reader to [1].
A total $k$-weighting $f$ of a graph $G$ is an assignment of integers from the set $\{1, \ldots, k\}$ to the vertices and edges of $G$. We say that $f$ is neighbor expanded sum distinguishing, or NESD for short, if

\begin{equation}
\sigma(v) = \sum_{u \in N(v)} (f(uv) + f(u))
\end{equation}

yields a proper vertex coloring of $G$. The minimum positive integer $k$ for which an NESD weighting of a graph $G$ exists is called the neighbor expanded sum distinguishing index of $G$ and denoted by $\text{egndi}_\Sigma(G)$. An NESD weighting was introduced and investigated by Flandrin et al. in [2], where they proposed the following conjecture.

**Conjecture 1.** For any graph $G$, $\text{egndi}_\Sigma(G) \leq 2$.

They examined some classes of graphs, including paths, cycles, complete graphs and trees, and proved a relaxed upper bound $\text{egndi}_\Sigma(G) \leq \chi(G) + 1$ for any graph $G$. Our main result is an improvement over this bound stated in the next theorem.

**Theorem 2.** For any graph $G$, $\text{egndi}_\Sigma(G) \leq 3$.

Proofs of our theorems are deferred to the Section 2. Flandrin et al. [2] proved the following theorem.

**Theorem 3.** Let $G = (X, Y, E)$ be a connected bipartite graph. If any of the bipartite sets $X$ and $Y$ has an even number of vertices, or there is a vertex of odd degree in $G$, then $\text{egndi}_\Sigma(G) \leq 2$.

In our next theorem we extend their claim to any bipartite graph.

**Theorem 4.** If $G$ is a bipartite graph, then $\text{egndi}_\Sigma(G) \leq 2$.

Furthermore, we show that for any 3-regular and any 4-regular graph there exists an NESD weighting using only the weights 1 and 2.

**Theorem 5.** If $G$ is a 3-regular or a 4-regular graph, then $\text{egndi}_\Sigma(G) \leq 2$.

Before we proceed with proofs of Theorems 2, 4 and 5, we will mention a closely related variation of weighting. An edge $k$-weighting $f$ of a graph $G$ is called neighbor sum distinguishing, or NSD for short, if a function $\phi(v) = \sum_{e \ni v} f(e)$ yields a proper vertex coloring of $G$. The minimum positive integer $k$ for which such a weighting exists is the neighbor sum distinguishing index of $G$, denoted by $\text{gndi}_\Sigma(G)$. The following problem, also known as the 1-2-3 conjecture, was proposed by Karonski et al. [3], and it received a significant attention.

**Conjecture 6.** For any graph $G$ without isolated edges, $\text{gndi}_\Sigma(G) \leq 3$.

Kalkowski et al. [4] proved that $\text{gndi}_\Sigma(G) \leq 5$ for any graph $G$ without isolated edges, which is by now the best known bound for $\text{gndi}_\Sigma(G)$. In the proof of Theorem 2 we use an approach similar to that used in [4].
2. Proofs of Our Theorems

**Proof of Theorem 2.** If the statement holds for any connected graph, an immediate consequence is that it also holds for any disconnected graph. Thus we may assume that $G$ is a connected graph.

We prove a slightly stronger claim, that is, there exists a total 3-weighting $f$ such that $\left\lceil \frac{\sigma(v)}{2} \right\rceil \neq \left\lceil \frac{\sigma(u)}{2} \right\rceil$ for any two adjacent vertices $v$ and $u$ of $G$. Let $V = V(G) = \{v_1, \ldots, v_k\}$ be the set of vertices of $G$ arranged in an arbitrary order. Let $E = E(G)$ be the set of edges of $G$ and $f : E \cup V \rightarrow \{1, 2, 3\}$.

For $1 \leq j \leq k$, let $S_1(V, E, f, j)$ and $S_2(V, E, f, j)$ denote the following two statements:

$$S_1(V, E, f, j) : \left\lceil \frac{\sigma(v_i)}{2} \right\rceil \neq \left\lceil \frac{\sigma(v_l)}{2} \right\rceil \text{ for every } v_i, v_l \in E \text{ and } 1 \leq i < l \leq j.$$  

$$S_2(V, E, f, j) : f(v_i) = 2, \text{ and } f(v_i u) = 2 \text{ for every } v_i \in V, j < i \leq k, \text{ and } v_i u \in E.$$  

We start by assigning the weight 2 to each vertex of $V$ and each edge of $E$. Hence $S_1(V, E, f, 1)$ and $S_2(V, E, f, 1)$ are trivially true. Suppose that $S_1(V, E, f, j)$ and $S_2(V, E, f, j)$ hold for some weighting $f$ and integer $j$ with $1 \leq j < k$. Let $d$ denote the value of $\sigma(v_{j+1})$ for the current weighting $f$. Let $U = \{u_1, \ldots, u_n\}$ be the subset of vertices from $\{v_1, \ldots, v_j\}$ that are adjacent to $v_{j+1}$ in $G$. Next, let $U_o$ and $U_e$ be the subsets of $U$, with $U_o \cup U_e = U$, such that $\sigma(v)$ is odd for every $v \in U_o$, and $\sigma(v)$ is even for every $v \in U_e$. Let $n_o = |U_o|$ and $n_e = |U_e|$, thus $n_o + n_e = n$. We now consider possible adjustments of $f(v_{j+1})$ and $f(u_l v_{j+1})$ for $u_l \in U$, where both $S_1(V, E, f, j)$ and $S_2(V, E, f, j + 1)$ remain satisfied.

1. Since $f(u_l v_{j+1}) = 2$ for every $l \in \{1, \ldots, n\}$, we can increase by 1 the weight of any of the $n_o$ edges joining $v_{j+1}$ with the vertices of $U_e$, while keeping $S_1(V, E, f, j)$ satisfied. Similarly, we can decrease by 1 the weight of any of the $n_o$ edges joining $v_{j+1}$ with the vertices of $U_o$. Hence we can adjust the weights of these edges to obtain that $\sigma(v_{j+1})$ equals any integer value from $[d - n_o, d + n_e]$.

2. By changing $f(v_{j+1})$ to 1, and $f(u_l v_{j+1})$ to 3, for every $l \in \{1, \ldots, n\}$, the value of $\sigma(v_{j+1})$ increases by $n$, while $\sigma(u_l)$ remains the same for every $l \in \{1, \ldots, n\}$. Now, we can decrease by 1 the weight of any of the $n_o$ edges joining $v_{j+1}$ with the vertices of $U_o$, while keeping $S_1(V, E, f, j)$ satisfied. Hence we can achieve that $\sigma(v_{j+1})$ equals any integer value from $[d + n_e, d + n]$.

3. By changing $f(v_{j+1})$ to 3 and $f(u_l v_{j+1})$ to 1, for every $l \in \{1, \ldots, n\}$, the value of $\sigma(v_{j+1})$ decreases by $n$, while $\sigma(u_l)$ remains the same for every $l \in \{1, \ldots, n\}$. Now, similarly to the previous case, we can achieve that $\sigma(v_{j+1})$ equals any integer value from $[d - n, d - n_o]$. 


Therefore, we can make adjustments to the weights of $v_{j+1}$ and $u_iv_{j+1}$, with $u_i \in U$, so that $\sigma(v_{j+1})$ equals any integer value from $[d-n, d+n]$, while keeping $S_1(V, E, f, j)$ and $S_2(V, E, f, j+1)$ satisfied. Since there are $n$ neighboring vertices of $v_{j+1}$ preceding it, and there are $2n+1$ reachable values for $\sigma(v_{j+1})$, we can adjust the weights of $v_{j+1}$ and $u_iv_{j+1}$, with $1 \leq l \leq n$, so that $S_1(V, E, f, j+1)$ holds. In the above described procedure we do not change the weight of any vertex $v_i \in V$, with $l > j + 1$, nor the weight of any edge incident with $v_i$. Thus $S_2(V, E, f, j+1)$ holds. Continuing in this manner until $j + 1 = k$, we obtain a desired weighting. 

In the proof of Theorem 4 we follow the idea used in [3], and later implemented in the proof of Theorem 3.

**Proof of Theorem 4.** As stated in the proof of Theorem 2, it suffices to show that the theorem holds for every connected graph. Thus we may assume that $G$ is a connected graph. If $X$ or $Y$ have an even number of vertices, or there exists a vertex of odd degree in $G$, the statement is true by Theorem 3. So we may assume that both $X$ and $Y$ have an odd number of vertices, and every vertex of $G$ has even degree.

Let $X = \{x_1, \ldots, x_{2k+1}\}$. We may assume, without loss of generality, that $X$ contains a vertex with degree equal to $\delta(G)$, and that $x_{2k+1}$ is such a vertex. First, we assign the weight 2 to every edge of $G$ and vertex of $X$, and the weight 1 to every vertex of $Y$. Since every vertex of $G$ has even degree, after the described assignment $\sigma(v)$ is even for every $v \in X \cup Y$. We now change the weights of some edges of $G$ so that $\sigma(x_i)$ becomes odd for every $i \in \{1, 2k\}$, while $\sigma(y)$ remains even for every $y \in Y$. We subsequently prove that $\sigma(x_{2k+1}) < \sigma(y)$ for every $y \in N_G(x_{2k+1})$, which implies the statement of the theorem.

Let $P_i$ denote a path from $x_i$ to $x_{i+k}$, for every integer $i$ with $1 \leq i \leq k$. For each path $P_i$, with $1 \leq i \leq k$, we change the weight of every edge on the path, that is, from 1 to 2, and from 2 to 1. This way the parity of $\sigma(u)$ stays the same for every vertex $u$ on $P_i$ different from $x_i$ and $x_{i+k}$. After this procedure the value of $\sigma(x_i)$ is odd for every integer $i$ with $1 \leq i \leq 2k$, while $\sigma(y)$ is even for every $y \in Y$. Next, let $y \in Y$ be an arbitrary neighbor of $x_{2k+1}$. Let $d = \deg(x_{2k+1})$. Since $d = \delta(G)$, we have $d \leq \deg(y)$. Since $f(u) = 1$ for every $u \in N_G(x_{2k+1})$, we have $\sigma(x_{2k+1}) \leq 3(d-1) + f(x_{2k+1}y) + 1$. On the other hand, since $f(u) = 2$ for every $u \in N_G(y)$, we have $\sigma(y) \geq 3(d-1) + f(x_{2k+1}y) + 2$. Therefore, $\sigma(x_{2k+1}) < \sigma(y)$ for every $y \in N_G(x_{2k+1})$, completing the proof.

The proof of Theorem 5 is organized as follows. We start from some proper vertex coloring $c$ of $G$. Then we define a total weighting $f$ of $G$ by assigning a weight from the set $\{1, 2\}$ to every vertex $v$ of $G$ depending on the value of $c(v)$,
and to every edge $uv$ of $G$ depending on the values of $c(u)$ and $c(v)$. Finally, we adjust some of these weights and prove that such a weighting is NESD.

**Proof of Theorem 5.** Let $G$ be a $k$-regular graph with $k \in \{3, 4\}$. Flandrin et al. proved in [2] that $\gamma_{\text{end}}(G) = 2$ for any complete graph $G$. Thus we may assume that $G$ is not a complete graph. Then $\chi(G) \leq k$ according to Brooks’ Theorem [5]. For $1 \leq i \leq k$, let $V_i$ be the color classes of $V(G)$. We may assume that every vertex $v$ of $V_j$, with $1 < j \leq k$, has at least one neighbor in every $V_i$, with $1 \leq i < j$. Otherwise, while there exists $v \in V_j$ that has no neighbor in $V_i$, $1 < i < j$, we move $v$ to $V_i$. This way sets $V_i$ remain independent for every $1 \leq i \leq k$, hence $c$ remains a proper coloring with $k$ colors.

First, we prove the case when $G$ is a 3-regular graph. As observed above, we may assume that $\chi(G) \leq 3$. Let $c$ be any proper vertex coloring of $G$ with colors from $\{1, 2, 3\}$. As noted earlier, we may assume that every vertex colored $j$, $1 < j \leq 3$, has a neighbor colored $i$ for every $1 < i < j$. We now define a total 2-weighting $f$ of $G$. We assign the weight 1 to every $v \in V_i$ with $i \in \{1, 3\}$, and the weight 2 to every $v \in V_2$. For every two adjacent vertices $u$ and $v$ of $G$, if $c(u) = 3$ or $c(v) = 3$, then we assign 2 to $f(uv)$; otherwise we assign 1 to $f(uv)$. For every $v \in V_2$, the vertex $v$ has a neighbor in every $V_i$ with $1 < i < j$, so we have the following:

1. for $j = 1$, we have $\sigma(v) = 9$,
2. for $j = 2$, we have $\sigma(v) < 9$,
3. for $j = 3$, we have $\sigma(v) > 9$.

Because $V_j$ is an independent set for every $j \in \{1, 2, 3\}$, it follows that $\sigma(v) \neq \sigma(u)$ for every two adjacent vertices $v$ and $u$ of $G$, completing the first part of the proof.

We now prove that the statement holds for every 4-regular graph $G$. As before, we may assume that $\chi(G) \leq 4$. Let $c$ be an arbitrary proper vertex coloring of $G$ with colors from $\{1, 2, 3, 4\}$. Again, we may assume that every $v \in V_j$, with $1 < j \leq 4$, has at least one neighbor in every $V_i$, with $1 < i < j$. We now define a total 2-weighting $f$. We assign the weight 1 to every vertex of $V_1$ and $V_3$, and the weight 2 to every vertex of $V_2$ and $V_4$. We assign the weight 1 to edges joining the vertices of $V_1$ with the vertices of $V_2$, and also to edges incident with the vertices of $V_4$, while to all other edges we assign the weight 2. Now, for every $v \in V_j$, the vertex $v$ has a neighbor in every $V_i$ with $1 < i < j$, so we have the following:

1. for $j = 1$, we have $\sigma(v) = 12$,
2. for $j = 2$, we have $\sigma(v) < 12$,
3. for $j = 3$, we have $\sigma(v) > 12$, 

4. for $j = 4$, we have $\sigma(v) = 12$. 

4. for $j = 4$, we have $\sigma(v) \in \{9, 10\}$.

Clearly, for any two adjacent vertices $v$ and $u$ of $G$, the values of $\sigma(v)$ and $\sigma(u)$ may be equal only when one of these two vertices is from $V_2$ and the other is from $V_4$. We now change the weights of some of the vertices and edges of $G$ to obtain an NESD coloring.

First, for every $v \in V_4$ with $\sigma(v) = 10$, we do the following. Since $\sigma(v) = 10$, the vertex $v$ has exactly one neighbor in both $V_1$ and $V_3$, and two neighbors in $V_2$. Denote by $\{v_1, v'_2, v''_2, v_3\}$ the neighbors of $v$, where $v_1 \in V_1, v'_2, v''_2 \in V_2$ and $v_3 \in V_3$. We consider two cases, depending on the value of $\sigma(v_3)$.

Case 1. $\sigma(v_3) = 14$. We adjust the weights as follows: $f(v) = 1, f(v_1v) = 2$. Now, we have $\sigma(v_1) = 12, \sigma(v'_2) \leq 10, \sigma(v''_2) \leq 10, \sigma(v_3) = 13$ and $\sigma(v) = 11$.

Case 2. $\sigma(v_3) = 13$. We make the following changes: $f(v) = 1, f(v_1v) = 2, f(v'_2v) = 2, f(v''_2v) = 2, f(v_3v) = 2$. Hence the values of $\sigma(v_1), \sigma(v'_2), \sigma(v''_2)$ and $\sigma(v_3)$ remain the same, while now $\sigma(v) = 14$. Thus $\sigma(v)$ differs from $\sigma(w)$ for every $w \in N(v)$.

Note that since the value of $\sigma(v_3)$ for $v_3 \in V_3$ never changes to 14, there is no conflict between the two adjustments above.

Next, for every $v \in V_4$ with $\sigma(v) = 9$, we do the following. In this case $v$ has only one neighbor $u$ in $V_2$. If $\sigma(u) \neq 9$, then $\sigma(v)$ is different from $\sigma(w)$ for every $w \in N_G(v)$, and we do not need to change any weight. Otherwise, since $f(v) = 2$ and $\sigma(u) = 9$, if follows that $u$ does not have any neighbor in $V_3$, and also every edge incident with $u$ has the weight 1. We now change the weight of $u$ to 1, and the weight of every edge incident with $u$ to 2. This way $\sigma(y)$ remains the same for every $y \in N_G(u)$, while the value of $\sigma(u)$ becomes 13. The vertex $u$ has no neighbor in $V_3$, while $\sigma(y) \neq 13$ for every $y \in V_4$. Thus $\sigma(u) \neq \sigma(w)$ for every $w \in N_G(u)$.

After the procedure above is finished, we have $\sigma(v) \neq \sigma(u)$ for every two adjacent vertices $v$ and $u$ of $G$, and the proof is completed.

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