ON TOTAL $H$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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Abstract

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the $H$-irregular total $k$-labeling of a graph $G$ admitting

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an $H$-covering if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$ is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling.

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

**Keywords:** $H$-covering, $H$-irregular labeling, total $H$-irregularity strength, copies of graphs, union of graphs.

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1. Introduction

Consider a simple and finite graph $G$ with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a total labeling. For a total $k$-labeling $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated total vertex-weight of a vertex $x$ is

$$wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

and the associated total edge-weight of an edge $xy$ is

$$wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y).$$

A total $k$-labeling $\psi$ is defined to be an edge irregular total $k$-labeling of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ there is $wt_\psi(xy) \neq wt_\psi(x'y')$ and to be a vertex irregular total $k$-labeling of $G$ if for every two distinct vertices $x$ and $y$ of $G$ there is $wt_\psi(x) \neq wt_\psi(y)$. This concept was given by Bača, Jendrol’, Miller and Ryan in [8].

The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of the graph $G$, tes($G$). Analogously, we define the total vertex irregularity strength of $G$, tvs($G$), as the minimum $k$ for which there exists a vertex irregular total $k$-labeling of $G$.

The following lower bound on the total edge irregularity strength of a graph $G$ is given in [8].

$$\text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\},$$

where $\Delta(G)$ is the maximum degree of $G$. This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$. 
Ivanˇco and Jendrol’ [12] posed a conjecture that for an arbitrary graph \(G\) different from \(K_5\) with maximum degree \(\Delta(G)\), \(\text{tes}(G) = \max \{\lceil|E(G)| + 2\rceil/3, \lceil(\Delta(G) + 1)/2\rceil\}\). This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with \((|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2\) in [10].

The bounds for the total vertex irregularity strength are given in [8] as follows.

\[
(2) \quad \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1,
\]

where \(\delta(G)\) is the minimum degree of \(G\).

Przybyło in [17] proved that \(\text{tvs}(G) < 32|V(G)|/\delta(G) + 8\) in general and \(\text{tvs}(G) < 8|V(G)|/\tau + 3\) for \(\tau\)-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

\[
(3) \quad \text{tvs}(G) \leq 3 \left\lceil \frac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4.
\]

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if \(\delta(G) \geq (|V(G)|)^{0.5}\ln |V(G)|\), then

\[
(4) \quad \text{tvs}(G) \leq \frac{(2+o(1))|V(G)|}{\delta(G)} + 4.
\]

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An edge-covering of \(G\) is a family of subgraphs \(H_1, H_2, \ldots, H_t\) such that each edge of \(E(G)\) belongs to at least one of the subgraphs \(H_i, i = 1, 2, \ldots, t\). Then it is said that \(G\) admits an \((H_1, H_2, \ldots, H_t)\)-edge covering. If every subgraph \(H_i\) is isomorphic to a given graph \(H\), then the graph \(G\) admits an \(H\)-covering.

Let \(G\) be a graph admitting an \(H\)-covering. For the subgraph \(H \subseteq G\) under the total \(k\)-labeling \(\psi\), we define the associated \(H\)-weight as

\[
wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).
\]

A total \(k\)-labeling \(\psi\) is called to be an \(H\)-irregular total \(k\)-labeling of the graph \(G\) if all subgraphs of \(G\) isomorphic to \(H\) have distinct weights. The total \(H\)-irregularity strength of a graph \(G\), denoted \(\text{ths}(G, H)\), is the smallest integer \(k\) such that \(G\) has an \(H\)-irregular total \(k\)-labeling. This definition was introduced by Ashraf, Baća, Lasciaková and Semaničová-Feňovčíková [7]. If \(H\) is isomorphic to \(K_2\), then the \(K_2\)-irregular total \(k\)-labeling is isomorphic to the edge irregular total \(k\)-labeling and thus the total \(K_2\)-irregularity strength of a graph \(G\) is equivalent to the total edge irregularity strength; that is \(\text{ths}(G, K_2) = \text{tes}(G)\).

The next theorem gives a lower bound for the total \(H\)-irregularity strength.
Theorem 1 [7]. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G, H) \geq 1 + \left\lceil \frac{t-1}{|V(H)|+|E(H)|} \right\rceil.$$  

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. Let $r, s, 2 \leq s \leq r$, be positive integers. Then

$$\text{ths}(P_r, P_s) = \left\lceil \frac{s+r-1}{2s-1} \right\rceil.$$  

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

Corollary 3. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \geq 1 + \left\lceil \frac{mt-1}{|V(H)|+|E(H)|} \right\rceil.$$  

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. Let $G$ be a graph having an $H$-irregular total $\text{ths}(G, H)$-labeling $f$. Let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{w_{\text{max}}^f(H)-w_{\text{min}}^f(H)+1}{|V(H)|+|E(H)|} \right\rceil,$$

where $w_{\text{max}}^f(H)$ and $w_{\text{min}}^f(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$.

Proof. Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$. The smallest
weight of a subgraph \( H \) under the total \( k \)-labeling \( f \) is denoted by the symbol \( \text{wt}^\text{min}_f(H) \). Evidently
\[
\text{wt}^\text{min}_f(H) \geq |V(H)| + |E(H)|.
\]
(5)

Analogously, the largest weight of a subgraph \( H \) under the total \( k \)-labeling \( f \) is denoted by the symbol \( \text{wt}^\text{max}_f(H) \). It holds that
\[
\text{wt}^\text{max}_f(H) \geq \text{wt}^\text{min}_f(H) + t - 1
\]
and
\[
\text{wt}^\text{max}_f(H) \leq (|V(H)| + |E(H)|)k.
\]
(6)
(7)

Thus \( f : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) and
\[
\{\text{wt}_f(H^i_j) : j = 1, 2, \ldots, t\} \subset \{\text{wt}^\text{min}_f(H), \text{wt}^\text{min}_f(H) + 1, \ldots, \text{wt}^\text{max}_f(H)\}.
\]
(8)

By the symbol \( x_i, i = 1, 2, \ldots, m \), we denote an element (a vertex or an edge) in the \( i \)-th copy of \( G \), denoted by \( G_i \), corresponding to the element \( x \) in \( G \), i.e., \( x \in V(G) \cup E(G) \). Analogously, let \( H^i_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, t \), be the subgraph in the \( i \)-th copy of \( G \) corresponding to the subgraph \( H^j \) in \( G \).

Let us define the total labeling \( g \) of \( mG \) in the following way. For \( i = 1, 2, \ldots, m \) let
\[
g(x_i) = f(x) + (i - 1) \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\]
Evidently, all the labels are at most
\[
k + (m - 1) \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\]

For the weight of every subgraph \( H^i_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, t \), isomorphic to the graph \( H \) under the labeling \( g \) we have
\[
\text{wt}_g(H^i_j) = \sum_{v \in V(H^i_j)} g(v) + \sum_{e \in E(H^i_j)} g(e)
\]
\[
= \sum_{v \in V(H^i_j)} \left( f(v) + (i - 1) \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right)
\]
\[
+ \sum_{e \in E(H^i_j)} \left( f(e) + (i - 1) \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right)
\]
\[
\begin{align*}
&= \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) + |V(H)|(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right] \\
&\quad + |E(H)|(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right] \\
&= wt_f(H^i) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right].
\end{align*}
\]

This means that in the given copy of \( G \) the \( H \)-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \( i^{th} \) copy of \( G \), \( i = 1, 2, \ldots, m \), denoted by \( w^\text{max}_g(H : H \subset G_i) \), is at most

\[
wt^\text{max}_g(H : H \subset G_i) \leq wt^\text{max}_f(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right]
\]

and the smallest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \((i + 1)^{th}\) copy of \( G \), \( i = 1, 2, \ldots, m - 1 \), denoted by \( wt^\text{min}_g(H : H \subset G_{i+1}) \), is at least

\[
wt^\text{min}_g(H : H \subset G_{i+1}) \geq wt^\text{min}_f(H) + (|V(H)| + |E(H)|)i \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right].
\]

After some manipulation we get

\[
wt^\text{min}_g(H : H \subset G_{i+1})
\geq wt^\text{min}_f(H) + (|V(H)| + |E(H)|)i \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right]
\]

\[
= wt^\text{min}_f(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right]
\]

\[
+ (|V(H)| + |E(H)|) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right].
\]

As

\[
\left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right] \geq \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|}
\]

we obtain

\[
wt^\text{min}_g(H : H \subset G_{i+1}) \geq wt^\text{min}_f(H)
\]

\[
+ (|V(H)| + |E(H)|)(i - 1) \left[ \frac{w^\text{max}_f(H) - w^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right]
\]

\[
+ (w^\text{max}_f(H) - w^\text{min}_f(H) + 1)
\]
On Total $H$-Irregularity Strength of the Disjoint Union ...

\[
= wt_f^{\text{max}}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + 1
\]

\[\geq wt_g^{\text{max}}(H : H \subset G_i) + 1 > wt_g^{\text{max}}(H : H \subset G_i).\]

Thus in all components the $H$-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

**Corollary 5.** Let $G$ be a graph admitting an $H$-irregular total $\text{ths}(G,H)$-labeling $f$. Let $m$ be a positive integer. Then

\[
\text{ths}(mG, H) \leq m \text{ths}(G, H).
\]

**Proof.** Let $f$ be a $\text{ths}(G,H)$-labeling of a graph $G$ and let $\text{ths}(G, H) = k$. As $wt_f^{\text{min}}(H) \geq |V(H)| + |E(H)|$ and $wt_f^{\text{max}}(H) \leq (|V(H)| + |E(H)|)k$ we get

\[
\left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq \left\lceil \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rceil = k - 1 + \frac{1}{|V(H)| + |E(H)|} = k.
\]

Hence, by Theorem 4,

\[
\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq k + (m - 1)k = mk.
\]

Let $\{H^1, H^2, \ldots, H^t\}$ be the set of all subgraphs of $G$ isomorphic to $H$. Let $f$ be an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$ such that

\[
\{wt_f(H^j) : j = 1, 2, \ldots, t\} = \{wt_f^{\text{min}}(H), wt_f^{\text{min}}(H) + 1, \ldots, wt_f^{\text{min}}(H) + t - 1\}.
\]

Evidently, if the fraction

\[
\frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}
\]

is an integer then the weights of all $H$-weights in $mG$ under the total labeling $g$ of $mG$ defined in the proof of Theorem 4 constitute the set

\[
\{wt_f^{\text{min}}(H), wt_f^{\text{min}}(H) + 1, \ldots, wt_f^{\text{min}}(H) + mt - 1\}.
\]

In particular, this implies that the upper bound for $\text{ths}(mG, H)$ given in Theorem 4 is tight if $G$ is a graph that satisfies the conditions mentioned above.
Theorem 6. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total $\text{th}(G,H)$-labeling of $G$ such that $\{\text{wt}_f(H^j) : j = 1, 2, \ldots, t\} = \{\text{wt}^\min_f(H), \text{wt}^\min_f(H) + 1, \ldots, \text{wt}^\min_f(H) + t - 1\}$. If the fraction $\frac{t}{|V(H)|+|E(H)|}$ is an integer then

$$\text{th}(mG, H) \leq \text{th}(G, H) + \frac{(m-1)t}{|V(H)|+|E(H)|}.$$ 

Moreover, if $\text{th}(G, H) = \left[1 + \frac{t}{|V(H)|+|E(H)|}\right] = 1 + \frac{t}{|V(H)|+|E(H)|}$ then

$$\text{th}(mG, H) = \text{th}(G, H) + \frac{(m-1)t}{|V(H)|+|E(H)|} = 1 + \frac{mt}{|V(H)|+|E(H)|}.$$ 

Theorem 2 gives the exact value for the total $P_r$-irregularity strength for a path $P_r$. Moreover, the $P_r$-irregular total $\{(s + r - 1)/(2s - 1)\}$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_r$-weights consists of $t$ consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that

$$\text{th}(mP_r, P_s) = \text{th}(P_s, P_s) + (m - 1)\frac{r - s + 1}{2s - 1} = \left[\frac{s + r - 1}{2s - 1}\right] + (m - 1)\frac{r - s + 1}{2s - 1}$$

$$= \left[\frac{r - s + 1 + 2s - 1 - 1}{2s - 1}\right] + (m - 1)\frac{r - s + 1}{2s - 1}$$

$$= \left[\frac{r - s + 1}{2s - 1}\right] + 1 - \frac{1}{2s - 1} + (m - 1)\frac{r - s + 1}{2s - 1}$$

$$= \frac{r - s + 1}{2s - 1} + 1 + (m - 1)\frac{r - s + 1}{2s - 1} = m\frac{r - s + 1}{2s - 1} + 1.$$ 

Thus we obtain the following result.

Corollary 7. Let $m, r, s, m \geq 1, 2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then

$$\text{th}(mP_r, P_s) = \frac{m(r - s + 1)}{(2s - 1)} + 1.$$ 

If $H$ is isomorphic to $K_2$ then $\text{th}(G, K_2) = \text{tes}(G)$. Immediately from Theorem 4 the following corollary follows.

Corollary 8. Let $m$ be a positive integer. Then

$$\left\lceil\frac{m|E(G)| + 2}{3}\right\rceil \leq \text{th}(mG, K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m - 1)\left\lceil\frac{\text{wt}^\max_f - \text{wt}^\min_f + 1}{3}\right\rceil,$$

where $\text{wt}^\max_f$ and $\text{wt}^\min_f$ are the largest and smallest edge weights under a total $\text{tes}(G)$-labeling $f$ of $G$. 
3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total $H$-irregularity strength of two graphs $G_1$ and $G_2$ admitting an $H$-covering. From Theorem 1 we immediately obtain

**Corollary 9.** Let $G_i$, $i = 1, 2$, be a graph admitting an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G_1 \cup G_2, H) \geq 1 + \frac{t_1 + t_2 - 1}{|V(H)| + |E(H)|}.$$ 

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

**Theorem 10.** Let $G_i$, $i = 1, 2$, be a graph having an $H$-irregular total $\text{ths}(G_i, H)$-labeling $f_i$. Then

$$\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \frac{\text{wt}^\text{max}(H)_{f_2} - \text{wt}^\text{min}(H)_{f_1} + 1}{|V(H)| + |E(H)|} \right\}, \right.$$

$$\left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \frac{\text{wt}^\text{max}(H)_{f_1} - \text{wt}^\text{min}(H)_{f_2} + 1}{|V(H)| + |E(H)|} \right\} \right\},$$

where $\text{wt}_{f_i}^\text{max}(H)$ and $\text{wt}_{f_i}^\text{min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f_i$ of $G_i$.

**Proof.** Let $G_i$, $i = 1, 2$, be a graph that admits an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1_i, H^2_i, \ldots, H^{t_i}_i$. Assume that $f_i$ is an $H$-irregular total $k_i$-labeling of a graph $G_i$ with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $\text{wt}_{f_i}^\text{min}(H)$. Evidently

$$(10) \quad \text{wt}_{f_i}^\text{min}(H) \geq |V(H)| + |E(H)|.$$ 

Analogously, the largest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $\text{wt}_{f_i}^\text{max}(H)$. It holds that

$$(11) \quad \text{wt}_{f_i}^\text{max}(H) \geq \text{wt}_{f_i}^\text{min}(H) + t_i - 1$$

and

$$(12) \quad \text{wt}_{f_i}^\text{max}(H) \leq (|V(H)| + |E(H)|)k_i.$$ 

Thus $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, k_i\}$ and

$$(13) \quad \{\text{wt}_{f_i}(H^j_i) : j = 1, 2, \ldots, t_i\} \subset \{\text{wt}_{f_i}^\text{min}(H), \text{wt}_{f_i}^\text{min}(H) + 1, \ldots, \text{wt}_{f_i}^\text{max}(H)\}.$$
Let us define the total labeling $g$ of $G_1 \cup G_2$ in the following way.

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in V(G_1) \cup E(G_1), \\ f_2(x) + \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil & \text{if } x \in V(G_2) \cup E(G_2). \end{cases}$$

Evidently, all the labels are not greater than

$$\max \left\{ k_1, k_2 + \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}.$$

For the weight of the subgraph $H_j^1$, $j = 1, 2, \ldots, t_1$, isomorphic to the graph $H$ under the labeling $g$ we get

$$w_t^g(H_j^1) = \sum_{v \in V(H_j^1)} g(v) + \sum_{e \in E(H_j^1)} g(e) = \sum_{v \in V(H_j^1)} f_1(v) + \sum_{e \in E(H_j^1)} f_1(e) = w_{f_1}^t(H_j^1).$$

For the weight of the subgraph $H_j^2$, $j = 1, 2, \ldots, t_2$, isomorphic to the graph $H$ under the labeling $g$ we get

$$w_t^g(H_j^2) = \sum_{v \in V(H_j^2)} g(v) + \sum_{e \in E(H_j^2)} g(e) = \sum_{v \in V(H_j^2)} f_2(v) + \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + \sum_{e \in E(H_j^2)} f_2(e) + |V(H)| \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + |E(H)| \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil = w_{f_2}^t(H_j^2) + (|V(H)| + |E(H)|) \left\lceil \frac{w_{t_{f_1}}^{\max}(H) - w_{t_{f_2}}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

According to (13) we get that the largest weight of a subgraph $H$ under the total labeling $g$ in $G_1$, denoted by $w_{t_{g}^{\max}}(H : H \subset G_1)$, is at most

$$w_{t_{g}^{\max}}(H : H \subset G_1) = w_{t_{f_1}^{\max}}(H)$$

and the smallest weight of a subgraph $H$ under the total labeling $g$ in $G_2$, denoted by $w_{t_{g}^{\min}}(H : H \subset G_2)$, is at least

$$w_{t_{g}^{\min}}(H : H \subset G_2) \geq w_{t_{f_2}^{\min}}(H) + (|V(H)| + |E(H)|) \left\lceil \frac{w_{t_{f_1}^{\max}}(H) - w_{t_{f_2}^{\min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$


Note, that when writing $H_i$ we only consider subgraphs of $G_i$ isomorphic to $H$. As  
\[
\frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H) + 1}{|V(H)| + |E(H)|} \geq \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H) + 1}{|V(H)| + |E(H)|}
\]
we get
\[
wt_g^{min}(H : H \subseteq G_2) \geq wt_{f_2}^{min}(H) + (|V(H)| + |E(H)|) \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H) + 1}{|V(H)| + |E(H)|}
\geq wt_{f_2}^{min}(H) + (wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H) + 1) = wt_{f_1}^{max}(H) + 1
\geq wt_{f_1}^{max}(H) = wt_g^{max}(H : H \subseteq G_1).
\]
Thus all the $H$-weights in $G_1 \cup G_2$ are distinct.
Analogously we can define the total labeling $h$ of $G_1 \cup G_2$ such that
\[
h(x) = f_2(x) \quad \text{if } x \in V(G_2) \cup E(G_2),
\]
\[
h(x) = f_1(x) + \left[ \frac{wt_{f_2}^{max}(H) - wt_{f_1}^{min}(H) + 1}{|V(H)| + |E(H)|} \right] \quad \text{if } x \in V(G_1) \cup E(G_1).
\]
Using similar arguments we can also show that under the total labeling $h$ the $H$-weights in $G_1 \cup G_2$ are distinct.
Thus $g$ and $h$ are $H$-irregular total labelings of $G$. Immediately from this fact we get
\[
\ths(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \ths(G_2, H), \ths(G_1, H) + \left[ \frac{wt_{f_2}^{max}(H) - wt_{f_1}^{min}(H) + 1}{|V(H)| + |E(H)|} \right] \right\}, \right. \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H) + 1}{|V(H)| + |E(H)|} \left. \right) \right\}.
\]
Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

**Theorem 11** [18]. The total edge irregularity strength of the disjoint union of graphs $G_1, G_2, \ldots, G_m$, $m \geq 2$, is
\[
tes \left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.
\]
If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that

$$\text{ths}(G_1 \cup G_2, K_2) = \text{tes}(G_1 \cup G_2)$$

$$\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lceil \frac{3\text{tes}(G_2) - 2}{3} \right\rceil \right\}, \right.$$  

$$\max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lceil \frac{3\text{tes}(G_1) - 2}{3} \right\rceil \right\} \right\}$$

$$= \text{tes}(G_1) + \text{tes}(G_2)$$

which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total $\text{ths}(G, H)$-labeling $f$ and $m$ is a positive integer then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left( \frac{\text{wt}_{\max}(H) - \text{wt}_{\min}(H) + 1}{|V(H)| + |E(H)|} \right),$$

where $\text{wt}_{\max}(H)$ and $\text{wt}_{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

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