Discussiones Mathematicae
Graph Theory xx (xxxx) 1–17
doi:10.7151/dmgt.2153

DECOMPOSING THE COMPLETE GRAPH INTO HAMILTONIAN PATHS (CYCLES) AND 3-STARS

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Abstract

Let H be a graph. A decomposition of H is a set of edge-disjoint subgraphs of H whose union is H. A Hamiltonian path (respectively, cycle) of H is a path (respectively, cycle) that contains every vertex of H exactly once. A k-star, denoted by S_k, is a star with k edges. In this paper, we give necessary and sufficient conditions for decomposing the complete graph into α copies of Hamiltonian path (cycle) and β copies of S_3.

Keywords: decomposition, complete graph, Hamiltonian path, Hamiltonian cycle, star.

2010 Mathematics Subject Classification: 05C70, 05C38.

1. Introduction

For positive integers m and n, K_n denotes the complete graph with n vertices, and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. Let k be a positive integer. A *k-path*, denoted by P_k, is a path on k vertices. A *k-cycle*, denoted by C_k, is a cycle of length k. A *k-star*, denoted by S_k, is a star with k edges, i.e., $S_k = K_{1,k}$. Let H be a graph. A *spanning subgraph* of H is a subgraph of H containing every vertex of H. A spanning path (respectively, cycle) of H is called a Hamiltonian path (respectively, cycle) of H. A 1-*factor* of G is a spanning subgraph of G in which each vertex is incident with exactly one edge.

Let F, G, and H be graphs. A *decomposition* of H is a set of edge-disjoint subgraphs of H whose union is H. If H can be decomposed into α copies of F and β copies of G for nonnegative integers α and β, then we say that H has an $\{\alpha F, \beta G\}$-decomposition. Furthermore, if $\alpha \geq 1$ and $\beta \geq 1$, then we say that H has an (F, G)-*decomposition* or H is (F, G)-*decomposable*.

Study on the existence of an (F, G)-decomposition of a graph has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of (K_k, S_k)-decomposition of the complete graph K_n. Abueida and Daven [4] investigated the problem of the (C_4, E_2)-decomposition of several graph products where E_2 denotes two vertex disjoint edges. Abueida and O'Neil [8] studied the existence problem for (C_k, S_{k-1})-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [25, 26] investigated the existence of (G, H)-decompositions of λK_n and $\lambda K_{n,n}$ where $G, H \in \{C_n, P_n, S_{n-1}\}$. A *graph-pair* (G, H) *of order* m is a pair of non-isomorphic graphs G and H with $V(G) = V(H)$ such that both G and H contain no isolated vertices and $G \cup H$ is isomorphic to K_m. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of n for which λK_n admits a (G, H)-decomposition where (G, H) is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ into the graph-pairs of order 4 and 5, respectively, where F is a Hamiltonian cycle, a 1-factor, or an almost 1-factor. Lee [18, 19], Lee and Lin [22], and Lin [23] established necessary and sufficient conditions for the existence of (C_k, S_k)-decompositions of the complete bipartite graph, the complete bipartite multigraph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Abueida and Lian [7] and Beggas *et al.* [10] investigated the problems of (C_k, S_k)-decompositions of the complete graph K_n and λK_n, giving some necessary or sufficient conditions for such decompositions to exist. Lee and Chen [20] completely settled the existence problem of (P_{k+1}, S_k)-decompositions of the complete multigraph λK_n and the balanced complete bipartite multigraph $\lambda K_{n,n}$.

Recently, the existence problem of an $\{\alpha F, \beta G\}$-decomposition of a graph where α and β are essential is also studied. Shyu gave necessary and sufficient conditions for the decomposition of K_n into paths and stars (both with 3 edges) [27], paths and cycles (both with k edges where $k = 3, 4$) [28, 29], and cycles and stars (both with 4 edges) [30]. He [31] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges.

Jeevadoss and Muthusamy [14,15] considered the $\{\alpha P_{k+1}, \beta C_k\}$-decomposability of $K_{m,n}$, K_n and $\lambda K_{m,n}$, giving some necessary or sufficient conditions for such decompositions to exist. Jeevadoss and Muthusamy [16] gave necessary and sufficient conditions for the existence of $\{\alpha P_5, \beta C_4\}$-decomposition of tensor product and cartesian product of complete graphs. In [33], Tarsi gave necessary and sufficient conditions for the existence of $\{\alpha F, \beta S_k\}$-decomposition of λK_n, where F is any subgraph of K_n and $\alpha = 0$. In this paper, we consider the existence of an $\{\alpha F, \beta G\}$-decomposition of the complete graph K_n with $F \in \{P_n, C_n\}$ and $G = S_3$, giving necessary and sufficient conditions.

2. Preliminaries

We first collect some needed terminology and notation. Let G be a graph. The *degree* of a vertex x of G, denoted by $\deg_G x$, is the number of edges incident with x. For $k \geq 2$, the vertex of degree k in S_k is the *center* of S_k and any vertex of degree 1 is a *pendent vertex* of S_k. Let $v_1 v_2 \cdots v_k$ denote the k-path through vertices $v_1, v_2, \ldots, v_k$ in order. The vertices v_1 and v_k are referred to as its *origin* and *terminus*, respectively. In addition, $P_k(v_1, v_k)$ denotes a k-path with origin v_1 and terminus v_k. We use $(v_1, v_2, \ldots, v_k)$ to denote the k-cycle through vertices $v_1, v_2, \ldots, v_k, v_1$ in order, and $S(u; v_1, v_2, \ldots, v_k)$ to denote a star with center u and pendent vertices $v_1, v_2, \ldots, v_k$. For $U, W \subseteq V(G)$ with $U \cap W = \phi$, we use $G[U]$ and $G[U, W]$ to denote the subgraph of G induced by U, and the maximal bipartite subgraph of G with bipartition (U, W), respectively. When $G_1, G_2, \ldots, G_t$ are edge disjoint subgraphs of a graph, use $G_1 \cup G_2 \cup \cdots \cup G_t$ to denote the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$.

Before going into more details, we present some results which are useful for our discussions.

Proposition 1 [11,13]**.** *For an even integer n and $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$, the complete graph K_n can be decomposed into $n/2$ copies of P_n, $P(1), P(2), \ldots, P(n/2)$ with $P(i+1) = x_i x_{i-1} x_{i+1} x_{i-2} \cdots x_{i+\frac{n}{2}-2} x_{i+\frac{n}{2}+1} x_{i+\frac{n}{2}-1} x_{i+\frac{n}{2}}$ for $0 \leq i \leq \frac{n}{2} - 1$, where the subscripts of x's are taken modulo n in the set of numbers $\{0, 1, 2, \ldots, n-1\}$.*

The following results are attributed to Walecki, see [9].

Lemma 2. *For an odd integer n and $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$, the complete graph K_n can be decomposed into $(n-1)/2$ copies of C_n, $C(1), C(2), \ldots, C((n-1)/2)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2})$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$.*

Lemma 3. *For an even integer n and $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$, the complete graph K_n can be decomposed into $n/2-1$ copies of C_n, $C(1), C(2), \ldots, C(n/2-1)$, and a 1-factor F, where $E(F) = \{x_0x_{n-1}, x_1x_{n-2}, x_2x_{n-3}, \ldots, x_{n/2-2}x_{n/2+1}, x_{n/2-1}x_{n/2}\}$ and $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+n/2+1}, x_{i+n/2-2}, x_{i+n/2}, x_{i+n/2-1})$ for $i = 1, 2, \ldots, n/2-1$, where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$.*

3. Decomposition of K_n Into n-Paths and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing K_n into α copies of P_n and β copies of S_3.

Lemma 4. *Let n be an odd integer and let α be a nonnegative integer. If $\binom{n}{2} - (n-1)\alpha$ is a nonnegative integer and $\binom{n}{2} - (n-1)\alpha \equiv 0 \pmod 3$, then*

$$\alpha \in \begin{cases} \{0, 1, \ldots, (n-1)/2\} & \text{if } n \equiv 1 \pmod 6, \\ \{(n-3)/2 - 3t \mid t = 0, 1, \ldots, (n-3)/6\} & \text{if } n \equiv 3 \pmod 6, \\ \{(n-3)/2 - 3t \mid t = 0, 1, \ldots, (n-5)/6\} & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Proof. Since $\binom{n}{2} - (n-1)\alpha$ is a nonnegative integer and n is odd, $\alpha \le \lfloor \binom{n}{2}/(n-1) \rfloor = (n-1)/2$. Let $\alpha = (n-1)/2 - (3t+j)$ where t is a nonnegative integer and $j \in \{0, 1, 2\}$. Since $\binom{n}{2} - (n-1)\alpha = n(n-1)/2 - (n-1)\alpha = (n-2\alpha)(n-1)/2 = (6t+2j+1)(n-1)/2$, $\binom{n}{2} - (n-1)\alpha \equiv (2j+1)(n-1)/2 \pmod 3$. If $n \equiv 1 \pmod 6$, then $(2j+1)(n-1)/2 \equiv 0 \pmod 3$ for any integer j. Hence $\alpha \in \{0, 1, \ldots, (n-1)/2\}$ for $n \equiv 1 \pmod 6$. When $n \equiv 3 \pmod 6$ or $n \equiv 5 \pmod 6$, the condition $(2j+1)(n-1)/2 \equiv 0 \pmod 3$ holds if and only if $j = 1$. Thus $\alpha = (n-3)/2 - 3t$ for some integer t when $n \equiv 3 \pmod 6$ or $n \equiv 5 \pmod 6$. Since α is a nonnegative integer, we have $t \le (n-3)/6$ for $n \equiv 3 \pmod 6$, and $t \le (n-5)/6$ for $n \equiv 5 \pmod 6$. This completes the proof. ∎

Lemma 5. *Let n be an even integer, and let α be a nonnegative integer. If $\binom{n}{2} - (n-1)\alpha \equiv 0 \pmod 3$, then*

$$\alpha \in \begin{cases} \{n/2 - 3t \mid t = 0, 1, \ldots, n/6\} & \text{if } n \equiv 0 \pmod 6, \\ \{n/2 - 3t \mid t = 0, 1, \ldots, (n-2)/6\} & \text{if } n \equiv 2 \pmod 6, \\ \{0, 1, \ldots, n/2\} & \text{if } n \equiv 4 \pmod 6. \end{cases}$$

Proof. Since $\binom{n}{2} - (n-1)\alpha$ is a nonnegative integer and n is even, $\alpha \le \lfloor \binom{n}{2}/(n-1) \rfloor = n/2$. Let $\alpha = n/2 - (3t+j)$ where t is a nonnegative integer and $j \in \{0, 1, 2\}$. Since $\binom{n}{2} - (n-1)\alpha = n(n-1)/2 - (n-1)\alpha = (n-2\alpha)(n-1)/2 = (3t+j)(n-1)$, $\binom{n}{2} - (n-1)\alpha \equiv j(n-1) \pmod 3$. If $n \equiv 4 \pmod 6$, then $j(n-1) \equiv 0 \pmod 3$ for any integer j. Hence $\alpha \in \{0, 1, \ldots, n/2\}$ for $n \equiv 4 \pmod 6$. When $n \equiv 0$

(mod 6) or $n \equiv 2 \pmod 6$, the condition $j(n-1) \equiv 0 \pmod 3$ holds if and only if $j = 0$. Thus $\alpha = n/2 - 3t$ for some integer t when $n \equiv 0 \pmod 6$ or $n \equiv 2 \pmod 6$. Since α is a nonnegative integer, we have $t \leq n/6$ for $n \equiv 0 \pmod 6$, and $t \leq (n-2)/6$ for $n \equiv 2 \pmod 6$. This completes the proof. ■

The following indecomposable case is trivial.

Lemma 6. *The complete graph K_4 cannot be decomposed into*

(1) *one copy of P_4 and one copy of S_3, nor*

(2) *two copies of S_3.*

In addition, we exclude the possibility $n = 5$.

Lemma 7. *The complete graph K_5 cannot be decomposed into one copy of P_5 and two copies of S_3.*

Proof. Suppose, on the contrary, that K_5 can be decomposed into one copy of P_5, say $P_5(x, y)$, and two copies of S_3, say S and T. Note that the edge xy must be in either S or T. Without loss of generality, assume that xy is in S. Since the degree of every vertex of $K_n - E(P_5(x, y) \cup S)$ is less than 3, we have a contradiction. ■

In the remainder of the paper, we assume that $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$.

Lemma 8. *If n is an odd integer with $n \geq 7$, then the following hold:*

(1) *The complete graph K_n can be decomposed into $(n-1)/2$ copies of P_n and $(n-1)/6$ copies of S_3 when $n \equiv 1 \pmod 6$.*

(2) *The complete graph K_n can be decomposed into $(n-3)/2$ copies of P_n and $(n-1)/2$ copies of S_3.*

(3) *The complete graph K_n can be decomposed into $(n-5)/2$ copies of P_n and $5(n-1)/6$ copies of S_3 when $n \equiv 1 \pmod 6$.*

Proof. By Lemma 2, K_n can be decomposed into $(n-1)/2$ copies of C_n, $C(1)$, $C(2), \ldots, C((n-1)/2)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2})$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$.

(1) For $i = 1, 2, \ldots, (n-1)/2$, let $P(i) = C(i) - \{x_0 x_i\}$. Clearly, $P(i)$ is an n-path. Let G be the subgraph of K_n which is induced by the set of edges $x_0x_1, x_0x_2, \ldots, x_0x_{(n-1)/2}$. Obviously, $G = S_{(n-1)/2}$. Since n is odd and $n-1 \equiv 0 \pmod 3$, the graph G can be decomposed into $(n-1)/6$ copies of S_3. This settles (1).

(2) For $n = 7$, the complete graph K_7 can be decomposed into 2 copies of P_7 and 3 copies of S_3 as follows: $x_6x_2x_5x_3x_4x_0x_1$, $x_6x_4x_5x_0x_3x_1x_2$, $(x_1; x_4, x_5, x_6)$, $(x_2; x_0, x_3, x_4)$, $(x_6; x_0, x_3, x_5)$.

Now we consider the case $n \geq 9$. For $i \in \{1, 2, \ldots, (n-1)/2\} - \{(n-7)/2, (n-3)/2\}$, let $P(i) = C(i) - \{x_0x_i\}$. Note that $x_{n-1}x_{n-7} \in E(C((n-7)/2))$ and $P((n-3)/2) = C((n-3)/2) = (x_0, x_{(n-3)/2}, x_{(n-5)/2}, x_{(n-1)/2}, x_{(n-7)/2}, \ldots, x_{n-4}, x_{n-1}, x_{n-3}, x_{n-2})$. Let $P((n-7)/2) = C((n-7)/2) - \{x_{n-1}x_{n-7}\}$ and $C((n-3)/2) - \{x_0x_{(n-3)/2}, x_{(n-1)/2}x_{(n-7)/2}\} \cup \{x_0x_{(n-1)/2}\}$. Hence $P(i)$ is an n-path for $i = 1, 2, \ldots, (n-1)/2$. Moreover, $P((n-1)/2) = x_{(n-1)/2}x_{(n-3)/2}x_{(n+1)/2}x_{(n-5)/2} \cdots x_{n-3}x_1x_{n-2}x_{n-1}x_0$. For $i = 1, 2, \ldots, (n-3)/2$, let $S(i) = (x_{(n-1)/2-i}; x_{(n-1)/2+i-1}, x_{(n-1)/2+i})$ and $S = (x_{n-1}; x_{n-2}, x_0)$. Obviously, $S(i)$ and S are 2-stars, and $P((n-1)/2)$ can be decomposed into $S(1), S(2), \ldots, S((n-3)/2)$ and S. Furthermore, let $W(i) = S(i) \cup \{x_0x_i\}$ for $i = 1, 2, \ldots, (n-3)/2 - \{(n-7)/2\}$, let $W((n-7)/2) = S((n-7)/2) \cup \{x_{(n-1)/2}x_{(n-7)/2}\}$, and let $W((n-1)/2) = S \cup \{x_{n-1}x_{n-7}\}$. Clearly, $W(i)$ is a 3-star. This settles (2).

(3) We will remove one edge from $C(i)$ to obtain an n-path for $i \in \{1, 2, \ldots, (n-5)/2\}$, and use $C((n-3)/2)$ and $C((n-1)/2)$ together with the edges removed from $C(i)$'s to constitute $5(n-1)/3$ copies of S_3.

Let $S(i) = (x_{(n-1)/2+3i-3}; x_{(n-1)/2-3i+1}, x_{(n-1)/2-3i})$ for $i = 1, 2, \ldots, (n-1)/6$, $S'(i) = (x_{(n-1)/2-3i-1}; x_{(n-1)/2+3i-2}, x_{(n-1)/2+3i-1})$ for $i = 1, 2, \ldots, (n-7)/6$, and $S'((n-1)/6) = (x_{n-2}; x_{n-3}, x_0)$. Obviously, $S(i)$ and $S'(i)$ are 2-stars. Let $J = \{j | 2 \leq j \leq (n-1)/2 \text{ and } j \equiv 0, 2 \pmod 3\}$. For $j \in J$, let

$$e''_j = \begin{cases} x_{(n-1)/2-j}x_{(n-1)/2+j-2} & \text{if } j \equiv 0 \pmod 3, \\ x_{(n-1)/2-j}x_{(n-1)/2+j-3} & \text{if } j \equiv 2 \pmod 3, \end{cases}$$

where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$. It is easy to see that $\{S(i), S'(i) | i = 1, 2, \ldots, (n-1)/6\} \cup \{e''_j | j \in J\}$ is a decomposition of $C((n-3)/2) - \{x_{(n-3)/2}x_0\}$.

Note that $C((n-1)/2) = (x_0, x_{(n-1)/2}, x_{(n-3)/2}, x_{(n+1)/2}, x_{(n-5)/2}, \ldots, x_{n-3}, x_1, x_{n-2}, x_{n-1})$. Let $S''(j) = (x_{(n-1)/2-j}; x_{(n-1)/2+j-1}, x_{(n-1)/2+j})$ for $j = 1, 2, \ldots, (n-3)/2$ and $S''((n-1)/2) = (x_{n-1}; x_{n-2}, x_0)$ where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \ldots, n-1\}$. Obviously, $S''(j)$ is a 2-star, and $C((n-1)/2) - \{x_{(n-1)/2}x_0\}$ can be decomposed into $S''(1), S''(2), \ldots, S''((n-1)/2)$.

For $i = 2, 3, \ldots, (n-1)/6$, let e_i be an edge in $C(i-1)$ incident with the center of $S(i)$. Then $C(i-1) - \{e_i\}$ is an n-path and $S(i) \cup \{e_i\}$ is a 3-star. For $i = 1, 2, \ldots, (n-1)/6$, let e'_i be an edge in $C((n-1)/6 + i - 1)$ incident with the center of $S'(i)$. Then $C((n-1)/6 + i - 1) - \{e'_i\}$ is an n-path and $S'(i) \cup \{e'_i\}$ is a 3-star. Let $K = \{k | 4 \leq k \leq (n-5)/2 \text{ and } k \equiv 1 \pmod 3\}$. For $k \in K$, let e''_k be an edge in $C((k-1)/3 + (n-1)/3 - 1)$ incident with the center of $S''(k)$.

Then $C((k-1)/3+(n-1)/3-1)-\{e''_k\}$ is an n-path and $S''(i)\cup\{e''_k\}$ is a 3-star. For $j\in J$, $S''(j)\cup\{e''_j\}$ is a 3-star. Moreover, $S(1)\cup\{x_{(n-1)/2}x_0\}$ and $S''(1)\cup\{x_{(n-3)/2}x_0\}$ are also 3-stars. This completes the proof. ■

Lemma 9. *If n is an even integer with $n\geq 4$, then the following hold:*

(1) *The complete graph K_n can be decomposed into $n/2$ copies of n-paths.*

(2) *The complete graph K_n can be decomposed into $n/2-1$ copies of P_n and $(n-1)/3$ copies of S_3 when $n\equiv 4 \pmod 6$ and $n\geq 10$.*

(3) *The complete graph K_n can be decomposed into $n/2-2$ copies of P_n and $2(n-1)/3$ copies of S_3 when $n\equiv 4 \pmod 6$ and $n\geq 10$.*

Proof. By Proposition 1, we have (1).

(2) For $n=10$, the complete graph K_{10} can be decomposed into 4 copies of P_{10} and 3 copies of S_3 as follows: $x_8x_2x_7x_3x_6x_4x_5x_0x_1x_9$, $x_1x_3x_8x_4x_7x_5x_6x_0x_2x_9$, $x_0x_3x_2x_4x_1x_5x_8x_6x_7x_9$, $x_0x_7x_8x_9x_4x_3x_5x_2x_6x_1$, $(x_0;x_4,x_8,x_9)$, $(x_1;x_2,x_7,x_8)$, $(x_9;x_3,x_5,x_6)$.

Now we consider the case $n\geq 16$. Let $G=K_n[\{x_0,x_1,\ldots,x_{n-2}\}]$. Clearly G is isomorphic to K_{n-1}. By Lemma 2, the graph G can be decomposed into $n/2-1$ copies of C_{n-1}, $C(1),C(2),\ldots,C(n/2-1)$ with $C(i)=(x_0,x_i,x_{i-1},x_{i+1},x_{i-2},\ldots,x_{i+n/2-3},x_{i+n/2},x_{i+n/2-2},x_{i+n/2-1})$ for $i=1,2,\ldots,n/2-1$, where the subscripts of x's are taken modulo $n-2$ in the set of numbers $\{1,2,\ldots,n-2\}$. Note that $C(1)$ contains edges x_1x_{n-2} and $x_{n/2}x_0$, $C(2)$ contains edges x_2x_1 and $x_{n/2+1}x_0$, and $C(3)$ contains the edge x_4x_1. Let $P(1)=C(1)\cup\{x_1x_{n-1}x_{n/2}\}-\{x_1x_{n-2},x_{n/2}x_0\}$, $P(2)=C(2)\cup\{x_2x_{n-1}x_{n/2+1}\}-\{x_2x_1,x_{n/2+1}x_0\}$, and $P(3)=C(3)\cup\{x_4x_{n-1}\}-\{x_1x_4\}$. In addition, let $P(i)=C(i)\cup\{x_{i+n/2-1}x_{n-1}\}-\{x_{i+n/2-1}x_0\}$ for $i=4,5,\ldots,n/2-1$. Obviously, $P(i)$ ia an n-path for $i=1,2,\ldots,n/2-1$. Let $S(1)=(x_0;x_{n/2},x_{n/2+1},x_{n/2+3},x_{n/2+4},\ldots,x_{n-2})$ and $S(2)=(x_{n-1};x_0,x_3,x_5,x_6,\ldots,x_{n/2-2},x_{n/2-1},x_{n/2+2})$. It is easy to see that $K_n-E\left(\bigcup_{i=1}^{n/2-1}P(i)\right)=S(1)\cup S(2)\cup(x_1;x_2,x_4,x_{n-2})$. Note that $S(1)$ and $S(2)$ are $(n/2-2)$-stars. Since $n\equiv 4 \pmod 6$, each of $S(1)$ and $S(2)$ can be decomposed into $(n-4)/6$ copies of S_3. This settles (2).

(3) By Lemma 3, K_n can be decomposed into $n/2-1$ copies of C_n, $C(1),C(2),\ldots,C(n/2-1)$, and a 1-factor F, where $E(F)=\{x_0x_{n-1},x_1x_{n-2},x_2x_{n-3},\ldots,x_{n/2-2}x_{n/2+1},x_{n/2-1}x_{n/2}\}$ and $C(i)=(x_0,x_i,x_{i-1},x_{i+1},x_{i-2},\ldots,x_{i+n/2+1},x_{i+n/2-2},x_{i+n/2},x_{i+n/2-1})$ for $i=1,2,\ldots,n/2-1$, where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1,2,\ldots,n-1\}$.

We obtain $n/2-2$ copies of P_n by removing one edge from each of n-cycles $C(1),C(2),\ldots,C(n/2-2)$. For $i=1,2,\ldots,n/2-3$, let $P(i)=C(i)-\{x_0x_i\}$. In addition, let $P(n/2-2)=C(n/2-2)-\{x_{n/2-3}x_{n/2-1}\}$. Trivially, $P(i)$ is an n-path for $i=1,2,\ldots,n/2-2$.

In the following, $2(n-1)/3$ copies of S_3 are constructed. We first obtain $n/2$ copies of S_3 by using all of the edges of $C(n/2-1)$ and $n/2-1$ edges of F and the edge $x_{n/2-3}x_{n/2-1}$ removed from $C(n/2-2)$. Note that $C(n/2-1) = (x_0, x_{n/2-1}, x_{n/2-2}, x_{n/2}, x_{n/2-3}, \ldots, x_1, x_{n-3}, x_{n-1}, x_{n-2})$. For $i = 1, 2, \ldots, n/2-1$, let $S(i) = (x_{n/2-1+i}; x_{n/2-1-i}, x_{n/2-2-i})$ and $S = (x_{n/2-1}; x_{n/2-2}, x_0)$. Obviously, $S(i)$ and S are 2-stars, and $C(n/2-1)$ is decomposable into $S(1), S(2), \ldots, S(n/2-1)$ and S. Let $W(i) = S(i) \cup \{x_{n/2-1+i}x_{n/2-i}\}$ for $i = 1, 2, \ldots, n/2-1$, and let $W(n/2) = S \cup \{x_{n/2-3}x_{n/2-1}\}$. Clearly, $W(i)$ is a 3-star.

Now we obtain $(n-4)/6$ copies of S_3 by using one edge of F and the edges removed from $C(i)$'s in constructing n-paths for $i = 1, 2, \ldots, n/2-3$. Let G be the subgraph of K_n induced by the set of edges $x_0x_1, x_0x_2, \ldots, x_0x_{n/2-3}, x_0x_{n-1}$. Obviously, $G = S_{n/2-2}$. Since $n \equiv 4 \pmod 6$, the graph G can be decomposed into $(n-4)/6$ copies of S_3. This settles (3) and completes the proof. ■

Lemma 10. *Let n and t be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian paths of K_n, then $\bigcup_{i=1}^{t} Q_i$ is S_t-decomposable.*

Proof. Since each Q_i is a Hamiltonian path of K_n, we have $V(Q_i) = V(K_n)$. For each Q_i, we orient the edges of Q_i from x_0 along Q_i to the end (or ends) of the path, and use $\overrightarrow{Q_i}$ to denote the digraph obtained from Q_i for such an orientation. Note that there is exactly one arc directed into x_j for each $j \in \{1, 2, \ldots, n-1\}$. Let $\overrightarrow{G} = \bigcup_{i=1}^{t} \overrightarrow{Q_i}$. It is easy to check that $\deg^-_{\overrightarrow{G}} x_j = t$ for $j \neq 0$. Thus there exists an S_t-decomposition of $\bigcup_{i=1}^{t} Q_i$ such that x_j is a center of a t-star for $j \neq 0$. This completes the proof. ■

By Lemma 10, the union of $3t$ edge-disjoint n-paths can be decomposed into $n-1$ copies of S_{3t}, in turn, each S_{3t} can be decomposed in to t copies of S_3. Hence we have the following result.

Theorem 11. *Let n, p and t be positive integers with $p \geq 3t$, and let q be a nonnegative integer. If K_n can be decomposed into p copies of P_n and q copies of S_3, then K_n can be decomposed into $p-3t$ copies of P_n and $q+(n-1)t$ copies of S_3.*

Obviously, if K_n can be decomposed into α copies of n-paths and β copies of S_3, then $\binom{n}{2} = (n-1)\alpha + 3\beta$. Using Theorem 11 together with Lemmas 4 to 9, we have the main result of this section.

Theorem 12. *Let n be a positive integer with $n \geq 4$, and let α and β be nonnegative integers. The complete graph K_n can be decomposed into α copies of P_n and β copies of S_3 if and only if $\binom{n}{2} = (n-1)\alpha + 3\beta$ and $(n, \alpha, \beta) \notin \{(4,1,1), (4,0,2), (5,1,2)\}$.*

4. Decomposition of K_n Into n-Cycles and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing K_n into α copies of C_n and β copies of S_3. The first two lemmas in the following are from [17] and [32], respectively.

Lemma 13. *For an odd integer n and $V(K_{n,n}) = \{x_0, \dots, x_{n-1}\} \cup \{y_0, \dots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $(n-1)/2$ copies of C_{2n}, $C(0)$, $C(1), \dots, C((n-3)/2)$, and a 1-factor F, where $E(F) = \{x_0y_{n-1}, x_1y_0, \dots, x_{n-1}y_{n-2}\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \dots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \dots, (n-3)/2$.*

Lemma 14. *For an even integer n and $V(K_{n,n}) = \{x_0, \dots, x_{n-1}\} \cup \{y_0, \dots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $n/2$ copies of C_{2n}, $C(0)$, $C(1), \dots, C(n/2-1)$, where $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \dots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \dots, n/2-1$.*

Lemma 15. *Let n be an odd integer and let α be a nonnegative integer. If $\binom{n}{2} - n\alpha$ is a nonnegative integer and $\binom{n}{2} - n\alpha \equiv 0 \pmod 3$, then*

$$\alpha \in \begin{cases} \{0, 1, \dots, (n-1)/2\} & \text{if } n \equiv 0 \pmod 3, \\ \{(n-1)/2 - 3t \mid t = 0, 1, \dots, \lfloor (n-1)/6 \rfloor\} & \text{otherwise.} \end{cases}$$

Proof. Since $\binom{n}{2} - n\alpha$ is a nonnegative integer and n is odd, $\alpha \le \lfloor \binom{n}{2}/n \rfloor = (n-1)/2$. Let $\alpha = (n-1)/2 - (3t+i)$, where t is a nonnegative integer and $i \in \{0,1,2\}$. Since $\binom{n}{2} - n\alpha = n(n-1)/2 - n\alpha = n(n-1-2\alpha)/2 = n(3t+i)$, $\binom{n}{2} - n\alpha \equiv ni \pmod 3$. If n is a multiple of 3, then $ni \equiv 0 \pmod 3$ holds for any $i \in \{0,1,2\}$. Hence $\alpha \in \{0, 1, \dots, (n-1)/2\}$ for $n \equiv 0 \pmod 3$. Otherwise, the condition $ni \equiv 0 \pmod 3$ holds if and only if $i = 0$. This implies $\alpha = (n-1)/2 - 3t$. Moreover, $t \le \lfloor (n-1)/6 \rfloor$ since α is a nonnegative integer. This completes the proof. ■

Lemma 16. *Let n be an even integer and let α be a nonnegative integer. If $\binom{n}{2} - n\alpha$ is a nonnegative integer and $\binom{n}{2} - n\alpha \equiv 0 \pmod 3$, then*

$$\alpha \in \begin{cases} \{0, 1, \dots, n/2-1\} & \text{if } n \equiv 0 \pmod 3, \\ \{n/2 - 3t - 2 \mid t = 0, 1, \dots, \lfloor (n-4)/6 \rfloor\} & \text{otherwise.} \end{cases}$$

Proof. Since $\binom{n}{2} - n\alpha$ is a nonnegative integer and n is even, $\alpha \le \lfloor \binom{n}{2}/n \rfloor = n/2-1$. Let $\alpha = n/2-1-(3t+i)$, where t is a nonnegative integer and $i \in \{0,1,2\}$. Since $\binom{n}{2} - n\alpha = n(n-1-2\alpha)/2 = n(6t+2i+1)/2$, $\binom{n}{2} - n\alpha \equiv n(2i+1)/2 \pmod 3$. If $n \equiv 0 \pmod 3$, then $n/2 \equiv 0 \pmod 3$, this implies that $n(2i+1)/2 \equiv 0 \pmod 3$ holds for any $i \in \{0,1,2\}$. Hence $\alpha \in \{0, 1, \dots, n/2-1\}$ for $n \equiv 0$

(mod 3). Otherwise, the condition $n(2i+1)/2 \equiv 0 \pmod 3$ holds if and only if $i = 1$. This implies $\alpha = n/2 - 3t - 2$. Moreover, $t \le \lfloor (n-4)/6 \rfloor$ since α is a nonnegative integer. This completes the proof. ■

Let $m = (n-3)/2$ for odd n and $m = (n-2)/2$ for even n. Let $C(1), C(2), \dots, C(m)$ be edge-disjoint n-cycles in K_n, and let $G = K_n - \bigcup_{i=1}^{m} E(C(i))$. Since $\deg_G x = n - 1 - 2m \le 2$ for each vertex x, G has no S_3-decomposition. Thus we have the following result.

Lemma 17. *Let $n \equiv 0 \pmod 3$. The complete graph K_n cannot be decomposed into $(n-3)/2$ copy of C_n and $n/3$ copies of S_3 for odd n, and cannot be decomposed into $(n-2)/2$ copy of C_n and $n/6$ copies of S_3 for even n.*

Lemma 18. *If n is an odd integer with $n \ge 5$, then the following hold:*

(1) *The complete graph K_n can be decomposed into $(n-1)/2$ copies of C_n.*

(2) *The complete graph K_n can be decomposed into $(n-5)/2$ copies of C_n and $2n/3$ copies of S_3 when $n \equiv 3 \pmod 6$ and $n \ge 9$.*

(3) *The complete graph K_n can be decomposed into $(n-9)/2$ copies of C_n and $4n/3$ copies of S_3 when $n \equiv 3 \pmod 6$ and $n \ge 9$.*

Proof. By Lemma 2, the complete graph K_n can be decomposed into $(n-1)/2$ copies of C_n, $C(1), C(2), \dots, C((n-1)/2)$ with $C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, \dots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2})$ for $i = 1, 2, \dots, (n-1)/2$, where the subscripts of x's are taken modulo $n-1$ in the set of numbers $\{1, 2, \dots, n-1\}$. Hence we have (1).

(2) If there exist s and t $(1 \le s < t \le (n-1)/2)$ such that $C(s) \cup C(t)$ can be decomposed into $2n/3$ copies of S_3, then we have the result. Consider the case $s = (n+3)/6$ and $t = n/3$. Note that $C((n+3)/6) = (x_0, x_{(n+3)/6}, x_{(n-3)/6}, x_{(n+9)/6}, x_{(n-9)/6}, \dots, x_{n/3-1}, x_1, x_{n/3}, x_{n-1}, \dots, x_{2n/3-2}, x_{2n/3+1}, x_{2n/3-1}, x_{2n/3})$. For $i = 1, 2, \dots, n/3 - 1$, let $S_2(i) = (x_{n-i}; x_{n/3-1+i}, x_{n/3+i})$ and $S_2(n/3) = (x_{2n/3}; x_{2n/3-1}, x_0)$. For $j = 1, 2, \dots, (n-3)/6$, let $P_2(j) = x_j x_{n/3+1-j}$. For $j = (n+3)/6, (n+9)/6, \dots, n/3 - 1$, let $P_2(j) = x_j x_{n/3-j}$. In addition, let $P_2(0) = x_0 x_{(n+3)/6}$. Obviously, $S_2(i)$ is a 2-star for $i = 1, 2, \dots, n/3$, and $P_2(j)$ is a 2-path for $j = 0, 1, \dots, n/3 - 1$. One can see that $C((n+3)/6)$ can be decomposed into $S_2(1), S_2(2), \dots, S_2(n/3)$ and $P_2(0), P_2(1), \dots, P_2(n/3-1)$.

On the other hand, $C(n/3) = (x_0, x_{n/3}, x_{n/3-1}, x_{n/3+1}, x_{n/3-2}, x_{n/3+2}, \dots, x_{2n/3-2}, x_1, x_{2n/3-1}, x_{n-1}, \dots, x_{(5n-15)/6}, x_{(5n+3)/6}, x_{(5n-9)/6}, x_{(5n-3)/6})$. For $j = 1, 2, \dots, n/3 - 1$, let $S_2'(j) = (x_j; x_{2n/3-1-j}, x_{2n/3-j})$. For $i = 1, 2, \dots, (n+3)/6$, let $P_2'(i) = x_{n-i} x_{2n/3-2+i}$. For $i = (n+9)/6, (n+12)/6, \dots, n/3$, let $P_2'(i) = x_{n-i} x_{2n/3-1+i}$. In addition, let $P_2'(0) = x_0 x_{n/3}$ and $P_2''(0) = x_0 x_{(5n-3)/6}$. Obviously, $S_2'(j)$ is a 2-star for $i = 1, 2, \dots, n/3 - 1$, and $P_2''(0)$ and $P_2'(i)$ are 2-paths for $i = 0, 1, \dots, n/3$. One can see that $C(n/3)$ can be decomposed into $S_2'(1), S_2'(2), \dots, S_2'(n/3-1)$ and $P_2'(0), P_2'(1), \dots, P_2'(n/3)$ as well as $P_2''(0)$.

For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P'_2(i)$. For $j = 1, 2, \ldots, n/3 - 1$, let $S'_3(j) = S'_2(j) \cup P_2(j)$. Clearly, $S_3(i)$ and $S'_3(j)$ are 3-stars. In addition, $P_2(0) \cup P'_2(0) \cup P''_2(0)$ is also a 3-star. Hence $C((n+3)/6) \cup C(n/3)$ can be decomposed into $2n/3$ copies of S_3. This settles (2).

(3) According to the proof of (2), the result holds if there exist s' and t' ($s', t' \notin \{(n+3)/6, n/3\}$) such that $C(s') \cup C(t')$ can be decomposed into $2n/3$ copies of S_3. Consider the case $s' = (n+9)/6$ and $t' = n/3 + 1$. Note that $C((n+9)/6) = (x_0, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+15)/6}, x_{(n-3)/6}, \ldots, x_{n/3+1}, x_1, x_{n/3+2}, x_{n-1}, \ldots, x_{2n/3-1}, x_{2n/3+2}, x_{2n/3}, x_{2n/3+1})$. For $i = 1, 2, \ldots, n/3 - 1$, let $S_2(i) = (x_{n+1-i}; x_{n/3+i}, x_{n/3+1+i})$ with $x_n = x_1$ and $S_2(n/3) = (x_{2n/3+1}; x_{2n/3}, x_0)$. For $j = 2, 3, \ldots, (n+3)/6$, let $P_2(j) = x_j x_{n/3+3-j}$. For $j = (n+9)/6, (n+15)/6, \ldots, n/3$, let $P_2(j) = x_j x_{n/3+2-j}$. In addition, let $P_2(0) = x_0 x_{(n+9)/6}$. Obviously, $S_2(i)$ is a 2-star for $i = 1, 2, \ldots, n/3$, and $P_2(j)$ is a 2-path for $j = 0, 2, 3, \ldots, n/3$. One can see that $C((n+3)/6)$ can be decomposed into $S_2(1), S_2(2), \ldots, S_2(n/3)$ and $P_2(0), P_2(2), P_2(3), \ldots, P_2(n/3)$.

On the other hand, $C(n/3+1) = (x_0, x_{n/3+1}, x_{n/3}, x_{n/3+2}, x_{n/3-1}, \ldots, x_{2n/3}, x_1, x_{2n/3+1}, x_{n-1}, x_{2n/3+2}, x_{n-2}, \ldots, x_{(5n-9)/6}, x_{(5n+9)/6}, x_{(5n-3)/6}, x_{(5n+3)/6})$. For $j = 2, 3, \ldots, n/3$, let $S'_2(j) = (x_j; x_{2n/3+1-j}, x_{2n/3+2-j})$. For $i = 1, 2, \ldots, (n+3)/6$, let $P'_2(i) = x_{n+1-i} x_{2n/3-1+i}$, and for $i = (n+9)/6, (n+12)/6, \ldots, n/3$, let $P'_2(i) = x_{n+1-i} x_{2n/3+i}$ with $x_n = x_1$. In addition, let $P'_2(0) = x_0 x_{n/3+1}$ and $P''_2(0) = x_0 x_{(5n+3)/6}$. Obviously, $S'_2(j)$ is a 2-star for $i = 2, 3, \ldots, n/3$, and $P''_2(0)$ and $P'_2(i)$ are 2-paths for $i = 0, 1, \ldots, n/3$. One can see that $C(n/3+1)$ can be decomposed into $S'_2(2), S'_2(3), \ldots, S'_2(n/3)$ and $P'_2(0), P'_2(1), \ldots, P'_2(n/3)$ as well as $P''_2(0)$.

For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P'_2(i)$. For $j = 2, 3, \ldots, n/3$, let $S'_3(j) = S'_2(j) \cup P_2(j)$. Clearly, $S_3(i)$ and $S'_3(j)$ are 3-stars. In addition, $P_2(0) \cup P'_2(0) \cup P''_2(0)$ is also a 3-star. Hence $C((n+9)/6) \cup C(n/3+1)$ can be decomposed into $2n/3$ copies of S_3. This settles (3). ∎

For positive integers l and n with $1 \le l \le n$, the (n, l)-*crown* $C_{n,l}$ is the bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $B = \{y_0, y_1, \ldots, y_{n-1}\}$, and edge set $\{x_i y_j : i = 0, 1, \ldots, n-1,\ j \equiv i+1, i+2, \ldots, i+l \pmod{l}\}$.

Proposition 19 [24]. *$\lambda C_{n,l}$ is S_k-decomposable if and only if $k \le l$ and $\lambda n l \equiv 0 \pmod{k}$.*

Lemma 20. *If n is an even integer $n \ge 6$, then the following hold:*

(1) *The complete graph K_n can be decomposed into $n/2 - 2$ copies of C_n and $n/2$ copies of S_3.*

(2) *The complete graph K_n can be decomposed into $n/2 - 3$ copies of C_n and $5n/6$ copies of S_3 when $n \equiv 0 \pmod{6}$.*

(3) *The complete graph* K_n *can be decomposed into* $n/2-4$ *copies of* C_n *and* $7n/6$ *copies of* S_3 *when* $n \equiv 0 \pmod 6$ *and* $n \geq 12$.

Proof. Let $V(K_n) = X \cup Y$, where $X = \{x_0, \ldots, x_{n/2-1}\}$ and $Y = \{y_0, \ldots, y_{n/2-1}\}$. Note that $K_n = K_n[X] \cup K_n[Y] \cup K_n[X,Y]$ where $K_n[X]$ and $K_n[Y]$ are isomorphic to $K_{n/2}$ and $K_n[X,Y]$ is isomorphic to $K_{n/2,n/2}$. We distinguish two cases : Case 1. $n \equiv 0 \pmod 4$ and Case 2. $n \equiv 2 \pmod 4$.

Case 1. $n \equiv 0 \pmod 4$. By Lemma 14, $K_n[X,Y]$ can be decomposed into $n/4$ copies of C_n, $C(0)$, $C(1), \ldots, C(n/4-1)$, where $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n/2-2)}, x_{n/2-2}, y_{2i+(n/2-1)}, x_{n/2-1})$ for $i = 0, 1, 2, \ldots, n/4-1$. By Proposition 1, we have the following results. $K_n[X]$ can be decomposed into the following $n/4$ copies of $P_{n/2}$: $P_{n/2}(x_0, x_{n/4}), P_{n/2}(x_1, x_{1+n/4}), \ldots, P_{n/2}(x_{n/4-1}, x_{n/2}-1)$, and $K_n[Y]$ can be decomposed into the following $n/4$ copies of $P_{n/2}$: $P_{n/2}(y_0, y_{n/4})$, $P_{n/2}(y_1, y_{1+n/4}), \ldots, P_{n/2}(y_{n/4-1}, y_{n/2-1})$.

For $i = 0, 1, \ldots, n/4-1$, let $Q(i) = P_{n/2}(x_i, x_{i+n/4}) \cup P_{n/2}(y_i, y_{i+n/4}) \cup \{y_i x_i, y_{i+n/4} x_{i+n/4}\}$. Clearly, $Q(i)$ is an n-cycle, and $y_i x_i, y_{i+n/4} x_{i+n/4} \in E(C(0))$ for $i = 0, 1, \ldots, n/4-1$. For $1 \leq t \leq n/4-1$, let

$$R(t) = \left(\bigcup_{i=0}^{t} C(i) \right) - \left\{ y_i x_i, y_{i+n/4} x_{i+n/4} \,|\, 0 \leq i \leq n/4-1 \right\}.$$

It is easy to see that $R(t)$ is isomorphic to the crown $C_{n/2,2t+1}$. Therefore, K_n can be decomposed into $n/2-(t+1)$ copies of C_n, $Q(0), Q(1), \ldots, Q(n/4-1)$ and $C(t+1), C(t+2), \ldots, C(n/4-1)$, and one copy of $(n/2, 2t+1)$-crown $R(t)$. Note that $2t+1 \geq 3$ and $|E(R(t))| = |E(C_{n/2,2t+1})| = (2t+1)n/2$. If $(2t+1)n/2 \equiv 0 \pmod 3$, then $R(t)$ can be decomposed into $(2t+1)n/6$ copies of S_3 by Proposition 19. Hence for $n \equiv 0 \pmod 4$, we have the following.

If $t = 1$, then $(2t+1)n/2 = 3n/2 \equiv 0 \pmod 3$ for each n. Thus K_n can be decomposed into $n/2-2$ copies of C_n and $n/2$ copies of S_3.

If $t = 2$, then $(2t+1)n/2 = 5n/2 \equiv 0 \pmod 3$ for $n \equiv 0 \pmod 6$. Thus K_n can be decomposed into $n/2-3$ copies of C_n and $5n/6$ copies of S_3.

If $t = 3$, then $(2t+1)n/2 = 7n/2 \equiv 0 \pmod 3$ for $n \equiv 0 \pmod 6$. Thus K_n can be decomposed into $n/2-4$ copies of C_n and $7n/6$ copies of S_3. This settles Case 1.

Case 2. $n \equiv 2 \pmod 4$. Since $n \equiv 2 \pmod 4$, $n/2$ is odd. By Lemma 13, $K_n[X,Y]$ can decomposed into $(n-2)/4$ copies of C_n, $C(0), C(1), \ldots, C((n-6)/4)$, and a 1-factor F, where $E(F) = \{x_0 y_{n/2-1}, x_1 y_0, \ldots, x_{n/2-1} y_{n/2-2}\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n/2-1)}, x_{n/2-1})$ for $i = 0, 1, \ldots, (n-6)/4$.

Now we consider $K_n[X]$ and $K_n[Y]$. By Lemma 2, we have the following results. $K_n[X]$ can be decomposed into $(n-2)/4$ copies of $C_{n/2}$, $W(1), W(2), \ldots, W((n-2)/4)$ with $W(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-10)/4}, x_{i+(n+2)/4},$

$x_{i+(n-6)/4}, x_{i+(n-2)/4})$, and $K_n[Y]$ can be decomposed into $(n-2)/4$ copies of $C_{n/2}$, $W'(1), W'(2), \ldots, W'((n-2)/4)$ with $W'(i) = (y_0, y_i, y_{i-1}, y_{i+1}, y_{i-2}, \ldots, y_{i+(n-10)/4}, y_{i+(n+2)/4}, y_{i+(n-6)/4}, y_{i+(n-2)/4})$ for $i = 1, 2, \ldots, (n-2)/4$, where the subscripts of x's and y's are taken modulo $(n-2)/2$ in the set of numbers $\{1, 2, \ldots, (n-2)/2\}$. For $i = 1, 2, \ldots, (n-2)/4$, let

$$e(i) = \begin{cases} x_0x_1 & \text{if } i = 1, \\ x_ix_{i-1} & \text{if } i \text{ is odd and } i \geq 3, \\ x_{i+(n-6)/4}x_{i+(n-2)/4} & \text{if } i \text{ is even,} \end{cases}$$

and let

$$e'(i) = \begin{cases} y_0y_1 & \text{if } i = 1, \\ y_iy_{i-1} & \text{if } i \text{ is odd and } i \geq 3, \\ y_{i+(n-6)/4}y_{i+(n-2)/4} & \text{if } i \text{ is even.} \end{cases}$$

Let $P(i) = W(i) - \{e(i)\}$ and $P'(i) = W'(i) - \{e'(i)\}$. Trivially, $P(i)$ and $P'(i)$ are $(n/2)$-paths. Let $M = \{e(i) | 1 \leq i \leq (n-2)/4\}$ and $M' = \{e'(i) | 1 \leq i \leq (n-2)/4\}$. If $n \equiv 2 \pmod 8$, then $(n-2)/4$ is even. Hence $M = \{x_0x_1, x_2x_3, \ldots, x_{(n-10)/4}x_{(n-6)/4}, x_{(n+2)/4}x_{(n+6)/4}, \ldots, x_{n/2-2}x_{n/2-1}\}$ and $M' = \{y_0y_1, y_2y_3, \ldots, y_{(n-10)/4}y_{(n-6)/4}, y_{(n+2)/4}y_{(n+6)/4}, \ldots, y_{n/2-2}y_{n/2-1}\}$. If $n \equiv 6 \pmod 8$, then $(n-2)/4$ is odd. Hence $M = \{x_0x_1, x_2x_3, \ldots, x_{n/2-3}x_{n/2-2}\}$ and $M' = \{y_0y_1, y_2y_3, \ldots, y_{n/2-3}y_{n/2-2}\}$. Let H be the subgraph of $K_n[X]$ induced by M, and let H' be the subgraph of $K_n[Y]$ induced by M'. Clearly, $K_n[X]$ can be decomposed into H and $(n-2)/4$ copies of $P_{n/2}$, $P(1), P(2), \ldots, P((n-2)/4)$, and $K_n[Y]$ can be decomposed into H' and $(n-2)/4$ copies of $P_{n/2}$, $P'(1), P'(2), \ldots, P'((n-2)/4)$.

Let $Z = \{y_0x_0, y_1x_1\} \cup \{y_{i-1}x_{i-1}, y_ix_i | \, i \text{ is odd and } i \geq 3\} \cup \{y_{i+(n-6)/4} x_{i+(n-6)/4}, y_{i+(n-2)/4}x_{i+(n-2)/4} | \, i \text{ is even}\}$. Obviously, $Z \subseteq E(C(0))$. For $i = 1, 2, \ldots, (n-2)/4$, let $K = \{y_{i+(n-6)/4}x_{i+(n-6)/4}, y_{i+(n-2)/4}x_{i+(n-2)/4}\}$ and

$$Q(i) = \begin{cases} P(1) \cup P'(1) \cup \{y_0x_0, y_1x_1\} & \text{if } i = 1, \\ P(i) \cup P'(i) \cup \{y_{i-1}x_{i-1}, y_ix_i\} & \text{if } i \text{ is odd and } i \geq 3, \\ P(i) \cup P'(i) \cup K & \text{if } i \text{ is even,} \end{cases}$$

and let $Q((n+2)/4) = H \cup H' \cup C(0) - Z$. One can see that each $Q(i)$ is an n-cycle. Thus $K_n[X] \cup K_n[Y] \cup C(0)$ can be decomposed into $(n+2)/4$ copies of C_n. For $1 \leq t \leq (n-6)/4$, let

$$R(t) = \left(\bigcup_{i=1}^{t} C((n-6)/4 - i + 1) \right) \cup F.$$

It is easy to see that $R(t)$ is isomorphic to the crown $C_{n/2,2t+1}$. Hence $K_n[X, Y]$ can be decomposed into $n/2 - (t+1)$ copies of C_n, $Q(1), Q(2), \ldots, Q((n+2)/4)$

and $C(1), C(2), \ldots, C((n-6)/4-t)$, and one copy of $(n/2, 2t+1)$-crown $R(t)$. Note that $2t+1 \geq 3$ and $|E(R(t))| = |E(C_{n/2,2t+1})| = (2t+1)n/2$. If $(2t+1)n/2 \equiv 0 \pmod 3$, then $R(t)$ can be decomposed into $(2t+1)n/6$ copies of S_3 by Proposition 19.

If $t = 1$, then $(2t+1)n/2 = 3n/2 \equiv 0 \pmod 3$ for each n. Thus K_n can be decomposed into $n/2-2$ copies of C_n and $n/2$ copies of S_3.

If $t = 2$, then $(2t+1)n/2 = 5n/2 \equiv 0 \pmod 3$ for $n \equiv 0 \pmod 6$. Thus K_n can be decomposed into $n/2-3$ copies of C_n and $5n/6$ copies of S_3.

If $t = 3$, then $(2t+1)n/2 = 7n/2 \equiv 0 \pmod 3$ for $n \equiv 0 \pmod 6$. Thus K_n can be decomposed into $n/2-4$ copies of C_n and $7n/6$ copies of S_3. This settles Case 2. ■

Let x and y be distinct vertices of a multigraph G. We use $e_G(x, y)$ to denote the number of edges joining x and y. A star decomposition of G is *center balanced* if every vertex of G is the center of the same number of members in the decomposition.

Proposition 21 [21]**.** *Let G be an r-regular multigraph with $r \geq 1$. Then G has a center balanced S_t-decomposition if and only if $r \equiv 0 \pmod{2t}$ and $e_G(x, y) \leq r/t$ for all $x, y \in V(G)$ with $x \neq y$.*

Lemma 22. *Let n and t be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian cycles of K_n, then $\bigcup_{i=1}^{t} Q_i$ is S_t-decomposable.*

Proof. Since each $Q(i)$ is 2-regular and $V(Q(i)) = V(Q(j))$ for $i, j \in \{1, 2, \ldots, t\}$, $\bigcup_{i=1}^{t} Q_i$ is $2t$-regular. Since $2t \equiv 0 \pmod{2t}$ and $e_{\bigcup_{i=1}^{t} Q_i}(x, y) \leq 1 < (2t)/t$ for all $x, y \in V(\bigcup_{i=1}^{t} Q_i)$ with $x \neq y$, the result follows from Proposition 21. ■

By Lemma 22, the union of $3t$ copies of edge-disjoint n-cycles can be decomposed into n copies of S_{3t}, in turn, each S_{3t} can be decomposed in to t copies of S_3. Hence we have the following result.

Theorem 23. *Let n, p and t be positive integers with $p \geq 3t$, and let q be a nonnegative integer. If K_n can be decomposed into p copies of C_n and q copies of S_3, then K_n can be decomposed into $p-3t$ copies of C_n and $q+nt$ copies of S_3.*

Obviously, if K_n can be decomposed into α copies of C_n and β copies of S_3, then $\binom{n}{2} = n\alpha + 3\beta$. Using Theorem 23 together with Lemmas 15 to 20, we have the main result of this section.

Theorem 24. *Let n, α and β be positive integers. The complete graph K_n can be decomposed into α copies of C_n and β copies of S_3 if and only if $\binom{n}{2} = n\alpha + 3\beta$ and $\alpha \neq (n-3)/2$ for $n \equiv 3 \pmod 6$ and $\alpha \neq (n-2)/2$ for $n \equiv 0 \pmod 6$.*

Acknowledgment

We would like to thank the referees for their valuable comments and helpful suggestions.

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Received 12 January 2018
Revised 14 May 2018
Accepted 14 May 2018