DECOMPOSING THE COMPLETE GRAPH INTO HAMILTONIAN PATHS (CYCLES) AND 3-STARS

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Abstract

Let $H$ be a graph. A decomposition of $H$ is a set of edge-disjoint subgraphs of $H$ whose union is $H$. A Hamiltonian path (respectively, cycle) of $H$ is a path (respectively, cycle) that contains every vertex of $H$ exactly once. A $k$-star, denoted by $S_k$, is a star with $k$ edges. In this paper, we give necessary and sufficient conditions for decomposing the complete graph into $\alpha$ copies of Hamiltonian path (cycle) and $\beta$ copies of $S_3$.

Keywords: decomposition, complete graph, Hamiltonian path, Hamiltonian cycle, star.

2010 Mathematics Subject Classification: 05C70, 05C38.

1. Introduction

For positive integers $m$ and $n$, $K_n$ denotes the complete graph with $n$ vertices, and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. Let $k$ be a positive integer. A $k$-path, denoted by $P_k$, is a path on $k$ vertices. A $k$-cycle, denoted by $C_k$, is a cycle of length $k$. A $k$-star, denoted by $S_k$, is a star
with $k$ edges, i.e., $S_k = K_{1,k}$. Let $H$ be a graph. A spanning subgraph of $H$ is a subgraph of $H$ containing every vertex of $H$. A spanning path (respectively, cycle) of $H$ is called a Hamiltonian path (respectively, cycle) of $H$. A 1-factor of $G$ is a spanning subgraph of $G$ in which each vertex is incident with exactly one edge.

Let $F$, $G$, and $H$ be graphs. A decomposition of $H$ is a set of edge-disjoint subgraphs of $H$ whose union is $H$. If $H$ can be decomposed into $\alpha$ copies of $F$ and $\beta$ copies of $G$ for nonnegative integers $\alpha$ and $\beta$, then we say that $H$ has an $(\alpha F, \beta G)$-decomposition. Furthermore, if $\alpha \geq 1$ and $\beta \geq 1$, then we say that $H$ has an $(F, G)$-decomposition or $H$ is $(F, G)$-decomposable.

Study on the existence of an $(F, G)$-decomposition of a graph has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of $(K_k, S_k)$-decomposition of the complete graph $K_n$. Abueida and Daven [4] investigated the problem of the $(C_4, E_2)$-decomposition of several graph products where $E_2$ denotes two vertex disjoint edges. Abueida and O’Neil [8] studied the existence problem for $(C_k, S_{k-1})$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [25, 26] investigated the existence of $(G, H)$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$ where $G, H \in \{C_n, P_n, S_{n-1}\}$. A graph-pair $(G, H)$ of order $m$ is a pair of non-isomorphic graphs $G$ and $H$ with $V(G) = V(H)$ such that both $G$ and $H$ contain no isolated vertices and $G \cup H$ is isomorphic to $K_m$. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of $n$ for which $\lambda K_n$ admits a $(G, H)$-decomposition where $(G, H)$ is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ into the graph-pairs of order 4 and 5, respectively, where $F$ is a Hamiltonian cycle, a 1-factor, or an almost 1-factor. Lee [18, 19], Lee and Lin [22], and Lin [23] established necessary and sufficient conditions for the existence of $(C_k, S_k)$-decompositions of the complete bipartite graph, the complete bipartite multigraph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Abueida and Lian [7] and Beggas et al. [10] investigated the problems of $(C_k, S_k)$-decompositions of the complete graph $K_n$ and $\lambda K_n$, giving some necessary or sufficient conditions for such decompositions to exist. Lee and Chen [20] completely settled the existence problem of $(P_{k+1}, S_k)$-decompositions of the complete multigraph $\lambda K_n$ and the balanced complete bipartite multigraph $\lambda K_{n,n}$.

Recently, the existence problem of an $(\alpha F, \beta G)$-decomposition of a graph where $\alpha$ and $\beta$ are essential is also studied. Shyu gave necessary and sufficient conditions for the decomposition of $K_n$ into paths and stars (both with 3 edges) [27], paths and cycles (both with $k$ edges where $k = 3, 4$) [28, 29], and cycles and stars (both with 4 edges) [30]. He [31] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges.
Jeevadoss and Muthusamy [14, 15] considered the \( \{\alpha P_{k+1}, \beta C_k\}\)-decomposability of \( K_{m,n}, K_n \) and \( \lambda K_{m,n} \), giving some necessary or sufficient conditions for such decompositions to exist. Jeevadoss and Muthusamy [16] gave necessary and sufficient conditions for the existence of \( \{\alpha P_3, \beta C_4\}\)-decomposition of tensor product and cartesian product of complete graphs. In [33], Tarsi gave necessary and sufficient conditions for the existence of \( \{\alpha F, \beta S_k\}\)-decomposition of \( \lambda K_n \), where \( F \) is any subgraph of \( K_n \) and \( \alpha = 0 \). In this paper, we consider the existence of an \( \{\alpha F, \beta G\}\)-decomposition of the complete graph \( K_n \) with \( F \in \{P_n, C_n\} \) and \( G = S_3 \), giving necessary and sufficient conditions.

2. Preliminaries

We first collect some needed terminology and notation. Let \( G \) be a graph. The degree of a vertex \( x \) of \( G \), denoted by \( \deg_G x \), is the number of edges incident with \( x \). For \( k \geq 2 \), the vertex of degree \( k \) in \( S_k \) is the center of \( S_k \) and any vertex of degree 1 is a pendant vertex of \( S_k \). Let \( v_1v_2 \cdots v_k \) denote the \( k \)-path through vertices \( v_1, v_2, \ldots, v_k \) in order. The vertices \( v_1 \) and \( v_k \) are referred to as its origin and terminus, respectively. In addition, \( P_k(v_1, v_k) \) denotes a \( k \)-path with origin \( v_1 \) and terminus \( v_k \). We use \( (v_1, v_2, \ldots, v_k) \) to denote the \( k \)-cycle through vertices \( v_1, v_2, \ldots, v_k \) in order, and \( S(u; v_1, v_2, \ldots, v_k) \) to denote a star with center \( u \) and pendant vertices \( v_1, v_2, \ldots, v_k \). For \( U, W \subseteq V(G) \) with \( U \cap W = \phi \), we use \( G[U] \) and \( G[U, W] \) to denote the subgraph of \( G \) induced by \( U \), and the maximal bipartite subgraph of \( G \) with bipartition \( (U, W) \), respectively. When \( G_1, G_2, \ldots, G_t \) are edge disjoint subgraphs of a graph, use \( G_1 \cup G_2 \cup \cdots \cup G_t \) to denote the graph with vertex set \( \bigcup_{i=1}^{t} V(G_i) \) and edge set \( \bigcup_{i=1}^{t} E(G_i) \).

Before going into more details, we present some results which are useful for our discussions.

**Proposition 1** [11, 13]. For an even integer \( n \) and \( V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\} \), the complete graph \( K_n \) can be decomposed into \( n/2 \) copies of \( P_n, P(1), P(2), \ldots, P(n/2) \) with \( P(i+1) = x_ix_{i+1}x_{i+2} \cdots x_i+\frac{i}{2}2x_i+\frac{i}{2}+1x_i+\frac{i}{2} \) for \( 0 \leq i \leq \frac{n}{2}, \) where the subscripts of \( x \)’s are taken modulo \( n \) in the set of numbers \( \{0, 1, 2, \ldots, n-1\} \).

The following results are attributed to Walecki, see [9].

**Lemma 2.** For an odd integer \( n \) and \( V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\} \), the complete graph \( K_n \) can be decomposed into \( (n-1)/2 \) copies of \( C_n, C(1), C(2), \ldots, C((n-1)/2) \) with \( C(i) = (x_0, x_1, x_i-1, x_i+1, x_i-2, \ldots, x_i+(n-5)/2, x_i+(n+1)/2, x_i+(n-3)/2, x_i+(n-1)/2) \) for \( i = 1, 2, \ldots, (n-1)/2 \), where the subscripts of \( x \)’s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \).
Lemma 3. For an even integer \( n \) and \( V(K_n) = \{ x_0, x_1, \ldots, x_{n-1} \} \), the complete graph \( K_n \) can be decomposed into \( n/2 - 1 \) copies of \( C_n \), \( C(1), C(2), \ldots, C(n/2 - 1) \), and a \( 1 \)-factor \( F \), where \( E(F) = \{ x_0x_{n-1}, x_1x_{n-2}, x_2x_{n-3}, \ldots, x_{n/2 - 2}x_{n/2 + 1}, x_{n/2 - 1}x_n/2 \} \) and \( C(i) = (x_0, x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+n/2 + 1}, x_{i+n/2 + 2}, \ldots, x_{i+n/2 - 2}, x_{i+n/2}) \) for \( i = 1, 2, \ldots, n/2 - 1 \), where the subscripts of \( x \)'s are taken modulo \( n - 1 \) in the set of numbers \( \{ 1, 2, \ldots, n - 1 \} \).

3. Decomposition of \( K_n \) Into \( n \)-Paths and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing \( K_n \) into \( \alpha \) copies of \( P_n \) and \( \beta \) copies of \( S_3 \).

Lemma 4. Let \( n \) be an odd integer and let \( \alpha \) be a nonnegative integer. If \( \binom{n}{2} - (n - 1)\alpha \equiv 0 \pmod{3} \), then

\[
\alpha \in \begin{cases} 
0, 1, \ldots, (n - 1)/2 & \text{if } n \equiv 1 \pmod{6}, \\
(n - 3)/2 - 3t | t = 0, 1, \ldots, (n - 3)/6 & \text{if } n \equiv 3 \pmod{6}, \\
(n - 3)/2 - 3t | t = 0, 1, \ldots, (n - 5)/6 & \text{if } n \equiv 5 \pmod{6}.
\end{cases}
\]

Proof. Since \( \binom{n}{2} - (n - 1)\alpha \) is a nonnegative integer and \( n \) is odd, \( \alpha \leq \lfloor \binom{n}{2}/(n - 1) \rfloor = (n - 1)/2 \). Let \( \alpha = (n - 1)/2 - (3t + j) \) where \( t \) is a nonnegative integer and \( j \in \{0, 1, 2\} \). Since \( \binom{n}{2} - (n - 1)\alpha = n(n - 1)/2 - (n - 1)\alpha = (n - 2\alpha)(n - 1)/2 = (6t + 2j + 1)(n - 1)/2, \) \( \binom{n}{2} - (n - 1)\alpha \equiv (2j + 1)(n - 1)/2 \pmod{3} \). If \( n \equiv 1 \pmod{6} \), then \( (2j + 1)(n - 1)/2 \equiv 0 \pmod{3} \) for any integer \( j \). Hence \( \alpha \in \{0, 1, \ldots, (n - 1)/2\} \) for \( n \equiv 1 \pmod{6} \). When \( n \equiv 3 \pmod{6} \) or \( n \equiv 5 \pmod{6} \), the condition \( (2j + 1)(n - 1)/2 \equiv 0 \pmod{3} \) holds if and only if \( j = 1 \). Thus \( \alpha = (n - 3)/2 - 3t \) for some integer \( t \) when \( n \equiv 3 \pmod{6} \) or \( n \equiv 5 \pmod{6} \). Since \( \alpha \) is a nonnegative integer, we have \( t \leq (n - 3)/6 \) for \( n \equiv 3 \pmod{6} \), and \( t \leq (n - 5)/6 \) for \( n \equiv 5 \pmod{6} \). This completes the proof.

Lemma 5. Let \( n \) be an even integer, and let \( \alpha \) be a nonnegative integer. If \( \binom{n}{2} - (n - 1)\alpha \equiv 0 \pmod{3} \), then

\[
\alpha \in \begin{cases} 
{n/2 - 3t | t = 0, 1, \ldots, n/6} & \text{if } n \equiv 0 \pmod{6}, \\
{n/2 - 3t | t = 0, 1, \ldots, (n - 2)/6} & \text{if } n \equiv 2 \pmod{6}, \\
\{0, 1, \ldots, n/2\} & \text{if } n \equiv 4 \pmod{6}.
\end{cases}
\]

Proof. Since \( \binom{n}{2} - (n - 1)\alpha \) is a nonnegative integer and \( n \) is even, \( \alpha \leq \lfloor \binom{n}{2}/(n - 1) \rfloor = n/2 \). Let \( \alpha = n/2 - (3t + j) \) where \( t \) is a nonnegative integer and \( j \in \{0, 1, 2\} \). Since \( \binom{n}{2} - (n - 1)\alpha = n(n - 1)/2 - (n - 1)\alpha = (n - 2\alpha)(n - 1)/2 = (3t + j)(n - 1), \) \( \binom{n}{2} - (n - 1)\alpha \equiv j(n - 1) \pmod{3} \). If \( n \equiv 4 \pmod{6} \), then \( j(n - 1) \equiv 0 \pmod{3} \) for any integer \( j \). Hence \( \alpha \in \{0, 1, \ldots, n/2\} \) for \( n \equiv 4 \pmod{6} \). When \( n \equiv 0 \)
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(\text{mod } 6) or \( n \equiv 2 \) (\text{mod } 6), the condition \( j(n-1) \equiv 0 \) (\text{mod } 3) holds if and only if \( j = 0 \). Thus \( \alpha = n/2 - 3t \) for some integer \( t \) when \( n \equiv 0 \) (\text{mod } 6) or \( n \equiv 2 \) (\text{mod } 6). Since \( \alpha \) is a nonnegative integer, we have \( t \leq n/6 \) for \( n \equiv 0 \) (\text{mod } 6), and \( t \leq (n-2)/6 \) for \( n \equiv 2 \) (\text{mod } 6). This completes the proof.

The following indecomposable case is trivial.

**Lemma 6.** The complete graph \( K_4 \) cannot be decomposed into

1. one copy of \( P_4 \) and one copy of \( S_3 \), nor
2. two copies of \( S_3 \).

In addition, we exclude the possibility \( n = 5 \).

**Lemma 7.** The complete graph \( K_5 \) cannot be decomposed into one copy of \( P_5 \) and two copies of \( S_3 \).

**Proof.** Suppose, on the contrary, that \( K_5 \) can be decomposed into one copy of \( P_5 \), say \( P_5(x,y) \), and two copies of \( S_3 \), say \( S \) and \( T \). Note that the edge \( xy \) must be in either \( S \) or \( T \). Without loss of generality, assume that \( xy \) is in \( S \). Since the degree of every vertex of \( K_n - E(P_5(x,y) \cup S) \) is less than 3, we have a contradiction.

In the remainder of the paper, we assume that \( V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\} \).

**Lemma 8.** If \( n \) is an odd integer with \( n \geq 7 \), then the following hold:

1. The complete graph \( K_n \) can be decomposed into \( (n-1)/2 \) copies of \( P_n \) and \( (n-1)/6 \) copies of \( S_3 \) when \( n \equiv 1 \) (\text{mod } 6).
2. The complete graph \( K_n \) can be decomposed into \( (n-3)/2 \) copies of \( P_n \) and \( (n-1)/2 \) copies of \( S_3 \).
3. The complete graph \( K_n \) can be decomposed into \( (n-5)/2 \) copies of \( P_n \) and \( 5(n-1)/6 \) copies of \( S_3 \) when \( n \equiv 1 \) (\text{mod } 6).

**Proof.** By Lemma 2, \( K_n \) can be decomposed into \( (n-1)/2 \) copies of \( C_n \), \( C(1) \), \( C(2) \), \ldots, \( C((n-1)/2) \) with \( C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, x_{i-2}, \ldots, x_{i+(n-5)/2}, x_{i+(n+1)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2}) \) for \( i = 1, 2, \ldots, (n-1)/2 \), where the subscripts of \( x \)'s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \).

(1) For \( i = 1, 2, \ldots, (n-1)/2 \), let \( P(i) = C(i) - \{x_0x_i\} \). Clearly, \( P(i) \) is an \( n \)-path. Let \( G \) be the subgraph of \( K_n \) which is induced by the set of edges \( x_0x_1, x_0x_2, \ldots, x_0x_{(n-1)/2} \). Obviously, \( G = S_{(n-1)/2} \). Since \( n \) is odd and \( n-1 \equiv 0 \) (\text{mod } 3), the graph \( G \) can be decomposed into \( (n-1)/6 \) copies of \( S_3 \). This settles (1).
(2) For \( n = 7 \), the complete graph \( K_7 \) can be decomposed into 2 copies of \( P_7 \) and 3 copies of \( S_3 \) as follows: \( x_6x_2x_5x_3x_4x_0x_1, x_6x_4x_5x_0x_3x_1x_2, (x_1; x_4, x_5, x_6), (x_2; x_0, x_3, x_4), (x_6; x_0, x_3, x_5) \).

Now we consider the case \( n \geq 9 \). For \( i \in \{1, 2, \ldots, (n-1)/2\} - \{(n-7)/2, (n-3)/2\} \), let \( P(i) = C(i) - \{x_0x_i\} \). Note that \( x_{n-1}x_{n-7} \in E(C((n-7)/2)) \) and \( P((n-3)/2) = C((n-3)/2) - \{x_0x_{(n-3)/2}, x_{(n-5)/2}, x_{(n-1)/2}, x_{(n-7)/2}, \ldots, x_{n-4}, x_{n-1}, x_{n-3}, x_{n-2}\} \). Let \( P((n-7)/2) = C((n-7)/2) - \{x_{n-1}x_{n-7}\} \) and \( C((n-3)/2) - \{x_0x_{(n-3)/2}, x_{(n-1)/2}x_{(n-7)/2}\} \cup \{x_0x_{(n-1)/2}\} \). Hence \( P(i) \) is an n-path for \( i = 1, 2, \ldots, (n-1)/2 \). Moreover, \( P((n-1)/2) = x_{(n-1)/2}x_{(n-3)/2}x_{(n+1)/2}x_{(n-5)/2} \cdot \cdot \cdot x_{n-3}x_{n-2}x_{n-1}x_0 \). For \( i = 1, 2, \ldots, (n-3)/2 \), let \( S(i) = (x_{(n-1)/2-i}; x_{n-1}/2+i, x_{(n-1)/2+i}) \) and \( S = (x_{n-1}; x_n, x_0) \). Obviously, \( S(i) \) and \( S \) are 2-stars, and \( P((n-1)/2) \) can be decomposed into \( S(1), S(2), \ldots, S((n-3)/2) \) and \( S \). Furthermore, let \( W(i) = S(i) \cup \{x_0x_i\} \) for \( i = 1, 2, \ldots, (n-3)/2 - \{(n-7)/2\} \), let \( W((n-7)/2) = S((n-7)/2) \cup \{x_{(n-1)/2}x_{(n-1)/2}\} \), and let \( W((n-1)/2) = S \cup \{x_{n-1}x_{n-7}\} \). Clearly, \( W(i) \) is a 3-star. This settles (2).

(3) We will remove one edge from \( C(i) \) to obtain an n-path for \( i \in \{1, 2, \ldots, (n-5)/2\} \), and use \( C((n-3)/2) \) and \( C((n-1)/2) \) together with the edges removed from \( C(i) \)’s to constitute 5 \((n-1)/3\) copies of \( S_3 \).

Let \( S(i) = (x_{(n-1)/2+i}; x_{n-1}/2-3i, x_{(n-1)/2-3i}) \) for \( i = 1, 2, \ldots, (n-1)/6 \), \( S'(i) = (x_{(n-1)/2+i}; x_{(n-1)/2-3i}) \) for \( i = 1, 2, \ldots, (n-1)/6 \), and \( S''((n-1)/6) = (x_{n-2}; x_{n-3}, x_0) \). Obviously, \( S(i) \) and \( S'(i) \) are 2-stars.

Let \( J = \{j|2 \leq j \leq (n-1)/2 \text{ and } j \equiv 0, 2 \pmod{3}\} \). For \( j \in J \), let

\[ e_j'' = \begin{cases} x_{(n-1)/2-j}x_{(n-1)/2+j-2} & \text{if } j \equiv 0 \pmod{3}, \\ x_{(n-1)/2-j}x_{(n-1)/2+j-3} & \text{if } j \equiv 2 \pmod{3}, \end{cases} \]

where the subscripts of \( x \)'s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \). It is easy to see that \( \{S(i), S'(i)|i = 1, 2, \ldots, (n-1)/6\} \cup \{e_j''|j \in J\} \) is a decomposition of \( C((n-3)/2) - \{x_{(n-3)/2}x_0\} \).

Note that \( C((n-1)/2) = (x_0, x_{(n-1)/2}, x_{(n-3)/2}, x_{(n+1)/2}, x_{(n-5)/2}, \ldots, x_{n-3}, x_1, x_{n-2}, x_{n-1}) \). Let \( S''(j) = (x_{(n-1)/2-j}; x_{(n-1)/2+j-1}, x_{(n-1)/2+j}) \) for \( j = 1, 2, \ldots, (n-3)/2 \) and \( S''((n-1)/2) = (x_{n-1}; x_{n-2}, x_0) \) where the subscripts of \( x \)'s are taken modulo \( n-1 \) in the set of numbers \( \{1, 2, \ldots, n-1\} \). Obviously, \( S''(j) \) is a 2-star, and \( C((n-1)/2) - \{x_{(n-1)/2}x_0\} \) can be decomposed into \( S''(1), S''(2), \ldots, S''((n-1)/2) \).

For \( i = 2, 3, \ldots, (n-1)/6 \), let \( e_i \) be an edge in \( C(i) \) incident with the center of \( S(i) \). Then \( C(i-1) - \{e_i\} \) is an n-path and \( S(i) \cup \{e_i\} \) is a 3-star. For \( i = 1, 2, \ldots, (n-1)/6 \), let \( e'_i \) be an edge in \( C((n-1)/6 + i - 1) \) incident with the center of \( S'(i) \). Then \( C((n-1)/6 + i - 1) - \{e'_i\} \) is an n-path and \( S'(i) \cup \{e'_i\} \) is a 3-star. Let \( K = \{k|4 \leq k \leq (n-5)/2 \text{ and } k \equiv 1 \pmod{3}\} \). For \( k \in K \), let \( e_k'' \) be an edge in \( C((k-1)/3 + (n-1)/3 - 1) \) incident with the center of \( S''(k) \).
Then $C((k - 1)/3 + (n - 1)/3 - 1) - \{e''_k\}$ is an $n$-path and $S''(i) \cup \{e''_k\}$ is a 3-star. For $j \in J$, $S''(j) \cup \{e''_j\}$ is a 3-star. Moreover, $S(1) \cup \{x_{(n-1)/2}\}$ and $S''(1) \cup \{x_{(n-3)/2}\}$ are also 3-stars. This completes the proof.

**Lemma 9.** If $n$ is an even integer with $n \geq 4$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $n/2$ copies of $n$-paths.
2. The complete graph $K_n$ can be decomposed into $(n/2 - 1)$ copies of $P_n$ and $(n-1)/3$ copies of $S_3$ when $n \equiv 4 \pmod{6}$ and $n \geq 10$.
3. The complete graph $K_n$ can be decomposed into $(n/2 - 2)$ copies of $P_n$ and $2(n-1)/3$ copies of $S_3$ when $n \equiv 4 \pmod{6}$ and $n \geq 10$.

**Proof.** By Proposition 1, we have (1).

For $n = 10$, the complete graph $K_{10}$ can be decomposed into 4 copies of $P_4$ and 3 copies of $S_3$ as follows: $x_8x_9x_4x_5x_2x_1x_9, x_7x_4x_5x_2x_9, x_3x_4x_1x_5x_8x_7x_9, x_0x_7x_8x_9x_3x_5x_2x_6x_1, (x_0; x_4, x_8, x_9), (x_1; x_2, x_7, x_8), (x_9; x_3, x_5, x_6).

Now we consider the case $n \geq 16$. Let $G = K_n[\{x_0, x_1, \ldots, x_{n-2}\}]$. Clearly $G$ is isomorphic to $K_{n-1}$. By Lemma 2, the graph $G$ can be decomposed into $n/2 - 1$ copies of $C_{n-1}, C(1), C(2), \ldots, C(n/2 - 1)$ with $C(i) = \{(x_0, x_i, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+n/2-3}, x_{i+n/2-2}, x_{i+n/2-1})\}$ for $i = 1, 2, \ldots, n/2 - 1$, where the subscripts of $x$’s are taken modulo $n - 2$ in the set of numbers $\{1, 2, \ldots, n - 2\}$. Note that $C(1)$ contains edges $x_1x_{n-2}$ and $x_n/2x_0, C(2)$ contains edges $x_2x_1$ and $x_{n/2+1}x_0$, and $C(3)$ contains the edge $x_4x_1$. Let $P(1) = C(1) \cup \{x_1x_{n-1}x_{n/2}\} - \{x_1x_{n-2}, x_{n/2}x_0\}$, $P(2) = C(2) \cup \{x_2x_{n-1}x_{n/2+1}\} - \{x_2x_1, x_{n/2+1}x_0\}$, and $P(3) = C(3) \cup \{x_4x_{n-1}\} - \{x_4x_1\}$. In addition, let $P(i) = C(i) \cup \{x_{i+n/2-1}x_{i+n+1}\} - \{x_{i+n/2-1}x_0\}$ for $i = 4, 5, \ldots, n/2 - 1$. Obviously, $P(i)$ is an $n$-path for $i = 1, 2, \ldots, n/2 - 1$. Let $S(1) = (x_0; x_{n/2}, x_{n/2+1}, x_{n/2+3}, x_{n/2+4}, \ldots, x_{n-2})$ and $S(2) = (x_{n-1}; x_0, x_3, x_5, x_6, \ldots, x_{n-2}, x_{n-2}, x_{n-1}, x_{n+2})$. It is easy to see that $K_n - E\left(\bigcup_{i=1}^{n/2-1} P(i)\right) = S(1) \cup S(2) \cup (x_1; x_2, x_4, x_{n-2})$. Note that $S(1)$ and $S(2)$ are $(n/2 - 2)$-stars. Since $n \equiv 4 \pmod{6}$, each of $S(1)$ and $S(2)$ can be decomposed into $(n - 4)/6$ copies of $S_3$. This settles (2).

By Lemma 3, $K_n$ can be decomposed into $n/2 - 1$ copies of $C_{n-1}, C(1), C(2), \ldots, C(n/2 - 1)$, and a factor $F$, where $E(F) = \{x_0x_{n-1}, x_1x_{n-2}, x_2x_{n-3}, \ldots, x_{n/2-2}x_{n/2+1}, x_{n/2-1}x_{n/2}\}$ and $C(i) = \{(x_0, x_i, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+n/2-3}, x_{i+n/2-2}, x_{i+n/2-1})\}$ for $i = 1, 2, \ldots, n/2 - 1$, where the subscripts of $x$’s are taken modulo $n - 1$ in the set of numbers $\{1, 2, \ldots, n - 1\}$.

We obtain $n/2 - 2$ copies of $P_{n}$ by removing one edge from each of $n$-cycles $C(1), C(2), \ldots, C(n/2 - 2)$. For $i = 1, 2, \ldots, n/2 - 3$, let $P(i) = C(i) - \{x_0x_i\}$. In addition, let $P(n/2 - 2) = C(n/2 - 2) - \{x_{n/2-3}x_{n/2-1}\}$. Trivially, $P(i)$ is an $n$-path for $i = 1, 2, \ldots, n/2 - 2$. 

In the following, $2(n - 1)/3$ copies of $S_3$ are constructed. We first obtain $n/2$ copies of $S_3$ by using all of the edges of $C(n/2 - 1)$ and $n/2 - 1$ edges of $F$ and the edge $x_{n/2 - 3}x_{n/2 - 1}$ removed from $C(n/2 - 2)$. Note that $C(n/2 - 1) = (x_0, x_{n/2 - 1}, x_{n/2 - 2}, x_{n/2}, x_{n/2 - 3}, \ldots, x_1, x_{n - 3}, x_{n - 1}, x_{n - 2})$. For $i = 1, 2, \ldots, n/2 - 1$, let $S(i) = (x_{n/2 - 1 + i}; x_{n/2 - 1 - i}; x_{n/2 - 2 - i})$ and $S = (x_{n/2 - 1}; x_{n/2 - 2}; x_0)$. Obviously, $S(i)$ and $S$ are 2-stars, and $C(n/2 - 1)$ is decomposable into $S(1), S(2), \ldots, S(n/2 - 1)$ and $S$. Let $W(i) = S(i) \cup \{x_{n/2 - 1 + i}x_{n/2 - 1}\}$ for $i = 1, 2, \ldots, n/2 - 1$, and let $W(n/2) = S \cup \{x_{n/2 - 3}x_{n/2 - 1}\}$. Clearly, $W$ is a 3-star.

Now we obtain $(n - 4)/6$ copies of $S_3$ by using one edge of $F$ and the edges removed from $C(i)$’s in constructing $n$-paths for $i = 1, 2, \ldots, n/2 - 3$. Let $G$ be the subgraph of $K_n$ induced by the set of edges $x_0x_1, x_0x_2, \ldots, x_0x_{n/2 - 3}, x_0x_{n - 1}$. Obviously, $G = S_{n/2 - 2}$. Since $n \equiv 4 \pmod{6}$, the graph $G$ can be decomposed into $(n - 4)/6$ copies of $S_3$. This settles (3) and completes the proof.

**Lemma 10.** Let $n$ and $t$ be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian paths of $K_n$, then $\bigcup_{i=1}^t Q_i$ is $S_t$-decomposable.

**Proof.** Since each $Q_i$ is a Hamiltonian path of $K_n$, we have $V(Q_i) = V(K_n)$. For each $Q_i$, we orient the edges of $Q_i$ from $x_0$ along $Q_i$ to the end (or ends) of the path, and use $\overrightarrow{Q_i}$ to denote the digraph obtained from $Q_i$ for such an orientation. Note that there is exactly one arc directed into $x_j$ for each $j \in \{1, 2, \ldots, n - 1\}$. Let $\overrightarrow{G} = \bigcup_{i=1}^t \overrightarrow{Q_i}$. It is easy to check that $\deg_{\overrightarrow{G}} x_j = t$ for $j \neq 0$. Thus there exists an $S_t$-decomposition of $\bigcup_{i=1}^t Q_i$ such that $x_j$ is a center of a $t$-star for $j \neq 0$. This completes the proof.

By Lemma 10, the union of $3t$ edge-disjoint $n$-paths can be decomposed into $n - 1$ copies of $S_{3t}$, in turn, each $S_{3t}$ can be decomposed in to $t$ copies of $S_3$. Hence we have the following result.

**Theorem 11.** Let $n$, $p$ and $t$ be positive integers with $p \geq 3t$, and let $q$ be a nonnegative integer. If $K_n$ can be decomposed into $p$ copies of $P_n$ and $q$ copies of $S_3$, then $K_n$ can be decomposed into $p - 3t$ copies of $P_n$ and $q + (n - 1)t$ copies of $S_3$.

Obviously, if $K_n$ can be decomposed into $\alpha$ copies of $n$-paths and $\beta$ copies of $S_3$, then \(\binom{n}{2} = (n - 1)\alpha + 3\beta\). Using Theorem 11 together with Lemmas 4 to 9, we have the main result of this section.

**Theorem 12.** Let $n$ be a positive integer with $n \geq 4$, and let $\alpha$ and $\beta$ be nonnegative integers. The complete graph $K_n$ can be decomposed into $\alpha$ copies of $P_n$ and $\beta$ copies of $S_3$ if and only if \(\binom{n}{2} = (n - 1)\alpha + 3\beta\) and $n, \alpha, \beta \notin \{(4, 1, 1), (4, 0, 2), (5, 1, 2)\}$. 


4. Decomposition of $K_n$ Into $n$-Cycles and 3-Stars

In this section, we obtain necessary and sufficient conditions for decomposing $K_n$ into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$. The first two lemmas in the following are from [17] and [32], respectively.

**Lemma 13.** For an odd integer $n$ and $V(K_{n,n}) = \{x_0, \ldots, x_{n-1}\} \cup \{y_0, \ldots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $(n-1)/2$ copies of $C_{2n}$, $C(0)$, $C(1), \ldots, C((n-3)/2)$, and a 1-factor $F$, where $E(F) = \{x_0y_{n-1}, x_1y_0, \ldots, x_{n-1}y_{2n-2}\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \ldots, (n-3)/2$.

**Lemma 14.** For an even integer $n$ and $V(K_{n,n}) = \{x_0, \ldots, x_{n-1}\} \cup \{y_0, \ldots, y_{n-1}\}$, the complete bipartite graph $K_{n,n}$ can be decomposed into $n/2$ copies of $C_{2n}$, $C(0)$, $C(1), \ldots, C(n/2 - 1)$, where $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+(n-2)}, x_{n-2}, y_{2i+(n-1)}, x_{n-1})$ for $i = 0, 1, \ldots, n/2 - 1$.

**Lemma 15.** Let $n$ be an odd integer and let $\alpha$ be a nonnegative integer. If $(\binom{n}{2} - \alpha) = 1$ is a nonnegative integer and $(\binom{n}{2} - \alpha) \equiv 0 \pmod{3}$, then

$$
\alpha \in \begin{cases} 
0, 1, \ldots, (n-1)/2 & \text{if } n \equiv 0 \pmod{3}, \\
(n-1)/2 - 3t | t = 0, 1, \ldots, [(n-1)/6] & \text{otherwise}. 
\end{cases}
$$

**Proof.** Since $(\binom{n}{2}) = (n(n-1))/2$ is a nonnegative integer and $n$ is odd, $\alpha \leq [(\binom{n}{2})/n] = (n-1)/2$. Let $\alpha = (n-1)/2 - (3t+i)$, where $t$ is a nonnegative integer and $i \in \{0, 1, 2\}$. Since $(\binom{n}{2}) - \alpha = n(n-1)/2 - \alpha = n(n-1-2\alpha)/2 = n(3t+i)$, $(\binom{n}{2}) - \alpha \equiv ni \pmod{3}$. If $n$ is a multiple of 3, then $ni \equiv 0 \pmod{3}$ holds for any $i \in \{0, 1, 2\}$. Hence $\alpha \in \{0, 1, \ldots, (n-1)/2\}$ for $n \equiv 0 \pmod{3}$. Otherwise, the condition $ni \equiv 0 \pmod{3}$ holds if and only if $i = 0$. This implies $\alpha = (n-1)/2 - 3t$. Moreover, $t \leq [(n-1)/6]$ since $\alpha$ is a nonnegative integer. This completes the proof.

**Lemma 16.** Let $n$ be an even integer and let $\alpha$ be a nonnegative integer. If $(\binom{n}{2} - \alpha) = 1$ is a nonnegative integer and $(\binom{n}{2} - \alpha) \equiv 0 \pmod{3}$, then

$$
\alpha \in \begin{cases} 
0, 1, \ldots, n/2 - 1 & \text{if } n \equiv 0 \pmod{3}, \\
n/2 - 3t - 2 | t = 0, 1, \ldots, [(n-4)/6] & \text{otherwise}. 
\end{cases}
$$

**Proof.** Since $(\binom{n}{2}) = (n(n-1))/2$ is a nonnegative integer and $n$ is even, $\alpha \leq [(\binom{n}{2})/n] = n/2 - 1$. Let $\alpha = n/2 - 1 - (3t+i)$, where $t$ is a nonnegative integer and $i \in \{0, 1, 2\}$. Since $(\binom{n}{2}) - \alpha = n(n-1-2\alpha)/2 = n(6t+2i+1)/2$, $(\binom{n}{2}) - \alpha \equiv n(2i+1)/2 \pmod{3}$. If $n \equiv 0 \pmod{3}$, then $n/2 \equiv 0 \pmod{3}$, this implies that $n(2i+1)/2 \equiv 0 \pmod{3}$ holds for any $i \in \{0, 1, 2\}$. Hence $\alpha \in \{0, 1, \ldots, n/2 - 1\}$ for $n \equiv 0 \pmod{3}$.
(mod 3). Otherwise, the condition \(n(2i + 1)/2 \equiv 0 \pmod{3}\) holds if and only if \(i = 1\). This implies \(\alpha = n/2 - 3t - 2\). Moreover, \(t \leq \lfloor (n - 4)/6 \rfloor\) since \(\alpha\) is a nonnegative integer. This completes the proof. \(\square\)

Let \(m = (n-3)/2\) for odd \(n\) and \(m = (n-2)/2\) for even \(n\). Let \(C(1), C(2), \ldots, C(m)\) be edge-disjoint \(n\)-cycles in \(K_n\), and let \(G = K_n - \bigcup_{i=1}^{m} E(C(i))\). Since \(\deg_G x = n - 1 - 2m \leq 2\) for each vertex \(x\), \(G\) has no \(S_3\)-decomposition. Thus we have the following result.

**Lemma 17.** Let \(n \equiv 0 \pmod{3}\). The complete graph \(K_n\) cannot be decomposed into \((n-3)/2\) copy of \(C_n\) and \(n/3\) copies of \(S_3\) for odd \(n\), and cannot be decomposed into \((n-2)/2\) copy of \(C_n\) and \(n/6\) copies of \(S_3\) for even \(n\).

**Lemma 18.** If \(n\) is an odd integer with \(n \geq 5\), then the following hold:

1. The complete graph \(K_n\) can be decomposed into \((n-1)/2\) copies of \(C_n\).
2. The complete graph \(K_n\) can be decomposed into \((n-5)/2\) copies of \(C_n\) and \(2n/3\) copies of \(S_3\) when \(n \equiv 3 \pmod{6}\) and \(n \geq 9\).
3. The complete graph \(K_n\) can be decomposed into \((n-9)/2\) copies of \(C_n\) and \(4n/3\) copies of \(S_3\) when \(n \equiv 3 \pmod{6}\) and \(n \geq 9\).

**Proof.** By Lemma 2, the complete graph \(K_n\) can be decomposed into \((n-1)/2\) copies of \(C_n\), \(C(1), C(2), \ldots, C((n-1)/2)\) with \(C(i) = (x_0, x_i, x_{i-1}, x_{i+1}, \ldots, x_{i+(n-5)/2}, x_{i+(n-3)/2}, x_{i+(n-1)/2})\) for \(i = 1, 2, \ldots, (n-1)/2\), where the subscripts of \(x\)'s are taken mod \(n-1\) in the set of numbers \(\{1, 2, \ldots, n-1\}\). Hence we have (1).

(2) If there exist \(s\) and \(t\) \((1 \leq s < t \leq (n-1)/2)\) such that \(C(s) \cup C(t)\) can be decomposed into \(2n/3\) copies of \(S_3\), then we have the result. Consider the case \(s = (n+3)/6\) and \(t = n/3\). Note that \(C((n+3)/6) = (x_0, x_{(n+3)/6}, x_{(n-3)/6}, x_{(n+9)/6}, x_{(n-9)/6}, \ldots, x_{n-3/1}, x_1, x_{n/3}, x_{n-1}, \ldots, x_{2n/3-2}, x_{2n/3-1}, x_{2n/3})\). For \(i = 1, 2, \ldots, n/3 - 1\), let \(S_2(i) = (x_{i-n/3}, x_{i+1}, x_{i+n/3})\) and \(S_2(n/3) = (x_{n/3}, x_{n/3-1}, x_0)\). For \(j = 1, 2, \ldots, (n-3)/6\), let \(P_2(j) = x_{j-n/3} \cup x_{j+n/3}\). For \(j = (n+3)/6, (n+9)/6, \ldots, n/3 - 1\), let \(P_2(j) = x_{j-n/3} \cup x_{j+n/3}\). In addition, let \(P_2(0) = x_0 x_{(n+3)/6}\). Obviously, \(S_2(i)\) is a 2-star for \(i = 1, 2, \ldots, n/3, \) and \(P_2(j)\) is a 2-path for \(j = 0, 1, \ldots, n/3 - 1\). One can see that \(C((n+3)/6)\) can be decomposed into \(S_2(1), S_2(2), \ldots, S_2(n/3)\) and \(P_2(0), P_2(1), \ldots, P_2(n/3 - 1)\).

On the other hand, \(C(n/3) = (x_0, x_{n/3}, x_{n/3-1}, x_{n/3+1}, x_{n/3-2}, x_{n/3+2}, \ldots, x_{2n/3-2}, x_1, x_{2n/3-1}, x_{n-1}, \ldots, x_{(5n-15)/6}, x_{(5n+3)/6}, x_{(5n-9)/6}, x_{(5n-3)/6})\). For \(j = 1, 2, \ldots, n/3 - 1\), let \(S_2'(j) = (x_j, x_{2n/3-1-j}, x_{2n/3-j})\). For \(i = 1, 2, \ldots, (n+3)/6, \) let \(P_2'(i) = x_{i-n/3} x_{2n/3-2+i}\). For \(i = (n+9)/6, (n+12)/6, \ldots, n/3, \) let \(P_2'(i) = x_{i-n/3} x_{2n/3-1+i}\). In addition, let \(P_2'(0) = x_0 x_{n/3}\) and \(P_2'(0) = x_0 x_{(5n-3)/6}\). Obviously, \(S_2'(j)\) is a 2-star for \(i = 1, 2, \ldots, n/3 - 1,\) and \(P_2'(0)\) and \(P_2'(i)\) are 2-paths for \(i = 0, 1, \ldots, n/3\). One can see that \(C(n/3)\) can be decomposed into \(S_2'(1), S_2'(2), \ldots, S_2'(n/3 - 1)\) and \(P_2'(0), P_2'(1), \ldots, P_2'(n/3)\) as well as \(P_2'(0)\).
For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P''_2(i)$. For $j = 1, 2, \ldots, n/3 - 1$, let $S_3'(j) = S_2'(j) \cup P_3(j)$. Clearly, $S_3(i)$ and $S_3'(j)$ are 3-stars. In addition, $P_2(0) \cup P''_2(0) \cup P''_2(0)$ is also a 3-star. Hence $C((n + 3)/6) \cup C(n/3)$ can be decomposed into 2n/3 copies of $S_3$. This settles (2).

(3) According to the proof of (2), the result holds if there exist $s'$ and $t'$ ($s', t' \notin \{(n + 3)/6, n/3\}$) such that $C(s') \cup C(t')$ can be decomposed into 2n/3 copies of $S_3$. Consider the case $s' = (n + 9)/6$ and $t' = n/3 + 1$. Note that $C((n + 9)/6) = (x_0, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+9)/6}, x_{(n+3)/6}, x_{(n+9)/6}, x_{(n+3)/6})$. For $i = 1, 2, \ldots, n/3 - 1$, let $S_2'(i) = (x_{n+1-i}; x_{n+3+i}, x_{n+1+i})$ with $x_n = x_1$ and $S_2(n/3) = (x_{2n/3+1}; x_{2n/3}, x_0)$. For $j = 2, 3, \ldots, (n + 3)/6$, let $P_2(j) = x_jx_{n/3+j}$. For $j = (n + 9)/6, (n + 15)/6, \ldots, n/3$, let $P_2(j) = x_jx_{n/3+j-3}$. In addition, let $P_2(0) = x_0x_{(n+9)/6}$. Obviously, $S_2'(i)$ is a 2-star for $i = 1, 2, \ldots, n/3$, and $P_2(j)$ is a 2-path for $j = 0, 2, 3, \ldots, n/3$. One can see that $C((n + 3)/6)$ can be decomposed into $S_2(1), S_2(2), \ldots, S_2(n/3)$ and $P_2(0), P_2(2), P_2(3), \ldots, P_2(n/3)$.

On the other hand, $C(n/3 + 1) = (x_0, x_{n/3+1}, x_{n/3}, x_{n/3+2}, x_{n/3-1}; \ldots, x_{2n/3}, x_1, x_{2n/3+1}, x_{n-1}, x_{2n/3+2}, x_{n-2}; \ldots, x_{(5n-9)/6}, x_{(5n+9)/6}, x_{(5n-3)/6}, x_{(5n+3)/6})$. For $j = 2, 3, \ldots, n/3$, let $S_2'(j) = (x_j; x_{2n/3+1-j}, x_{2n/3+2-j})$. For $i = 1, 2, \ldots, (n + 3)/6$, let $P_2'(i) = x_{n+1-i}x_{2n/3+1+i}$, and for $i = (n + 9)/6, (n + 12)/6, \ldots, n/3$, let $P_2'(i) = x_{n+1-i}x_{2n/3+i}$ with $x_n = x_1$. In addition, let $P''_2(0) = x_0x_{n/3+1}$ and $P''_2(0) = x_0x_{(5n+3)/6}$. Obviously, $S_2'(j)$ is a 2-star for $i = 2, 3, \ldots, n/3$, and $P''_2(0)$ and $P''_2(i)$ are 2-paths for $i = 0, 1, \ldots, n/3$. One can see that $C(n/3 + 1)$ can be decomposed into $S_2'(2), S_2'(3), \ldots, S_2'(n/3)$ and $P_2'(0), P_2'(1), \ldots, P_2'(n/3)$ as well as $P''_2(0)$.

For $i = 1, 2, \ldots, n/3$, let $S_3(i) = S_2(i) \cup P''_2(i)$. For $j = 2, 3, \ldots, n/3$, let $S_3'(j) = S_2'(j) \cup P_3(j)$. Clearly, $S_3(i)$ and $S_3'(j)$ are 3-stars. In addition, $P_2(0) \cup P_2'(0) \cup P''_2(0)$ is also a 3-star. Hence $C((n + 9)/6) \cup C(n/3 + 1)$ can be decomposed into 2n/3 copies of $S_3$. This settles (3).

For positive integers $l$ and $n$ with $1 \leq l \leq n$, the $(n,l)$-crown $C_{n,l}$ is the bipartite graph with bipartition $(X, Y)$, where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, and edge set $\{x_iy_j : i = 0, 1, \ldots, n-1, j \equiv i + 1, i + 2, \ldots, i + l \pmod l\}$.

**Proposition 19** [24]. $\lambda C_{n,l}$ is $S_k$-decomposable if and only if $k \leq l$ and $\lambda nl \equiv 0 \pmod k$.

**Lemma 20.** If $n$ is an even integer $n \geq 6$, then the following hold:

1. The complete graph $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.
2. The complete graph $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$ when $n \equiv 0 \pmod 6$. 

The complete graph $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$ when $n \equiv 0 \pmod{6}$ and $n \geq 12$.

**Proof.** Let $V(K_n) = X \cup Y$, where $X = \{x_0, \ldots, x_{n/2-1}\}$ and $Y = \{y_0, \ldots, y_{n/2-1}\}$. Note that $K_n = K_n[X] \cup K_n[Y] \cup K_n[X,Y]$ where $K_n[X]$ and $K_n[Y]$ are isomorphic to $K_{n/2}$ and $K_n[X,Y]$ is isomorphic to $K_{n/2,n/2}$. We distinguish two cases: Case 1. $n \equiv 0 \pmod{4}$ and Case 2. $n \equiv 2 \pmod{4}$.

**Case 1.** $n \equiv 0 \pmod{4}$. By Lemma 14, $K_n[X,Y]$ can be decomposed into $n/4$ copies of $C_n$, $C(0)$, $C(1)$, $C(n/4 - 1)$, where $C(i) = (y_2i, x_0, y_{2i+1}, x_1, \ldots, y_{2i+1}, x_{n/2-2}, y_{2i+1}, x_{n/2-1}, x_{n/2})$ for $i = 0, 1, 2, \ldots, n/4 - 1$. By Proposition 1, we have the following results. $K_n[X]$ can be decomposed into the following $n/4$ copies of $P_{n/2}: P_{n/2}(x_0, x_{n/4}), P_{n/2}(x_2, x_{n/4+2}), \ldots, P_{n/2}(x_{n/4-1}, x_{n/2-1})$, and $K_n[Y]$ can be decomposed into the following $n/4$ copies of $P_{n/2}: P_{n/2}(y_0, y_{n/4}), P_{n/2}(y_1, y_{n/4+1}), \ldots, P_{n/2}(y_{n/4-1}, y_{n/2-1})$.

For $i = 0, 1, \ldots, n/4 - 1$, let $Q(i) = P_{n/2}(x_i, x_{i+n/4}) \cup P_{n/2}(y_i, y_{i+n/4}) \cup \{y_ix_i, y_{i+n/4}x_{i+n/4}\}$. Clearly, $Q(i)$ is an $n$-cycle, and $y_ix_i, y_{i+n/4}x_{i+n/4} \in E(C(0))$ for $i = 0, 1, \ldots, n/4 - 1$. For $1 \leq t \leq n/4 - 1$, let

$$R(t) = \left( \bigcup_{i=0}^{t} C(i) \right) - \{y_ix_i, y_{i+n/4}x_{i+n/4} | 0 \leq i \leq n/4 - 1\}.$$

It is easy to see that $R(t)$ is isomorphic to the crown $C_{n/2,2t+1}$. Therefore, $K_n$ can be decomposed into $n/2 - (t + 1)$ copies of $C_n$, $Q(0), Q(1), \ldots, Q(n/4 - 1)$ and $C(t+1), C(t+2), \ldots, C(n/4 - 1)$, and one copy of $(n/2, 2t+1)$-crown $R(t)$. Note that $2t + 1 \geq 3$ and $|E(R(t))| = |E(C_{n/2,2t+1})| = (2t + 1)n/2$. If $(2t + 1)n/2 \equiv 0 \pmod{3}$, then $R(t)$ can be decomposed into $(2t + 1)n/2$ copies of $S_3$ by Proposition 19. Hence for $n \equiv 0 \pmod{4}$, we have the following.

If $t = 1$, then $(2t + 1)n/2 = 3n/2 \equiv 0 \pmod{3}$ for each $n$. Thus $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.

If $t = 2$, then $(2t + 1)n/2 = 5n/2 \equiv 0 \pmod{6}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$.

If $t = 3$, then $(2t + 1)n/2 = 7n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$. This settles Case 1.

**Case 2.** $n \equiv 2 \pmod{4}$. Since $n \equiv 2 \pmod{4}$, $n/2$ is odd. By Lemma 13, $K_n[X,Y]$ can be decomposed into $(n-2)/4$ copies of $C_n$, $C(0), C(1), \ldots, C((n-5)/4)$, and a 1-factor $F$, where $E(F) = \{x_0y_{n/2-1}, x_{1}y_{0}, \ldots, x_{n/2-1}y_{n/2-2}\}$ and $C(i) = (y_{2i}, x_0, y_{2i+1}, x_1, \ldots, y_{2i+1}, x_{n/2-1}, x_{n/2})$ for $i = 0, 1, \ldots, (n-6)/4$.

Now we consider $K_n[X]$ and $K_n[Y]$. By Lemma 2, we have the following results. $K_n[X]$ can be decomposed into $(n-2)/4$ copies of $C_{n/2}$, $W(1), W(2), \ldots, W((n-2)/4)$ with $W(i) = (x_0, x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+(n-10)/4}, x_{i+(n+2)/4}, x_{i+(n+4)/4}, \ldots, x_{i+(n+2)/4})$. 

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$x_{i+(n-6)/4}, x_{i+(n-2)/4}$, and $K_n[Y]$ can be decomposed into $(n-2)/4$ copies of $C_{n/2}, W'(1), W'(2), \ldots, W'((n-2)/4)$ with $W'(i) = (y_0, y_i, y_{i-1}, y_{i+1}, y_{i-2}, \ldots, y_{i+(n-10)/4}, y_{i+(n+2)/4}, y_{i+(n-6)/4}, y_{i+(n-2)/4})$ for $i = 1, 2, \ldots, (n-2)/4$, where the subscripts of $x$'s and $y$'s are taken modulo $(n-2)/2$ in the set of numbers $\{1, 2, \ldots, (n-2)/2\}$. For $i = 1, 2, \ldots, (n-2)/4$, let

$$e(i) = \begin{cases} x_0x_1 & \text{if } i = 1, \\ x_{i-1}x_i & \text{if } i \text{ is odd and } i \geq 3, \\ x_{i+(n-6)/4}x_{i+(n-2)/4} & \text{if } i \text{ is even}, \end{cases}$$

and let

$$e'(i) = \begin{cases} y_0y_1 & \text{if } i = 1, \\ y_{i-1}y_i & \text{if } i \text{ is odd and } i \geq 3, \\ y_{i+(n-6)/4}y_{i+(n-2)/4} & \text{if } i \text{ is even}. \end{cases}$$

Let $P(i) = W(i) - \{e(i)\}$ and $P'(i) = W'(i) - \{e'(i)\}$. Trivially, $P(i)$ and $P'(i)$ are $(n/2)$-paths. Let $M = \{e(i)\} | 1 \leq i \leq (n-2)/4$ and $M' = \{e'(i)\} | 1 \leq i \leq (n-2)/4$. If $n \equiv 2 \pmod{8}$, then $(n-2)/4$ is even. Hence $M = \{x_0x_1, x_2x_3, \ldots, x_{(n-10)/4}x_{(n-6)/4}, x_{(n+2)/4}x_{(n+6)/4}, \ldots, x_{n/2-2}x_{n/2-1}\}$ and $M' = \{y_0y_1, y_2y_3, \ldots, y_{(n-10)/4}y_{(n-6)/4}, y_{(n+2)/4}y_{(n+6)/4}, \ldots, y_{n/2-2}y_{n/2-1}\}$. If $n \equiv 6 \pmod{8}$, then $(n-2)/4$ is odd. Hence $M = \{x_0x_1, x_2x_3, \ldots, x_{n/2-2}x_{n/2-1}\}$ and $M' = \{y_0y_1, y_2y_3, \ldots, y_{n/2-2}y_{n/2-1}\}$. Let $H$ be the subgraph of $K_n[X]$ induced by $M$, and let $H'$ be the subgraph of $K_n[Y]$ induced by $M'$. Clearly, $K_n[X]$ can be decomposed into $H$ and $(n-2)/4$ copies of $P_{n/2}, P(1), P(2), \ldots, P((n-2)/4)$, and $K_n[Y]$ can be decomposed into $H'$ and $(n-2)/4$ copies of $P_{n/2}, P'(1), P'(2), \ldots, P'((n-2)/4)$.

Let $Z = \{y_0x_0, y_1x_1\} \cup \{y_{i-1}x_{i-1}, y_ix_i\} | i \text{ is odd and } i \geq 3 \} \cup \{y_{i+(n-6)/4}x_{i+(n-6)/4}, y_{i+(n-2)/4}x_{i+(n-2)/4}\} | i \text{ is even}\}$. Obviously, $Z \subseteq E(C(0))$. For $i = 1, 2, \ldots, (n-2)/4$, let $K = \{y_{i+(n-6)/4}x_{i+(n-6)/4}, y_{i+(n-2)/4}x_{i+(n-2)/4}\}$ and

$$Q(i) = \begin{cases} P(1) \cup P(1) \cup \{y_0x_0, y_1x_1\} & \text{if } i = 1, \\ P(i) \cup P'(i) \cup \{y_{i-1}x_{i-1}, y_ix_i\} & \text{if } i \text{ is odd and } i \geq 3, \\ P(i) \cup P'(i) \cup K & \text{if } i \text{ is even}, \end{cases}$$

and let $Q((n+2)/4) = H \cup H' \cup C(0) - Z$. One can see that each $Q(i)$ is an $n$-cycle. Thus $K_n[X] \cup K_n[Y] \cup C(0)$ can be decomposed into $(n+2)/4$ copies of $C_n$. For $1 \leq t \leq (n-6)/4$, let

$$R(t) = \left( \bigcup_{i=1}^{t} C((n-6)/4 - i + 1) \right) \cup F.$$

It is easy to see that $R(t)$ is isomorphic to the crown $C_{n/2t+1}$. Hence $K_n[X, Y]$ can be decomposed into $n/2 - (t + 1)$ copies of $C_n, Q(1), Q(2), \ldots, Q((n+2)/4)$.
and $C(1), C(2), \ldots, C((n - 6)/4 - t)$, and one copy of $(n/2, 2t + 1)$-crown $R(t)$.

Note that $2t + 1 \geq 3$ and $|E(R(t))| = |E(C_{n/2,2t+1})| = (2t + 1)n/2$. If $(2t + 1)n/2 \equiv 0 \pmod{3}$, then $R(t)$ can be decomposed into $(2t + 1)n/6$ copies of $S_3$ by Proposition 19.

If $t = 1$, then $(2t + 1)n/2 = 3n/2 \equiv 0 \pmod{3}$ for each $n$. Thus $K_n$ can be decomposed into $n/2 - 2$ copies of $C_n$ and $n/2$ copies of $S_3$.

If $t = 2$, then $(2t + 1)n/2 = 5n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 3$ copies of $C_n$ and $5n/6$ copies of $S_3$.

If $t = 3$, then $(2t + 1)n/2 = 7n/2 \equiv 0 \pmod{3}$ for $n \equiv 0 \pmod{6}$. Thus $K_n$ can be decomposed into $n/2 - 4$ copies of $C_n$ and $7n/6$ copies of $S_3$. This settles Case 2.

Let $x$ and $y$ be distinct vertices of a multigraph $G$. We use $e_G(x, y)$ to denote the number of edges joining $x$ and $y$. A star decomposition of $G$ is center balanced if every vertex of $G$ is the center of the same number of members in the decomposition.

**Proposition 21** [21]. Let $G$ be an $r$-regular multigraph with $r \geq 1$. Then $G$ has a center balanced $S_t$-decomposition if and only if $r \equiv 0 \pmod{2t}$ and $e_G(x, y) \leq r/t$ for all $x, y \in V(G)$ with $x \neq y$.

**Lemma 22.** Let $n$ and $t$ be positive integers. If $Q_1, Q_2, \ldots, Q_t$ are edge-disjoint Hamiltonian cycles of $K_n$, then $\bigcup_{i=1}^t Q_i$ is $S_t$-decomposable.

**Proof.** Since each $Q(i)$ is $2$-regular and $V(Q(i)) = V(Q(j))$ for $i, j \in \{1, 2, \ldots, t\}$, $\bigcup_{i=1}^t Q_i$ is $2t$-regular. Since $2t \equiv 0 \pmod{2t}$ and $e_{\bigcup_{i=1}^t Q_i}(x, y) \leq 1 < (2t)/t$ for all $x, y \in V(\bigcup_{i=1}^t Q_i)$ with $x \neq y$, the result follows from Proposition 21.

By Lemma 22, the union of $3t$ copies of edge-disjoint $n$-cycles can be decomposed into $n$ copies of $S_{3t}$, in turn, each $S_{3t}$ can be decomposed in to $t$ copies of $S_3$. Hence we have the following result.

**Theorem 23.** Let $n$, $p$ and $t$ be positive integers with $p \geq 3t$, and let $q$ be a nonnegative integer. If $K_n$ can be decomposed into $p$ copies of $C_n$ and $q$ copies of $S_3$, then $K_n$ can be decomposed into $p - 3t$ copies of $C_n$ and $q + nt$ copies of $S_3$.

Obviously, if $K_n$ can be decomposed into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$, then $\left(\begin{array}{c}n \\ 2\end{array}\right) = n\alpha + 3\beta$. Using Theorem 23 together with Lemmas 15 to 20, we have the main result of this section.

**Theorem 24.** Let $n$, $\alpha$ and $\beta$ be positive integers. The complete graph $K_n$ can be decomposed into $\alpha$ copies of $C_n$ and $\beta$ copies of $S_3$ if and only if $\left(\begin{array}{c}n \\ 2\end{array}\right) = n\alpha + 3\beta$ and $\alpha \neq (n - 3)/2$ for $n \equiv 3 \pmod{6}$ and $\alpha \neq (n - 2)/2$ for $n \equiv 0 \pmod{6}$.
Acknowledgment

We would like to thank the referees for their valuable comments and helpful suggestions.

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   doi:10.1016/0012-365X(79)90034-7

Received 12 January 2018
Revised 14 May 2018
Accepted 14 May 2018