DOMINATION NUMBER, INDEPENDENT DOMINATION NUMBER AND 2-INDEPENDENCE NUMBER IN TREES

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Abstract
For a graph $G$, let $\gamma(G)$ be the domination number, $i(G)$ be the independent domination number and $\beta_2(G)$ be the 2-independence number. In this paper, we prove that for any tree $T$ of order $n \geq 2$, $4\beta_2(T) - 3\gamma(T) \geq 3i(T)$, and we characterize all trees attaining equality. Also we prove that for every tree $T$ of order $n \geq 2$, $i(T) \leq \frac{3\beta_2(T)}{4}$, and we characterize all extreme trees.

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1. Introduction

In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The minimum degree and the maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v) \setminus S$, and the closed neighborhood of $S$ is the set $N_G[S] = N[S] = N(S) \cup S$. A leaf of a tree $T$ is a vertex of degree 1, a support vertex is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. We denote the set of all leaves of a tree $T$ by $L(T)$. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to $r$ leaves and the other to $s$ leaves. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$. Let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. We denote the set of leaves adjacent to a vertex $v$ by $L_v$.

A set $S$ of vertices in a graph $G$ is a dominating set if every vertex of $V \setminus S$ is adjacent to some vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. A dominating set of minimum cardinality of $G$ is called a $\gamma(G)$-set. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [10, 11].

A subset $S \subseteq V(G)$ is said to be independent if $E(G[S]) = \emptyset$, where $G[S]$ is the subgraph induced by $S$. The independent domination number (respectively, the independence number) of $G$ denoted by $i(G)$ (respectively, $\beta(G)$) is the size of the smallest (respectively, the largest) maximal independent set in $G$. It is well known that an independent set is maximal if and only if it is also dominating. Hence, we can say that the domination, which is defined even for non-independent sets, is the property which makes an independent set maximal. Furthermore, every set which is both independent and dominating is a minimal dominating set of $G$. This leads to the well known inequality chain

$$\gamma(G) \leq i(G) \leq \beta(G).$$
Fink and Jacobson [7, 8] generalized the concepts of independent and dominating sets. Let \( k \) be a positive integer. A set \( S \) of vertices in a graph \( G \) is \( k \)-independent if the maximum degree of the subgraph induced by \( S \) is at most \( k-1 \). The maximum cardinality of a \( k \)-independent set of \( G \) is the \( k \)-independence number of \( G \) and is denoted \( \beta_k(G) \). A \( k \)-independent set of \( G \) with maximum cardinality is called a \( \beta_k(G) \)-set. The subset \( S \) is \( k \)-dominating if every vertex of \( V \setminus S \) has at least \( k \) neighbors in \( S \). The \( k \)-domination number \( \gamma_k(G) \) is the minimum cardinality of a \( k \)-dominating set of \( G \).

Relationships between two parameters \( \gamma_k(G) \) and \( \beta_k(G) \) have been studied by several authors. Favaron [5] proved that for any graph \( G \) and positive integer \( k \), \( \gamma_k(G) \leq \beta_k(G) \). Also, Favaron [6] proved that for every graph \( G \) and positive integer \( k \leq \Delta \), \( \beta_k(G) + \gamma_{\Delta-k+1}(G) \geq n \). Jacobson, Peters and Rall [12] showed that for every graph \( G \) and positive integer \( k \leq \delta \), \( \beta_k(G) + \gamma_{\delta-k+1}(G) \leq n \). Hansberg, Meierling and Volkmann [9] showed that if \( G \) is a connected \( r \)-partite graph and \( k \) is an integer such that \( \Delta \geq k \), then \( \gamma_k(G) \leq \frac{\beta(G)}{r}(r(r-1)+k-1) \).

For more information on \( k \)-independence number and \( k \)-domination see [2].

The relation between 2-independent set and some domination parameters have been studied by several authors (see for example [1, 3, 4, 13]).

Motivated by the aforementioned works, we consider the difference of \( \beta_2(T) - \gamma(T) \) for trees and prove that for any tree \( T \) of order \( n \geq 2 \), \( \frac{4\beta_2(T)}{3} - \gamma(T) \geq i(T) \) and characterize all extreme trees. Also we prove that for every \( T \) of order \( n \geq 2 \), \( i(T) \leq \frac{3\beta_2(T)}{4} \), and we classify all trees attaining this inequality.

2. A Lower Bound on the Difference \( \frac{4\beta_2(T)}{3} - \gamma(T) \)

In this section we show that for every tree \( T \) of order \( n \geq 2 \), \( \frac{4\beta_2(T)}{3} - \gamma(T) \geq i(T) \) and we characterize all extreme trees. We proceed with some definitions and lemmas.

A subdivision of an edge \( uv \) is obtained by replacing the edge \( uv \) with a path \( uvw \), where \( w \) is a new vertex. The subdivision graph \( S(G) \) is the graph obtained from \( G \) by subdividing each edge of \( G \) once. The subdivision star \( S(K_{1,t}) \) for \( t \geq 1 \), is called a healthy spider \( S_t \). A wounded spider \( S_{t,q} \) (\( 0 \leq q \leq t-1 \) is the tree obtained from \( K_{1,t} \) (\( t \geq 1 \) by subdividing \( q \) edges of \( K_{1,t} \). Note that stars are wounded spiders. A spider is a healthy or a wounded spider.

**Lemma 1.** Let \( T' \) be a tree and \( v \in V(T') \). If \( T \) is the tree obtained from \( T' \) by adding a path \( P_4 = u_1u_2u_3u_4 \) and joining \( v \) to \( u_2 \), then \( \gamma(T) + i(T) \leq \gamma(T') + i(T') + 4 \) and \( \beta_2(T) = \beta_2(T') + 3 \).

**Proof.** Clearly, any (independent) dominating set of \( T' \) can be extended to a (independent) dominating set of \( T \) by adding \( u_1, u_3 \) and this implies that \( \gamma(T) +
\[i(T) \leq \gamma(T') + i(T') + 4.\]

Also, obviously any \(\beta_2(T')\)-set can be extended to an 2-independent set of \(T\) by adding \(u_1, u_3, u_4\) yielding \(\beta_2(T) \geq \beta_2(T') + 3\). On the other hand, if \(S\) is a \(\beta_2(T')\)-set then clearly \(|S \cap \{u_1, u_2, u_3, u_4\}| \leq 3\) and so \(S \cap V(T')\) is a 2-independent set of \(T'\) of size at least \(\beta_2(T) - 3\) implying that \(\beta_2(T) \leq \beta_2(T') + 3\). Thus \(\beta_2(T) = \beta_2(T') + 3\). □

**Lemma 2.** Let \(T'\) be a tree and \(v \in V(T')\). If \(T\) is the tree obtained from \(T'\) by adding a path \(P_3 = u_1u_2u_3\) and joining \(v\) to \(u_1\), then \(\gamma(T) \leq \gamma(T') + 1, i(T) \leq i(T') + 1\) and \(\beta_2(T) = \beta_2(T') + 2\).

**Proof.** Clearly, any (independent) dominating set of \(T'\) can be extended to a (independent) dominating set of \(T\) by adding \(u_2\) and this implies that \(\gamma(T) \leq \gamma(T') + 1\) and \(i(T) \leq i(T') + 1\).

Also, obviously any \(\beta_2(T')\)-set can be extended to an 2-independent set of \(T\) by adding \(u_2, u_3\) yielding \(\beta_2(T) \geq \beta_2(T') + 2\). On the other hand, if \(S\) is a \(\beta_2(T')\)-set then clearly \(|S \cap \{u_1, u_2, u_3\}| \leq 2\) and hence \(S \cap V(T')\) is a 2-independent set of \(T'\) of size at least \(\beta_2(T) - 2\) implying that \(\beta_2(T) \leq \beta_2(T') + 2\). Thus \(\beta_2(T) = \beta_2(T') + 2\). □

**Lemma 3.** If \(T\) is a spider of order \(n \geq 2\), then \(\gamma(T) + i(T) \leq \frac{4\beta_2(T)}{3}\) with equality if and only if \(T = P_4\).

**Proof.** If \(T = S_t\) is a healthy spider for some \(t \geq 1\), then obviously \(\gamma(T) + i(T) = 2t\) because \(\gamma(T) = t\) and \(i(T) = t\). Also \(\beta_2(T) = 2t\). Hence \(\gamma(T) + i(T) = \beta_2(T) < \frac{4\beta_2(T)}{3}\). Now let \(T = S_{t, q}\) be a wounded spider. If \(q = 0\), then \(T\) is a star and we have \(\gamma(T) + i(T) = 2 \leq t = \beta_2(T) < \frac{4\beta_2(T)}{3}\). Suppose \(q > 0\). If \(t = 2\), then \(T = P_4\) and clearly \(\gamma(T) + i(T) = \frac{4\beta_2(T)}{3}\). If \(t \geq 3\), then clearly \(\gamma(T) + i(T) = 2q + 2\) and \(\beta_2(T) = t + q\) and so \(\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}\). □

Next we introduce a family \(\mathcal{T}\) of trees \(T\) that can be obtained from a sequence \(T_1, T_2, \ldots, T_k\) of trees such that \(T_1 = P_4\), and if \(k \geq 2\), then \(T_{i+1}\) can be obtained recursively from \(T_i\) by the operation \(\mathcal{T}_i\) for \(1 \leq i \leq k - 1\).

**Operation \(\mathcal{T}_i\).** If \(v \in T_i\) is a support vertex, then \(\mathcal{T}_i\) adds a path \(P_4 = u_1u_2u_3u_4\) and joins \(v\) to \(u_2\).

**Observation 4.** Let \(T \in \mathcal{T}\). Then the following conditions are satisfied.

1. Every support vertex is adjacent to exactly one leaf.
2. Every vertex of \(T\) is a leaf or support vertex.
3. Both of \(L(T)\) and \(V(T) - L(T)\) are \(\gamma(T)\)-set.
4. \(L(T)\) is an \(i(T)\)-set.
5. \( L(T) \subset \beta_2(T) \)-set.

6. \( \beta_2(T) = |L(T)| + |V(T) - L(T)|/2 = 3\gamma(T)/2 \).

**Theorem 5.** If \( T \) is a tree of order \( n \geq 2 \), then

\[
\gamma(T) + i(T) \leq \frac{4\beta_2(T)}{3}
\]

with equality if and only if \( T \in \mathcal{T} \).

**Proof.** The proof is by induction on \( n \). The results are trivial for trees of order \( n = 2, 3, 4 \). Let \( n \geq 5 \) and suppose that for every non-trivial tree \( T \) of order less than \( n \) the results are true. Let \( T \) be a tree of order \( n \). If \( \text{diam}(T) = 2 \), then \( T \) is a star and clearly \( \gamma(T) + i(T) = 2 < \frac{4\beta_2(T)}{3} \) by Lemma 3. If \( \text{diam}(T) = 3 \), then \( T \) is a double star \( DS_{rs} \). Since \( r + s \geq 3 \), if we suppose \( r \geq s \), then we have \( r \geq 2 \). If \( r \geq s \geq 2 \), then \( \gamma(T) + i(T) = s + 3 < \frac{4(r+s)}{3} = \frac{4\beta_2(T)}{3} \). If \( s = 1 \), then \( \gamma(T) + i(T) = 4 < \frac{4(r+2)}{3} = \frac{4\beta_2(T)}{3} \). Hence, we may assume that \( \text{diam}(T) \geq 4 \).

Let \( v_1v_2 \cdots v_D \) be a diametrical path in \( T \) such that \( \text{deg}(v_2) \) is as large as possible. Root \( T \) at \( v_D \). Consider the following cases.

**Case 1.** \( \text{deg}_T(v_2) \geq 4 \). Suppose \( T' = T - \{v_1\} \). Clearly, any \( \gamma(T) \)-set and any \( \gamma(T') \)-set contains \( v_2 \) and this implies that \( \gamma(T) = \gamma(T') \). Let \( S \) be a \( i(T') \)-set. If \( v_2 \in S \), then \( S \) is an independent dominating set of \( T \) and if \( v_2 \notin S \), then \( S \cup \{v_1\} \) is an independent dominating set of \( T \) yielding \( i(T) \leq i(T') + 1 \). On the other hand, if \( S \) is a \( \beta_2(T') \)-set such that \( |S \cap L(T')| \) is as large as possible, then clearly \( v_2 \notin S \) and \( S \cup \{v_1\} \) is a 2-independent set of \( T \) implying that \( \beta_2(T) \geq |S| + 1 = \beta_2(T') + 1 \). By the induction hypothesis, we have

\[
\gamma(T) + i(T) \leq \gamma(T') + i(T') + 1 \leq \frac{4\beta_2(T')}{3} + 1 \leq \frac{4\beta_2(T) - 1}{3} < \frac{4\beta_2(T)}{3}.
\]

**Case 2.** \( \text{deg}_T(v_2) = 3 \). Assume that \( L_{v_2} = \{v_1, z\} \). First let \( \text{deg}(v_3) = 2 \). Suppose \( T' = T - T_{v_3} \). As Case 1, we have \( \gamma(T) = \gamma(T') + 1 \) and \( i(T) \leq i(T') + 1 \). On the other hand, if \( S \) is a \( \beta_2(T') \)-set, then \( S \cup \{v_1, v_2\} \) is a 2-independent set of \( T \) yielding \( \beta_2(T) \geq |S| + 2 = \beta_2(T') + 2 \). By the induction hypothesis, we have

\[
\gamma(T) + i(T) \leq \gamma(T') + i(T') + 2 \leq \frac{4\beta_2(T')}{3} + 2 \leq \frac{4\beta_2(T) - 2}{3} < \frac{4\beta_2(T)}{3}.
\]

Now let \( \text{deg}(v_3) \geq 3 \). Let \( L_{v_3} = \{x_1, \ldots, x_t\} \). If \( L_{v_3} \neq \emptyset \), then let \( C_2 = \{y_1, \ldots, y_k\} \) be the children of \( v_3 \) with depth 1 and degree 2, if any, and redlet \( z_1, \ldots, z_t \) be the children of \( v_3 \) with depth 1 and degree 3 where \( z_1 = v_2 \). Let \( T' = T - T_{v_3} \). Clearly, any \( \gamma(T) \)-set can be extended to a dominating set of \( T' \) by adding \( v_3 \) and its children of depth 1 and this yields \( \gamma(T) \leq \gamma(T') + |C_2| + t + 1 \). Also, any \( i(T') \)-set can be extended to an independent dominating set of \( T \) by
adding all children of $v_3$ implying that $i(T) \leq i(T') + |C_2| + t + |L_{v_3}|$. On the other hand, any $\beta_2(T')$-set, can be extended to a 2-independent set of $T$ by adding $L_{v_3}$, $y_1, \ldots, y_k$ and their children, if any, and the children of $z_1, \ldots, z_t$ yielding $\beta_2(T) \geq \beta_2(T') + |L_{v_3}| + 2t + 2|C_2|$. It follows from the induction hypothesis that

$$\gamma(T) + i(T) \leq \gamma(T') + i(T') + 2|C_2| + 2t + |L_{v_3}| + 1$$
$$\leq \frac{4\beta_2(T')}{3} + 2|C_2| + 2t + |L_{v_3}| + 1$$
$$\leq \frac{4\beta_2(T) - 8|C_2| - 8t - 4|L_{v_3}|}{3} + 2|C_2| + 2t + |L_{v_3}| + 1$$
$$\leq \frac{4\beta_2(T) - 2|C_2| - 2t - |L_{v_3}| + 3}{3} \leq \frac{4\beta_2(T)}{3}.$$ 

We claim that the equality does not hold. Suppose, to the contrary, that $\gamma(T) + i(T) = \frac{4\beta_2(T)}{3}$. Then all inequalities occurring the above chain must be equalities and this holds if and only if $\gamma(T') + i(T') = \frac{4\beta_2(T')}{3}$, $|C_2| = 0$, $t = 1$ and $|L_{v_3}| = 1$. Thus $\deg_T(v_3) = 3$ and $v_3$ is adjacent with a leaf $w$. By the induction hypothesis, we have $T' \in T$. It follows from Observation 4 that $v_4$ is either a leaf or is a weak support vertex. We distinguish the following subcases.

**Subcase 2.1.** $\deg_T(v_4) = 2$. If $\text{diam}(T) = 4$, then clearly $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$, which is a contradiction. Let $\text{diam}(T) \geq 5$. Let $T' = T - v_3$. Clearly, any $\gamma(T')$-set can be extended to a dominating set of $T$ by adding $v_3, v_2$, any $i(T')$-set can be extended to a dominating set of $T$ by adding $v_3, v_1, z$, and any $\beta_2(T')$-set can be extended to a 2-independent set of $T$ by adding $v_3, w, v_1, z$. By the induction hypothesis, we obtain $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$ a contradiction.

**Subcase 2.2.** $v_4$ is a support vertex. Let $T' = T - v_2$. Clearly, any $\gamma(T')$-set can be extended to a dominating set of $T$ by adding $v_2$ and any $\beta_2(T')$-set can be extended to a 2-independent set of $T$ by adding $v_1, z$. Let $S'$ be a $i(T')$-set. If $v_2 \not \in S'$, then let $S = S' \cup \{v_2\}$ and if $v_3 \in S'$, then let $S = (S' \setminus \{v_3\}) \cup \{w, v_2\}$. Obviously, $S$ is an independent dominating set of $T$ yielding $i(T) \leq i(T') + 1$. By the induction hypothesis, we obtain $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$, a contradiction. This proved our claim.

**Case 3.** $\deg_T(v_2) = 2$. If $\deg_T(v_3) = 2$, then let $T' = T - v_3$. By Lemma 2 and the induction hypothesis, we have $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$. Let $\deg_T(v_4) \geq 3$. By the choice of diametrical path we may assume that all children of $v_3$ with depth 1, have degree 2. First we suppose that there is a pendant path $v_3z_2z_1$. Let $T' = T - v_3$. Clearly, any $\gamma(T')$-set and any $i(T')$-set can be extended to a dominating set of $T$ by adding $v_1$ yielding $\gamma(T) \leq \gamma(T') + 1$ and $i(T) \leq i(T') + 1$. Let $S'$ be a $\beta_2(T')$-set. If $v_3 \not \in S'$, then let $S = S' \cup \{v_1, v_2\}$ and if $v_3 \in S'$, then let $S = (S' \setminus \{v_3\}) \cup \{v_1, v_2, z_1, z_2\}$. Obviously, $S$ is a 2-independent set of $T$ yielding
\( \beta_2(T) \geq \beta_2(T') + 2 \). By the induction hypothesis, we obtain \( \gamma(T) + i(T) < \frac{4\beta_2(T)}{3} \). Now let all children of \( v_3 \) with exception \( v_2 \) are leaves. If \( \deg_T(v_3) \geq 4 \), then as above we can see that \( \gamma(T) + i(T) < \frac{4\beta_2(T)}{3} \). Henceforth, we assume that \( \deg_T(v_3) = 3 \). Let \( w \) be the leaf adjacent to \( v_3 \). Suppose \( T' = T - v_3 \). By the induction hypothesis and Lemma 1 we have

\[
\gamma(T) + i(T) = \gamma(T') + i(T') + 4 \leq \frac{4\beta_2(T')}{3} + 4 \leq \frac{4\beta_2(T) - 3}{3} + 4 = \frac{4\beta_2(T)}{3}.
\]

If the equality holds, then we must have \( \gamma(T') + i(T') = \frac{4\beta_2(T')}{3} \) and it follows from the induction hypothesis that we have \( T' \in \mathcal{T} \). Thus each vertex of \( T' \) is either a leaf or a support vertex. We claim that \( v_4 \) is not a leaf in \( T' \). Suppose, to the contrary, that \( v_4 \) is a leaf in \( T' \). If \( \text{diam}(T) = 4 \), then \( T \) is a wounded spider and by Lemma 3 we have \( \gamma(T) + i(T) < \frac{4\beta_2(T)}{3} \), a contradiction. Let \( \text{diam}(T) \geq 5 \).

Since \( v_5 \) is not a strong support vertex, we observe that \( v_6 \) is a support vertex too. We consider two subcases.

**Subcase 3.1.** \( \deg(v_5) = 2 \). Let \( T'' = T - v_4 \) and let \( w, v_5 \) be two leaves adjacent to \( v_6 \) in \( T'' \). It follows from the induction hypothesis that \( T'' \notin \mathcal{T} \) and so \( \gamma(T'') + i(T'') < \frac{4\beta_2(T'')}{3} \). As above cases, we can see that \( \gamma(T) \leq \gamma(T'') + 2 \), \( i(T) \leq i(T'') + 2 \) and \( \beta_2(T) \geq \beta_2(T'') + 3 \). This implies that

\[
\gamma(T) + i(T) \leq \gamma(T'') + i(T'') + 4 < \frac{4\beta_2(T'')}{3} + 4 = \frac{4\beta_2(T)}{3},
\]

which is a contradiction.

**Subcase 3.2.** \( \deg(v_5) \geq 3 \). Since \( T' \notin \mathcal{T} \) and \( v_4 \) is a leaf, every vertex \( z \in N_T(v_5) \setminus \{v_4\} \) is a support vertex. Let \( T'' = T - v_4 \) and let \( u \) be a leaf adjacent to \( v_6 \) in \( T'' \). As above, we have \( \gamma(T) \leq \gamma(T'') + 2 \) and \( i(T) \leq i(T'') + 2 \). Let \( S' \) be a \( \beta_2(T'') \)-set. If \( v_5 \notin S' \) or \( v_5 \in S \) and \( z \notin S' \) for each \( z \in N_T(v_5) \setminus \{v_4\} \), then \( S = S' \cup \{v_4, v, v_2, v_1\} \) is a 2-independent set of \( T \) yielding \( \beta_2(T) \geq \beta_2(T'') + 4 \) and by the induction hypothesis we obtain

\[
\gamma(T) + i(T) \leq \gamma(T'') + i(T'') + 4 \leq \frac{4\beta_2(T'')}{3} + 4 < \frac{4\beta_2(T)}{3},
\]

a contradiction again. Assume that \( v_5 \in S' \) and \( z \in S' \) for some \( z \in N_T(v_5) \setminus \{v_4\} \).

We may assume, without loss of generality, that \( z = v_6 \). Then \( u \notin S' \) and the set \( S = (S' \setminus \{v_5\}) \cup \{u, v_1, v, v_2, v_1\} \) is a 2-independent set of \( T \) yielding \( \beta_2(T) \geq \beta_2(T'') + 4 \) and as above we get a contradiction.

Consequently, \( v_4 \) is a support vertex of \( T' \). Now \( T \) can be obtained from \( T' \) by operation \( T_1 \) and so \( T \in \mathcal{T} \). This completes the proof.

The next result is an immediate consequence of Theorem 5.

**Corollary 6.** If \( T \) is a tree of order \( n \geq 2 \), then \( \gamma(T) \leq \frac{2\beta_2(T)}{3} \).
3. Independent Domination and 2-Independence of Trees

In this section we show that for any $T$ of order $n \geq 2$, $i(T) \leq \frac{3\beta_2(T)}{4}$ and we characterize all extreme trees. First we introduce a family $\mathcal{F}$ of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = DS_{2,2}$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by the operation $\mathcal{O}$ for $1 \leq i \leq k - 1$.

**Operation $\mathcal{O}$**. If $v \in V(T_i)$ is a strong support vertex with $|L_v| = 2$, then operation $\mathcal{O}$ adds a double star $DS_{2,2}$ and joins a support vertex of $DS_{2,2}$ to $v$.

**Observation 7.** If $T \in \mathcal{F}$, then
1. $L(T)$ is a $\beta_2(T)$-set of $T$ and so $\beta_2(T) = \frac{2n(T)}{3}$,
2. every strong support vertex is adjacent with exactly two leaves,
3. $|L(T)| = 2|V(T) - L(T)|$,
4. $i(T) = \frac{n(T)}{2}$,
5. $i(T) = \frac{3\beta_2(T)}{4}$.

**Theorem 8.** If $T$ is a tree of order $n \geq 2$, then

\[
(2) \quad i(T) \leq \frac{3\beta_2(T)}{4},
\]

with equality if and only if $T \in \mathcal{F}$.

**Proof.** The proof is by induction on $n$. The statements clearly hold for all trees of order $n = 2, 3, 4$. Let $n \geq 5$, and suppose that for every nontrivial tree $T$ of order less than $n$ the results are true. Let $T$ be a tree of order $n$. If diam$(T) = 2$, then $T$ is a star and clearly $i(T) = 1 < \frac{3\beta_2(T)}{4}$. If diam$(T) = 3$, then $T$ is a double star $DS_{r,s}$ for some $r \geq s \geq 1$. If $r \geq s \geq 2$, then

\[
i(T) = s + 1 \leq \frac{3(r + s)}{4} = \frac{3\beta_2(T)}{4},
\]

with equality if and only if $r = s = 2$ and this if and only if $T \in \mathcal{F}$. If $s = 1$, then $i(T) = 2 < \frac{3(r + 2)}{4} = \frac{3\beta_2(T)}{4}$. Hence we may assume that diam$(T) \geq 4$.

Let $v_1v_2\ldots v_D$ be a diametrical path in $T$ such that $t = \text{deg}(v_2)$ is as large as possible. Let $L_{v_3} = \{z_1 = v_1, z_2, \ldots, z_{t-1}\}$. Let $k_1$ be the number of children of $v_3$ with depth 0, $k_2$ be the number of children of $v_3$ with depth 1 and degree at most three and $k_3$ be the number of children of $v_3$ with depth 1 and degree at least four. First let $2k_2 + 5k_3 > k_1$. Assume that $T' = T - v_3$. Clearly any $i(T')$-set can be extended by adding all children of $v_3$ to an independent dominating set of $T$ and so $i(T) \leq i(T') + k_1 + k_2 + k_3$. On the other hand, any $\beta_2(T')$-set can be extended to a 2-independent set of $T$ by adding all leaves in...
$L_{v_3}$, all children of $v_3$ with degree at most three and one of their children, and all leaves adjacent to the children of $v_3$ with degree at least four implying that $eta_2(T) \geq \beta_2(T') + k_1 + 2k_2 + 3k_3$. By the induction hypothesis, we obtain

$$i(T) \leq i(T') + k_1 + k_2 + k_3 \leq \frac{3\beta_2(T')}{4} + k_1 + k_2 + k_3$$

$$\leq \frac{3\beta_2(T) - 3k_1 - 6k_2 - 2k_3}{4} + k_1 + k_2 + k_3$$

$$\leq \frac{3\beta_2(T)}{4} + \frac{k_1 - 2k_2 - 5k_3}{4} < \frac{3\beta_2(T)}{4}.$$

Henceforth, we assume that $2k_2 + 5k_3 \leq k_1$. This implies that $v_3$ is a strong support vertex, that is $k_1 \geq 2$. Consider the following cases.

**Case 1.** $t \geq 4$. Let $w_1, w_2 \in L_{v_3}$ and let $T' = T - \{z_1, z_2, w_1, w_2\}$. If $S'$ is a $\beta_2(T')$-set, then the set $S = (S' \setminus \{v_2, v_3\}) \cup L_{v_2} \cup L_{v_3}$ if $|S' \cap \{v_2, v_2\}| = 2$, and $S = (S' \setminus \{v_2, v_3\}) \cup \{z_1, z_2, w_1, w_2\}$ if $|S' \cap \{v_2, v_3\}| \leq 1$, is a 2-independent set of $T$ yielding $\beta_2(T) \geq \beta_2(T') + 3$. Now we show that $i(T) \leq i(T') + 2$. Let $D'$ be a $i(T')$-set. Since $D'$ is independent, we have $|D' \cap \{v_3, v_2\}| \leq 1$. If $|D' \cap \{v_3, v_2\}| = 0$, then $(D' - L_{v_2}) \cup \{v_2\}$ is a $i(T')$-set. Hence we may assume that $|D' \cap \{v_3, v_2\}| = 1$. Let $D = D' \cup \{z_1, z_2\}$ if $v_3 \in D'$, and $D = D' \cup \{w_1, w_2\}$ if $v_2 \in D'$. Clearly, $D$ is an independent dominating set of $T$ and so $i(T) \leq i(T') + 2$. By the induction hypothesis, we obtain

$$i(T) \leq i(T') + 2 \leq \frac{3\beta_2(T')}{4} + 2 < \frac{3\beta_2(T)}{4}.$$  

**Case 2.** $t = 3$ and $k_1 \geq 3$. Let $w_1, w_2, w_3 \in L_{v_3}$ and $T' = T - \{z_1, z_2, w_1, w_2\}$. If $S'$ is a $\beta_2(T')$-set, then the set $S = (S' \setminus \{v_2, v_3\}) \cup L_{v_2} \cup L_{v_3}$ if $|S' \cap \{v_2, v_3\}| = 2$, and $S = (S' \setminus \{v_2, v_3\}) \cup \{z_1, z_2, w_1, w_2\}$ if $|S' \cap \{v_2, v_3\}| \leq 1$, is a 2-independent set of $T$ yielding $\beta_2(T) \geq \beta_2(T') + 3$. As above, we can see that $i(T) \leq i(T') + 2$ and by the induction hypothesis, we have $i(T) \leq i(T') + 2 \leq \frac{3\beta_2(T')}{4} + 2 < \frac{3\beta_2(T)}{4}.$

**Case 3.** $t = 3$ and $k_1 = 2$. We deduce from $2k_2 + 5k_3 \leq k_1$ that $k_2 \leq 1$ and $k_3 = 0$. Since $t = \deg(v_2) = 3$, then $k_2 = 1$. This yields $\deg(v_3) = 4$ and $T_{v_3} = DS_{2,2}$. Let $L_{v_3} = \{w_1, w_2\}$ and $T' = T - T_{v_3}$. Clearly, every $i(T')$-set can be extended to an independent dominating set of $T$ by adding $v_2, w_1, w_2$ yielding $i(T) \leq i(T') + 3$. On the other hand, any $\beta_2(T')$ can be extended to a 2-independent set by adding $z_1, z_2, w_1, w_2$ and so $\beta_2(T) \geq \beta_2(T') + 4$. By the induction hypothesis we have

$$i(T) \leq i(T') + 3 \leq \frac{3\beta_2(T')}{4} + 3 \leq \frac{3(\beta_2(T) - 4)}{4} + 3 \leq \frac{3\beta_2(T)}{4}.$$  

If the equality holds, then we must have $i(T') = \frac{3\beta_2(T)}{4}$ and this if and only if $T' \in F$. Hence each vertex of $T'$ is either a leaf or a strong support vertex. Now
we show that \( v_4 \) is a support vertex of \( T' \). Assume that \( v_4 \) is not a support vertex of \( T' \). Then \( v_4 \) is a leaf of \( T' \) and \( v_5 \) is its support vertex in \( T' \). Let \( T'' = T - T_{v_4} \). Then clearly \( T'' \notin F \) and so \( i(T'') < \frac{3\beta_2(T'')}{4} \). Obviously, every \( i(T') \)-set can be extended to an independent dominating set of \( T \) by adding \( v_2, z_1, z_2 \) yielding \( i(T) \leq i(T') + 3 \), and any \( \beta_2(T') \) can be extended to a 2-independent set by adding \( z_1, z_2, w_1, w_2 \) and so \( \beta_2(T) \geq \beta_2(T') + 4 \). Therefore

\[
i(T) \leq i(T') + 3 < \frac{3\beta_2(T')}{4} + 3 \leq \frac{3(\beta_2(T) - 4)}{4} + 3 \leq \frac{3\beta_2(T)}{4},
\]

which is a contradiction. Thus \( v_4 \) is a support vertex. Now \( T \) can be obtained from \( T' \) by operation \( O \) and so \( T \in F \).

Case 4. \( t = 2 \). Let \( w_1, w_2 \in L_{v_3} \) and \( T' = T - \{v_1, v_2\} \). Clearly, any \( i(T') \)-set can be extended to an independent dominating set of \( T \) by adding \( v_1 \), and this implies that \( i(T) \leq i(T') + 1 \). On the other hand, for any \( \beta_2(T') \)-set \( S' \), the set \( S = (S' \setminus \{v_3\}) \cup L_{v_3} \cup \{v_1, v_2\} \) is a 2-independent set of \( T \) yielding \( \beta_2(T) \geq \beta_2(T') + 2 \). It follows from the induction hypothesis that

\[
i(T) \leq i(T') + 1 \leq \frac{3\beta_2(T')}{4} + 1 < \frac{3\beta_2(T)}{4},
\]

and the proof is complete. \( \blacksquare \)

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