

## ON $k$ -TRESTLES IN POLYHEDRAL GRAPHS

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### Abstract

A  $k$ -trestle of a graph  $G$  is a 2-connected spanning subgraph of  $G$  of maximum degree at most  $k$ . We show that a polyhedral graph  $G$  has a 3-trestle, if the separator-hypergraph of  $G$  contains no two different cycles joined by a path of 3-separators of length  $\geq 0$ . There are graphs not satisfying this condition that have no 3-trestles. Further, for each integer  $k$  every graph with toughness smaller than  $\frac{2}{k}$  has no  $k$ -trestle.

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### 1. INTRODUCTION

By Steinitz's theorem a polyhedral graph is a planar and 3-connected graph. Let  $G$  be a connected graph. A subset  $S$  of the vertex set of  $G$  *separates*  $G$  if the graph  $G - S$  obtained from  $G$  by deleting the vertices of  $S$  is disconnected. If  $|S| = k$ ,  $S$  is said to be a  $k$ -separator of  $G$ . If no

$S_p \subset S$  (a proper subset of the set  $S$ ) separates  $G$  then the  $S$  is said to be a *proper  $k$ -separator of  $G$* . A subgraph of  $G$  is a *spanning subgraph of  $G$*  if it contains all vertices of  $G$ . 2-connected spanning subgraphs in which all vertices have degree at most  $k$  are called  *$k$ -trestles*. We will say that a graph  $G$  is  *$k$ -trestled* if  $G$  has a  $k$ -trestle [6]. Note that a graph  $G$  has a 2-trestle if and only if  $G$  is Hamiltonian.

A graph  $G$  is said to be  *$t$ -tough* if for every separating set  $S \subseteq V(G)$  the number  $\omega(G-S)$  of components of  $G-S$  is at most  $\frac{|S|}{t}$ . The toughness  $\tau(G)$  of a non-complete graph  $G$  is defined to be the largest integer  $t > 0$  such that  $G$  is  $t$ -tough. For a complete graph  $G$  let  $\tau(G) = \infty$ . The concept of toughness was introduced by Chvátal [4]. It is easy to see that every graph with toughness less than one has no 2-trestles. The following Lemma shows that every graph has a similar property with respect to  $k$ -trestles,  $k \geq 3$ .

**Lemma 1.** *Every graph  $G$  with toughness  $\tau(G) < \frac{2}{k}$  (where the integer  $k$  is greater than one) has no  $k$ -trestle.*

In [4] Chvátal conjectured:

**Conjecture 1** (Chvátal). There is a real number  $t_0 > 0$  such that every  $t_0$ -tough graph has a Hamiltonian cycle, i.e., a 2-trestle.

It seems to be interesting to consider relations between  $t$ -tough and  $k$ -trestled graphs in general. We pose the following conjecture.

**Conjecture 2.** For every integer  $k$  greater than one there is a real number  $t_k > 0$  such that every  $t_k$ -tough graph has a  $k$ -trestle.

There are several papers which deal with  $k$ -trestled polyhedral graphs. In [1] Barnette showed that there is a polyhedral graph with no 5-trestles. In [5] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle.

The well known theorem of Tutte [8] states that every 4-connected planar graph contains a Hamiltonian cycle, which means that every polyhedral graph with no 3-separators has a 2-trestle. Moreover, Tutte [8] proved

**Theorem 1.** *Let  $G$  be a 4-connected planar graph and let  $e$  and  $f$  be two edges of a facial cycle of  $G$ . Then  $G$  has a Hamiltonian cycle through  $e$  and  $f$ .*

Let  $H_1$  and  $H_2$  be two disjoint subsets of the vertex set  $V(G)$  of a graph  $G$ . The length of a minimal path in  $G$  with one end in  $H_1$  and the second in  $H_2$  is said to be *the distance of  $H_1$  and  $H_2$  in  $G$* .

Böhme, Harant and Tkáč in [3] showed that every maximal planar graph  $G$  in which no 3-separator has any common vertex with a proper 4-separator and every two distinct 3-separators have distance at least three, has a 2-trestle. In [2] Böhme and Harant presented examples of maximal planar graphs with no 2-trestles in which the minimal distances between two 3-separators are arbitrarily large.

Our next theorems partially supplement these results but in a more general case.

For each polyhedral graph  $G$  we will construct a separator-hypergraph  $\mathcal{H}(G)$  with the same set of vertices, such that the edges of  $\mathcal{H}(G)$  are the 3-separators of  $G$ . A cycle (and a path) of a hypergraph is a sequence  $P_1e_1P_2e_2\cdots P_ke_kP_{k+1}$ , where  $P_1, P_2, \dots, P_k$  are pairwise distinct vertices,  $e_1, e_2, \dots, e_k$  are pairwise distinct edges, the edge  $e_i$  is incident with both  $P_i$  and  $P_{i+1}$ ,  $1 \leq i \leq k$ , and  $P_{k+1} = P_1$  (and  $P_{k+1} \notin \{P_1, P_2, \dots, P_k\}$ , respectively).

**Theorem 2.** *Let  $G$  be a polyhedral graph. Let each component of the separator-hypergraph  $\mathcal{H}(G)$  have at most one cycle. Then  $G$  has a 3-trestle.*

**Theorem 3.** *There are polyhedral graphs with more than one cycle in their separator-hypergraph which have no 3-trestles.*

The polyhedral graphs constructed for Theorem 3 have separator-hypergraphs with many cycles; even 2-cycles are present.

## 2. PROOFS OF THEOREMS

**The Proof of Lemma 1.** Let  $G$  be a graph with toughness  $\tau(G) < \frac{2}{k}$  (where the integer  $k$  is greater than one). Suppose that  $G$  has a  $k$ -trestle  $H$ . Since  $\tau(G) < \frac{2}{k}$  there exists a subset  $S_0$  of the vertex set of  $G$  ( $S_0 \subset V(G)$ ) with

$$\frac{|S_0|}{\omega(G - S_0)} = \tau(G) < \frac{2}{k}.$$

So  $G$  contains a vertex set  $S_0$  such that

$$2\omega(G - S_0) > k|S_0|.$$

If  $G$  has a  $k$ -trestle  $H$  then  $S_0 \subset V(G) = V(H)$  and every vertex from  $S_0$  has in  $H$  a degree at most  $k$ . Since  $H$  is 2-connected, every component of  $G - S_0$  is adjacent with at least two vertices from  $S_0$ . This means that the following inequality holds

$$2\omega(G - S_0) \leq k|S_0|.$$

But this contradicts the before stated inequality.

Instead of Theorem 2 we shall prove the slightly stronger but more technical Theorem 4.

**Theorem 4.** *Let  $G$  be a polyhedral graph. Let each component of the separator-hypergraph  $\mathcal{H}(G)$  have at most one cycle. Label a vertex in each cycle-free component of  $\mathcal{H}(G)$ . Then  $G$  has a 3-trestle  $H$  such that every 3-valent vertex of  $H$  is an unlabelled vertex of a 3-separator in  $G$ .*

**The Proof of Theorem 4.** The proof is by induction on the number of 3-separators of the considered graphs. If  $G$  has no 3-separator then  $G$  is 4-connected and by Tutte's Theorem 1 the graph  $G$  has a Hamiltonian cycle. Thus  $G$  has a special 3-trestle with the required properties.

Assume that Theorem 4 is true for all polyhedral graphs with at most  $m$  3-separators,  $m \geq 0$ . Let  $G$  be a polyhedral graph with  $m + 1$  3-separators such that each component of the "separator"-hypergraph  $\mathcal{H}(G)$  has at most one cycle.

A 3-separator  $S = \{x, y, z\}$  is called *elementary* if one component  $I(S)$  of  $G - S$  has no 3-separators. W.l.o.g. we may suppose that  $G$  is mapped into the plane so that  $I(S)$  is the interior of the cycle  $(x, y, z)$ . Now we prove the following

**Claim 1.** If  $S = \{x, y, z\}$  is an elementary 3-separator of  $G$  then  $\langle I(S) \cup S \rangle_G$ , the subgraph induced by  $I(S) \cup S$  in  $G$ , contains an  $x, y$ -path through all vertices of  $I(S) \cup S \setminus \{z\}$  avoiding  $z$ .

**Proof of Claim 1.** Since  $S = \{x, y, z\}$  is elementary the subgraph  $H := \langle I(S) \cup S \rangle_G \cup (x, y, z)$  has no 3-separators and  $H$  is 4-connected or  $K_4$  (a complete graph on four vertices). By Tutte's Theorem 1 the subgraph  $H$  has a Hamiltonian cycle  $h$  through the edges  $(x, z)$  and  $(z, y)$ . The path  $p = h \setminus \{z\}$  has the required properties, and the proof of Claim 1 is complete.

The graph  $G$  obviously contains an elementary 3-separator  $S = \{x, y, z\}$ . This 3-separator  $S$  is a hyperedge of a component  $K$  of  $\mathcal{H}(G)$ .

*Case 1.* Let  $K$  have no cycle in  $\mathcal{H}(G)$ .

The subhypergraph  $K \setminus \{S\}$  of  $\mathcal{H}(G)$  has at most three cycle-free components  $K_x, K_y$  and  $K_z$  containing  $x, y$ , and  $z$ , respectively. Note that some of these components can be trivial. W.l.o.g. let  $K_x$  have the vertex with the label of  $K$  (it may be that  $x$  has this label). In  $K_y$  and  $K_z$  we label the vertices  $y$  and  $z$ , respectively.

*Case 2.* Let  $K$  have a cycle  $C$  in  $\mathcal{H}(G)$ .

Note that  $K$  has no label.

*Case 2.1.* Let  $S \notin C$ .

The subhypergraph  $K \setminus \{S\}$  of  $\mathcal{H}(G)$  has at most three components  $K_x, K_y$  and  $K_z$  containing  $x, y$  and  $z$ , respectively. W.l.o.g. let  $C \subseteq K_x$ , and  $K_y, K_z$  are cycle-free in  $\mathcal{H}(G)$ . In  $K_y$  and  $K_z$  we label the vertices  $y$  and  $z$ , respectively.

*Case 2.2.* Let  $S \in C$ .

Two vertices of  $S$  belong to  $C$ , say,  $x$  and  $y$ . The subhypergraph  $K \setminus S$  of  $\mathcal{H}(G)$  has at most two components  $K_{x,y}$  and  $K_z$  containing  $\{x, y\}$  or  $\{z\}$ , respectively. The path  $C \setminus \{S\} \subseteq K_{x,y}$  and both components  $K_{x,y}$  and  $K_z$  are cycle-free in  $\mathcal{H}(G)$ . We label  $y$  and  $z$ .

In all cases we proceed in the same way.

The graphs  $G_1$  and  $G_2$  are obtained from  $G$  by deleting the interior or the exterior of  $(x, y, z)$ , respectively, and adding the cycle  $(x, y, z)$ . Thus  $G$  has a separation:  $G = G_1 \cup G_2$ ,  $G_1 \cap G_2 = (x, y, z)$ ,  $K \setminus \{S\} \subseteq G_1$ .

By the induction hypothesis  $G_1$  contains a 3-trestle  $T_1$  with the required properties. The degrees  $\deg_{T_1}(y) = \deg_{T_1}(z) = 2$ .

By Claim 1 the subgraph  $G_2$  contains a  $y, z$ -path  $T_2$  through all vertices of  $G_2 \setminus \{x\}$  avoiding  $x$ . Then  $T_1 \cup T_2$  is a 3-trestle of  $G$  with the required properties. ■

**The Proof of Theorem 3.** Theorem 3 will be proved by constructing an appropriate graph. A double-cube is obtained from two disjoint copies  $C_1$  and  $C_2$  of the cube by identifying a face of  $C_1$  with a face of  $C_2$ . This polyhedral graph has  $n = 12$  vertices and  $f = 10$  quadrangles. In each quadrangle with bounding 4-cycle  $(v_0, v_1, v_2, v_3)$  we introduce a 4-cycle

$(w_0, w_1, w_2, w_3)$  so that for every  $i \pmod{4}$  a vertex  $v_i$  is connected with  $w_i$  and  $w_{i+1}$  by an edge, introduce a new vertex  $\alpha_i$  in each triangle face with bounding cycle  $(v_i, v_{i+1}, w_{i+1})$  and join  $\alpha_i$  to each vertex of the bounding 3-cycle  $(v_i, v_{i+1}, w_{i+1})$  by an edge.

The resulting graph  $H$  is polyhedral and its connected separator-hypergraph has more than one cycle.

We claim that  $H$  has no 3-trestle.

Suppose  $H$  has a 3-trestle  $T$ . By construction each vertex  $\alpha_i$  is joined to the vertex  $v_i$  or  $v_{i+1}$  of the double-cube by at least one edge of  $T$ . Thus the subgraph  $T$  has at least  $4f$  such edges. Consequently, the double-cube has at least one vertex  $v$  of degree

$$\deg_T(v) \geq \frac{4f}{n} = \frac{40}{12} > 3.$$

Hence  $v$  has a degree  $\deg_T(v) \geq 4$  and  $T$  is no 3-trestle. This contradiction shows that  $H$  has no 3-trestle.

Starting our construction with  $l \geq 3$  cubes results in an infinite sequence of graphs satisfying Theorem 3. ■

#### REFERENCES

- [1] D. Barnette, *2-connected spanning subgraphs of planar 3-connected graphs*, J. Combin. Theory (B) **61** (1994) 210–216.
- [2] T. Böhme and J. Harant, *On hamiltonian cycles in 4- and 5-connected planar triangulations*, Discrete Math. **191** (1998) 25–30.
- [3] T. Böhme, J. Harant and M. Tkáč, *On certain Hamiltonian cycles in planar graphs*, J. Graph Theory **32** (1999) 81–96.
- [4] V. Chvátal, *Tough graphs and Hamiltonian circuits*, Discrete Math. **5** (1973) 215–228.
- [5] Z. Gao, *2-connected coverings of bounded degree in 3-connected graphs*, J. Graph Theory **20** (1995) 327–338.
- [6] D.P. Sanders and Y. Zhao, *On 2-connected spanning subgraphs with low maximum degree*, J. Combin. Theory (B) **74** (1998) 64–86.
- [7] C. Thomassen, *A theorem on paths in planar graphs*, J. Graph Theory **7** (1983) 169–176.
- [8] W.T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. **82** (1956) 99–116.

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