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ON GENERATING SETS OF INDUCED-HEREDITARY PROPERTIES

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Abstract

A natural generalization of the fundamental graph vertex-colouring problem leads to the class of problems known as generalized or improper colourings. These problems can be very well described in the language of reducible (induced) hereditary properties of graphs. It turned out that a very useful tool for the unique determination of these properties are generating sets. In this paper we focus on the structure of specific generating sets which provide the base for the proof of The Unique Factorization Theorem for induced-hereditary properties of graphs.

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1. Introduction and Motivation

The fundamental graph colouring problem deals with the partitioning of the vertex set of a graph G into classes according to the rule that no pair of adjacent vertices can appear in the same class. To distinguish the classes we use a finite set of colours $\mathcal{C}$, and the divisions into classes is given as a mapping $f : V(G) \to \mathcal{C}$ satisfying that $f(u)$ differs from $f(v)$ whenever $\{u, v\}$ is an edge of G. According to our colouring rule, each colour class

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forms an *independent set of vertices* of G. Such a colouring is called a *proper* colouring.

In *improper* colourings (sometimes called *generalized*, *defective* or *relaxed*) adjacent vertices may be assigned the same colour, but some other constraint is placed on colour classes (see e.g. [1, 2, 5, 6, 7, 8, 9, 11] and [14]).

A convenient language that may be used for formulating problems of graph colouring in a general setting is the language of reducible hereditary properties. The concept of reducible hereditary properties was introduced in [6] and [11] (see also [3]).

A *graph property* is any non-empty isomorphism closed class of graphs. Since we have, in general, no reason to distinguish between isomorphic copies of a graph, we use the notation $\mathcal{I}$ to denote the set of all finite unlabelled loopless undirected graphs, one from each isomorphism class, and we can consider a graph property to be a subset of $\mathcal{I}$. We count a graph G and its isomorphic images as one graph. By saying that H is a subgraph of G, we mean that H is isomorphic to a subgraph of G. If G belongs to a property $\mathcal{P} \subseteq \mathcal{I}$ then we also say that G has the property $\mathcal{P}$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of G is a partition $(V_1, V_2, \ldots, V_n)$ of the vertex set $V(G)$ such that the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$ for $i = 1, 2, \ldots, n$. If a graph G has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition, then we say that G has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$ we simply write $\mathcal{P}^n$ instead of $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If there are properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, then the property $\mathcal{P}$ is called *reducible*. If such properties do not exist, the property $\mathcal{P}$ is called *irreducible*. If we denote by $\mathcal{O}$ the set of all the edgeless graphs then it is obvious that a graph G is k-colourable if and only if $G \in \mathcal{O}^k$.

Since it is very difficult to deal with properties in such a general setting, we need an additional reasonable requirement. It seems to be fruitful to consider some partial order $\preceq$ on the set $\mathcal{I}$, for example "to be a subgraph", "to be an induced subgraph", "to be a minor" etc. We say that a property $\mathcal{P}$ is $\preceq$-*hereditary* if $G \in \mathcal{P}$ implies that $H \in \mathcal{P}$, for all $H \preceq G$. In particular, we shall deal with $\subseteq$-*hereditary* (in short *hereditary*) and $\leq$-*hereditary* (we use also the term *induced-hereditary*) properties of graphs, meaning those which are closed under taking subgraphs and induced subgraphs, respectively. It is easy to observe that hereditary properties are special examples of induced-hereditary properties. A number of problems refer to specific types of hereditary properties which are called *additive*. Those properties are closed under taking the disjoint union of graphs with the given property. It is not difficult to see that many interesting and important properties (e.g. to be an acyclic graph, to be a planar graph, to be a k-colourable graph, to be a graph with maximum degree at most k, to be homomorphic to a given graph etc.) studied in graph theory are additive and induced hereditary or even hereditary. For much more details, many applications and open problems concerning hereditary and induced-hereditary properties of graphs we refer the reader to [2].

One of the most important problems concerning induced-hereditary properties is the problem of the unique factorization of an induced-hereditary property into irreducible factors (see [10] — Problem 17.9). This problem is solved by so-called unique factorizations theorems in [12] (for induced-hereditary properties) and [13] (for hereditary properties). Both proofs require detailed analyses of the structure of reducible properties which is the aim of our paper.

2. Four Different Types of Generating Sets

In the case of hereditary properties it is natural to characterize a property $\mathcal{P}$ by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs. To be more accurate, let us define the set of $\mathcal{P}$-*maximal graphs* in the following way:

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}.$$

One can observe that a graph G belongs to a property $\mathcal{P}$ if and only if it is a subgraph of some graph $H \in \boldsymbol{M}(\mathcal{P})$. Unfortunately, in the case of induced-hereditary properties we have no natural description of such a type. It inspires us to introduce a more general concept — the *generating set* of a hereditary property (see also [2]).

Given an arbitrary set $\mathcal{G}$, a subset of $\mathcal{I}$. It is quite easy to see that the properties

$$[\mathcal{G}] = \{G \in \mathcal{I} : G \text{ is a subgraph of some graph } H \in \mathcal{G}\},$$

$$[\mathcal{G}]^a = \{G \in \mathcal{I} : \text{ each component of } G \text{ is a subgraph of some } H \in \mathcal{G}\}$$

are hereditary and the properties

$$[\mathcal{G}]^i = \{G \in \mathcal{I} : G \text{ is an induced subgraph of some graph } H \in \mathcal{G}\},$$

$$[\mathcal{G}]^{ia} = \{G \in \mathcal{I} : \text{each component of } G \text{ is an induced-subgraph of some } H \in \mathcal{G}\}$$

are induced-hereditary. Moreover the properties $[\mathcal{G}]^a$ and $[\mathcal{G}]^{ia}$ are also additive.

In such cases we say that the set $\mathcal{G}$ is an **H**-*generating set* of the property $[\mathcal{G}]$, an $\mathbf{H_a}$-*generating* set of the property $[\mathcal{G}]^a$, an **I**-*generating set* of the property $[\mathcal{G}]^i$ and an $\mathbf{I_a}$-*generating set* of the property $[\mathcal{G}]^{ia}$. The members of $\mathcal{G}$ shall be called *generators*. It is obvious that the set $\boldsymbol{M}(\mathcal{P})$ is an **H**-generating set for the property $\mathcal{P}$. If $\mathcal{P}$ is additive, then $\boldsymbol{M}(\mathcal{P})$ is also an $\mathbf{H_a}$-generating set of $\mathcal{P}$.

It is not so difficult to see, that the property

$$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has order at most } k+1\}$$

has the set $\mathcal{G} = \{K_{k+1}\}$ as an $\mathbf{H_a}$-generating set. It is interesting that only one graph is sufficient for the description of this hereditary property. On the other hand, we can observe that the property $\mathcal{D}_1$ "to be an acyclic graph" has no finite **H**-generating or $\mathbf{H_a}$-generating set.

The following two results provide relationships between the number of graphs of a given property and the cardinality of the corresponding generating sets. Their proofs are almost straightforward applications of the definitions.

Lemma 2.1. *Let $\mathcal{P}$ be a hereditary property and $\mathcal{Q}$ an induced-hereditary property of graphs. Then*

(i) *$\mathcal{P}$ contains only a finite number of graphs if and only if some* **H**-*generating set of $\mathcal{P}$ is finite.*

(ii) *$\mathcal{Q}$ contains only a finite number of graphs if and only if some* **I**-*generating set of $\mathcal{Q}$ is finite.*

Moreover, it is obvious that if $\mathcal{P}$ ($\mathcal{Q}$) contains only a finite number of graphs then it is not additive and therefore there exists no $\mathbf{H_a}$-generating set ($\mathbf{I_a}$-generating set) of $\mathcal{P}$ ($\mathcal{Q}$).

Lemma 2.2. *Let $\mathcal{P}$ be an additive hereditary property and let $\mathcal{Q}$ be an induced-hereditary property of graphs. Then*

(i) *There exists a finite $\mathbf{H_a}$-generating set of $\mathcal{P}$ if and only if there exists a non-negative integer n such that every component of each graph $G \in \mathcal{P}$ has at most $n+1$ vertices (i.e., $\mathcal{P} \subseteq \mathcal{O}_n$).*

(ii) *There exists a finite* $\mathbf{I_a}$*-generating set of* $\mathcal{Q}$ *if and only if there exists a non-negative integer* n *such that every component of each graph* $G \in \mathcal{Q}$ *has at most* $n+1$ *vertices* (*i.e.*, $\mathcal{Q} \subseteq \mathcal{O}_n$).

Corollary 2.3. *If the property* $\mathcal{D}_1$ *of* 1*-degenerate graphs* (*acyclic graphs*) *is a subclass of a hereditary property* $\mathcal{P}$ *then* $\mathcal{P}$ *has neither a finite* $\mathbf{H_a}$*-generating set nor a finite* $\mathbf{H}$*-generating set of* $\mathcal{P}$.

Corollary 2.4. *If all the* $\mathcal{P}$*-maximal graphs are connected then there exists no finite* $\mathbf{H_a}$*-generating set of* $\mathcal{P}$.

In order to present the next property of generating sets we need a new concept. We say that a graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is uniquely $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$-*partitionable* if it has only one $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$-partition of $V(G)$ up to the order of the sets in this partition. The following theorem was established in [4].

Theorem 2.5. *Let* n *be a positive integer and* P_i, $i = 1, 2, \ldots, n$ *be additive hereditary properties of graphs. If* $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ *and there exists at least one uniquely* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$*-partitionable graph, then* H *is and induced subgraph of some uniquely* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$*-partitionable graph.*

The previous theorem immediately implies the following result.

Theorem 2.6. *Let* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ *be an additive reducible hereditary property of graphs. If there exists an uniquely* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$*-partitionable graph then there exists also an* $\mathbf{I}$*-generating set of* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ *containing only uniquely* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$*-partitionable graphs.*

It was proved in [13] that for any $\mathbf{H}$-generating set $\mathcal{G}$ of a property $\mathcal{P}$ there exist $\mathbf{H}$-generating sets $\mathcal{G}', \mathcal{G}^* \subseteq \mathcal{G}$ such that every graph $G' \in \mathcal{G}'$ contains a prescribed graph $H \in \mathcal{P}$ as a subgraph and all the graphs of $\mathcal{G}^*$ have the same number of components in their complements.

3. Induced-Hereditary Properties

The proof of The Unique Factorization Theorem for hereditary properties is based on the constructions and examinations of $\mathbf{H}$-generating sets derived from the sets of $\mathcal{P}$-maximal graphs (for details see [13]). When we tried to prove The Unique Factorization Theorem for induced-hereditary properties we were unsuccessful for a long time because we did not have analogues for the set of $\mathcal{P}$-maximal graphs and the join operation. Eventually we found the set of $\mathcal{P}$-strict graphs and the star operation to be the suitable analogues.

Definition 3.1. For given graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, we denote by $G_1 * G_2 * \cdots * G_n$ the set

$$\left\{ G \in \mathcal{I} : \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i \right\},$$

where $\bigcup_{i=1}^{n} G_i$ denotes the disjoint union and $\sum_{i=1}^{n} G_i$ the join of graphs $G_1, G_2, \ldots, G_n$, respectively.

One can immediately see that the operation $*$ is rather complicated and its result is not one graph but a class of graphs. All the graphs belonging to the result of the operation $*$ are of the same order and therefore they are mutually incomparable with respect to the relation "to be an induced subgraph". However, such a definition allows us to work with different vertex partitions of a graph and disregard the edges between the colour classes.

Definition 3.2. A graph $G \in \mathcal{P}$ is said to be $\mathcal{P}$-strict if $G * K_1 \not\subseteq \mathcal{P}$. The class of all $\mathcal{P}$-strict graphs is denoted by $\mathbf{S}(\mathcal{P})$.

We shall show that the sets of $\mathcal{P}$-strict graphs play an important role in the characterization of induced-hereditary properties. The next result states that the set $\mathbf{S}(\mathcal{P})$ can be used as a natural generating set of an induced-hereditary property $\mathcal{P}$.

Theorem 3.3. *If $\mathcal{P} \neq \mathcal{I}$ is an induced-hereditary property of graphs then $\mathbf{S}(\mathcal{P})$ is an $\mathbf{I}$-generating set of $\mathcal{P}$. Moreover, if $\mathcal{P}$ is additive then $\mathbf{S}(\mathcal{P})$ is an $\mathbf{I_a}$-generating set of $\mathcal{P}$.*

Proof. Since $\mathcal{P} \neq \mathcal{I}$, there exists a graph $F \notin \mathcal{P}$.
Let $f(\mathcal{P}) = \min\{|V(F)| : F \notin \mathcal{P}\}$. It is not difficult to see that for any graph $G \in \mathcal{P}$ the class $G * \underbrace{K_1 * \cdots * K_1}_{f(\mathcal{P})-1}$ is not a subset of $\mathcal{P}$. Let us put $\mathcal{H}_0 = \{G\}$ and $\mathcal{H}_i = G * \underbrace{K_1 * \cdots * K_1}_{i}$, for $i = 1, \ldots, f(\mathcal{P}) - 1$. Then it

is obvious, that for some $j < f(\mathcal{P}) - 1$ there exists a graph $H_j \in \mathcal{H}_j$ such that $H_j \in \mathcal{P}$ and $H_j * K_1 \notin \mathcal{P}$. It means that H_j is a $\mathcal{P}$-strict graph and obviously $G \leq H_j$.

In addition, if $\mathcal{P}$ is additive then the disjoint union of an arbitrary finite number of subgraphs of some $\mathcal{P}$-strict graphs belongs to $\mathcal{P}$. On the other hand, any component of a graph $G \in \mathcal{P}$ is, according to the previous, an induced subgraph of some $\mathcal{P}$-strict graph. ∎

Definition 3.4. Let $\mathcal{R}$ be an induced-hereditary property. For each $G \in \mathcal{R}$ put $\mathrm{dec}_{\mathcal{R}}(G) = \max\{n :$ there exist a partition $(V_1, V_2, \ldots, V_n), V_i \neq \emptyset$ of $V(G)$ such that for each $k \geq 1$, $k.G[V_1] * k.G[V_2] * \cdots * k.G[V_n] \subseteq \mathcal{R}\}$. If $G \notin \mathcal{R}$ we set $\mathrm{dec}_{\mathcal{R}}(G)$ to be zero. A graph G is said to be $\mathcal{R}$-decomposable if $\mathrm{dec}_{\mathcal{R}}(G) \geq 2$, otherwise G is $\mathcal{R}$-indecomposable.

The invariant $\mathrm{dec}_{\mathcal{R}}(G)$ describes the variability of a graph G with respect to different partitions of its vertex set. The following useful lemma related to the $\mathcal{P}$-decomposability number of $\mathcal{P}$-strict graphs is proved in [12].

Lemma 3.5. *Let G be an $\mathcal{R}$-strict graph and $G^* \in \mathcal{R}$ be an induced supergraph of G (i.e., $G \leq G^*$). Then G^* is $\mathcal{R}$-strict and* $\mathrm{dec}_{\mathcal{R}}(G) \geq \mathrm{dec}_{\mathcal{R}}(G^*)$.

From the next lemma it follows that if it is possible to generate an induced-hereditary property $\mathcal{R}$ with an $\mathbf{I}$-generating set consisting of $\mathcal{R}$-decomposable graphs then all the $\mathcal{R}$-strict graphs must also be $\mathcal{R}$-decomposable. This feature can be used as a criterion for reducibility of induced-hereditary properties.

Lemma 3.6. *Let $\mathcal{R}$ be an induced-hereditary property of graphs and let $\mathcal{G}$ be an $\mathbf{I}$-generating set of $\mathcal{R}$. If all the graphs belonging to $\mathcal{G}$ are $\mathcal{R}$-decomposable, then all the $\mathcal{R}$-strict graphs are $\mathcal{R}$-decomposable as well.*

Proof. Suppose to the contrary that there exists an $\mathcal{R}$-strict graph H which is $\mathcal{R}$-indecomposable. Since $H \in \mathcal{R}$, there exists $G \in \mathcal{G}$ such that $H \leq G$. Since G is $\mathcal{R}$-decomposable, there exists a partition $(V_1, V_2, \ldots, V_n)$, $n \geq 2$, of the vertex set $V(G)$ such that $k.G[V_1] * k.G[V_2] * \cdots * k.G[V_n] \subseteq \mathcal{R}$ for each $k \geq 1$.

If the intersections $V(H) \cap V_i$ are non-empty for the indices $i_1, i_2, \ldots, i_q$, $n \geq q \geq 1$ then it is evident that each class $k.H[V(H) \cap V_{i_1}] * k.H[V(H) \cap V_{i_2}] * \cdots * k.H[V(H) \cap V_{i_q}]$ is a subclass of $\mathcal{R}$ (because $\mathcal{R}$ is induced-hereditary and each $k.G[V_{i_1}] * k.G[V_{i_2}] * \cdots * k.G[V_{i_q}] \subseteq \mathcal{R}$). The assumption that H is $\mathcal{R}$-indecomposable immediately implies that q should be equal to one. But then there exists at least one vertex $v \in V(G)\setminus V(H)$ such that all the graphs from $H * K_1 = H * (\{v\}, \emptyset)$ are induced subgraphs of some graphs from the class $G[V_1] * G[V_2] * \cdots * G[V_n] \subseteq \mathcal{R}$ which contradicts our assumption that H is $\mathcal{R}$-strict. ∎

The following theorem allows us to assume that each graph of our generating set has a prescribed substructure.

Theorem 3.7. *Let $\mathcal{P}$ be an additive induced-hereditary property and $\mathcal{G}$ be an* **I***-generating set of $\mathcal{P}$. If G is an arbitrary graph with the property $\mathcal{P}$ then there exists an* **I***-generating set $\mathcal{G}' \subseteq \mathcal{G}$ such that each graph $H \in \mathcal{G}'$ contains at least one copy of the graph G as an induced subgraph.*

Proof. Let $\mathcal{S} \subseteq \mathcal{G}$ be the set of graphs which do not contain G as an induced subgraph. Since the property $\mathcal{P}$ is additive, for any choice of the graph $H \in \mathcal{S}$, the graph $H' = H \cup G$ also belongs to $\mathcal{P}$. Therefore there exists a graph $H^* \in \mathcal{G}$ such that H' is an induced subgraph of H^* (in such a case H^* evidently contains a copy of G).

Further, it is easy to observe that the set $\mathcal{G}' = \mathcal{G} \setminus \mathcal{S} \subseteq \mathcal{G}$ is an **I**-generating set with the required properties. ∎

Definition 3.8. Let $\mathcal{R}$ be an induced hereditary property. The $\mathcal{R}$-decomposability number of an **I**-generating set $\mathcal{G}$ of $\mathcal{R}$ is defined as follows: $\mathrm{dec}_{\mathcal{R}}(\mathcal{G}) = \min\{\mathrm{dec}_{\mathcal{R}}(G) : G \in \mathcal{G}\}$. The decomposability number of the property $\mathcal{R}$ is the $\mathcal{R}$-decomposability number of the set $\mathbf{S}(\mathcal{R})$.

The proof of The Unique Factorization Theorem shows that the decomposability number of a property uniquely determines the number of irreducible factors in the factorization of the property. The next theorem states that the $\mathcal{R}$-decomposability number is the same for all generating sets contained in the set of $\mathcal{R}$-strict graphs.

Theorem 3.9. *Let $\mathcal{P}$ be an induced-hereditary property of graphs. Let $\mathcal{G}$ be an* **I***-generating set of $\mathcal{P}$ such that $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$. Then $\mathrm{dec}_{\mathcal{P}}(\mathcal{G}) = \mathrm{dec}_{\mathcal{P}}(\mathcal{P})$.*

Proof. Since $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$ we immediately have:

$$\mathrm{dec}_{\mathcal{P}}(\mathcal{P}) = \mathrm{dec}_{\mathcal{P}}(\mathbf{S}(\mathcal{P})) \leq \mathrm{dec}_{\mathcal{P}}(\mathcal{G}).$$

Let $G \in \mathbf{S}(\mathcal{P})$ be a graph such that $\operatorname{dec}_{\mathcal{P}}(G) = \operatorname{dec}_{\mathcal{P}}(\mathbf{S}(\mathcal{P})) = q$. Since $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{G}$ such that $G \leq H$. According to our assumptions G and H are $\mathcal{P}$-strict graphs. Hence, by an application of Lemma 3.5, we obtain the inequality $\operatorname{dec}_{\mathcal{P}}(G) \geq \operatorname{dec}_{\mathcal{P}}(H)$. This implies that

$$\begin{aligned} \operatorname{dec}_{\mathcal{P}}(\mathcal{P}) &= \operatorname{dec}_{\mathcal{P}}(\mathbf{S}(\mathcal{P})) = \operatorname{dec}_{\mathcal{P}}(G) \geq \operatorname{dec}_{\mathcal{P}}(H) \geq \operatorname{dec}_{\mathcal{P}}(\mathcal{G}) \\ &= \min\{\operatorname{dec}_{\mathcal{P}}(H^*) : H^* \in \mathcal{G}\} \end{aligned}$$

and the proof is complete. ■

The next theorem guarantees the existence of a generating set, contained in the set of $\mathcal{P}$-strict graphs, with a prescribed value of the invariant $\operatorname{dec}_{\mathcal{R}}$ for each of its members. Generating sets of this type are very important for the proof of The Unique Factorization Theorem (see [12]), because they enable us to construct the factors of a reducible induced-hereditary property.

Theorem 3.10. *Let $\mathcal{P}$ be an additive induced-hereditary property of graphs. Let $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$ be any generating set of $\mathcal{P}$. Then there exists a set $\mathcal{G}^* \subseteq \mathcal{G}$, which is a generating set of $\mathcal{P}$ and contains only graphs of $\mathcal{P}$-decomposability number equal to* $\operatorname{dec}(\mathcal{P})$.

Proof. Let G be an arbitrary graph from $\mathcal{G}$ with $\operatorname{dec}_{\mathcal{P}}(G) = \operatorname{dec}(\mathcal{G}) = q$. Then there exists a partition $(V_1, V_2, \ldots, V_q)$ of $V(G)$ such that $k.G[V_1] * k.G[V_2] * \cdots * k.G[V_q] \subseteq \mathcal{P}$ for each positive integer k. By Lemma 3.7 there is a generating set $\mathcal{G}^* \subseteq \mathcal{G}$ such that each graph $H \in \mathcal{G}^*$ contains G as an induced subgraph.

Let H be any graph from $\mathcal{G}^*$. Similar arguments as in the proof of Lemma 3.6 show that $\operatorname{dec}_{\mathcal{P}}(H) = \operatorname{dec}(\mathcal{P})$ and the proof is complete. ■

Using The Unique Factorization Theorem it is also possible to prove the following interesting result (see [12]).

Theorem 3.11. *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $n \geq 2$ be the factorization of a reducible induced-hereditary property $\mathcal{R}$ into irreducible factors. Then there exists an $\mathbf{I}$-generating set of $\mathcal{R}$ which contains only uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs.*

References

[1] B. Bollobás and A.G. Thomason, *Hereditary and monotone properties of graphs*, in: R.L. Graham and J. Nešetřil, eds., The mathematics of Paul Erdős, II, Algorithms and Combinatorics **14** (Springer-Verlag, 1997) 70–78.

[2] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *Survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[3] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli, ed., Advances in Graph Theory (Vishwa International Publication, Gulbarga, 1991) 42–69.

[4] I. Broere and J.Bucko, *Divisibility in additive hereditary graph properties and uniquely partitionable graphs*, Tatra Mountains Math. Publications **18** (1999) 79–87.

[5] G. Chartrand, D. Geller and S. Hedetniemi, *Graphs with forbidden subgraphs*, J. Combin. Theory (B) **10** (1971) 12–41.

[6] E.J. Cockayne, *Color classes for r-graphs*, Canad. Math. Bull. **15** (3) (1972) 349–354.

[7] E.J. Cockayne, G.G. Miller and G. Prins, *An interpolation theorem for partitions which are complete with respect to hereditary properties*, J. Combin. Theory (B) **13** (1972) 290–297.

[8] M. Frick, *A survey of (m, k)-colorings*, in: J. Gimbel c.a, ed., Quo Vadis, Graph Theory? A source book for challenges and directions, Annals of Discrete Mathematics **55** (North-Holland, Amsterdam, 1993) 45–57.

[9] S.T. Hedetniemi, *On hereditary properties of graphs*, J. Combin. Theory (B) **14** (1973) 94–99.

[10] T.R. Jensen and B. Toft, Graph colouring problems (Wiley–Interscience Publications, New York, 1995).

[11] P. Mihók, *Additive hereditary properties and uniquely partitionable graphs*, in: M. Borowiecki and Z. Skupień, eds., Graphs, hypergraphs and matroids (Zielona Góra, 1985) 49–58.

[12] P. Mihók, *Unique factorization theorem*, Discuss. Math. Graph Theory **20** (2000) 143–154.

[13] P. Mihók, G. Semanišin and R. Vasky, *Additive and Hereditary Properties of Graphs are Uniquely Factorizable into Irreducible Factors*, J. Graph Theory **33** (2000) 44–53.

[14] J. Mitchem, *Maximal k-degenerate graphs*, Utilitas Math. **11** (1977) 101–106.

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