Discussiones Mathematicae
Graph Theory 22 (2002) 173–182

THREE EDGE-COLORING CONJECTURES

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Abstract

The focus of this article is on three of the author's open conjectures. The article itself surveys results relating to the conjectures and shows where the conjectures are known to hold.

Keywords: edge-coloring, Ramsey number, vertex-distinguishing edge-coloring, strong chromatic index, balanced edge-coloring, local coloring, mean coloring.

2000 Mathematics Subject Classifications: 05C15, 05C55, 05C78.

The purpose of this article is to give additional attention to three edge-coloring conjectures that have been considered over the last several years. Each conjecture together with closely related results appear in a separate section of the paper. For the reader's convenience each conjecture itself is set off in boldface type.

1. A Conjecture on the Classical Ramsey Number

The classical Ramsey number $r_k(G)$ is well known and is the smallest positive integer m such that any edge-coloring of K_m by k colors contains a monchromatic copy of G. The conjecture to be posed allows a less restrictive edge-coloring of K_m with the same consequence, a monochromatic copy of G in the colored K_m. To be precise two special colorings need to be defined, the k-local coloring and the k-mean coloring.

The concept of local colorings first appeared in a paper of Erdős and Sós [10] and was later studied extensively in [18]. It has also been more recently rediscovered by Galluccio, Simonovits, and Simonyi [17]. A *k-local coloring* of a graph G is an edge-coloring of G, by any number of colors, as long as each vertex is incident to edges colored by at most k different colors. A k-mean coloring of G is an edge-coloring, by any number of colors, such that "on average" each vertex is incident to edges colored by at most k colors. For preciseness if f is an edge-coloring of G by any number of colors and $k_f(v)$ is the number of distinct colors that appear on edges of G incident to v under the coloring f, then a *k-mean coloring* f of G is one where $1/|G|\sum_v k_f(v) \le k$. Note that f is a k-local coloring if $k_f(v) \le k$ for all v in G, i.e. every k-local coloring is a k-mean coloring.

The *k-local Ramsey number* $r_{k-\ell oc}$ (G) (*k-mean Ramsey number* $r_{k-mean}(G)$) is defined as the smallest positive integer m, such that any k-local coloring of K_m (any k- mean coloring of K_m) contains a monochromatic copy of G. Both of these Ramsey numbers are known to exist which follows from the following theorem.

Theorem 1.1.

(i) (Gyárfás, Lehel, Schelp, Tuza [18]). *Let $n, k \ge 2$ be integers. Then* $r_{k-\ell oc}(K_n) \le \lceil (k^{k(n-2)+1}/(k-1)\rceil$.

(ii) (Caro [11]). *Let $n, k \ge 2$ be integers. Then* $r_{k-mean}(K_n) \le k \cdot r_{(k+1)-\ell oc}(K_n)$.

Since every k edge-coloring is a k-local coloring and since every k-local coloring is a k-mean coloring, it is clear that $r_k(G) \le r_{k-\ell oc}(G) \le r_{k-mean}(G)$.

A natural question is whether either or both of the inequalities are in fact equalities. This leads to the first of the three conjectures of the paper.

Conjecture 1. For all integers $n \ge 3$, $k \ge 2$, $r_k(K_n) = r_{k-mean}(K_n)$.

In the remainder of this section results are presented which form a basis for the conjecture.

Comparisons between $r_2(G)$ and $r_{2-\ell oc}(G)$ have been studied extensively in [18]. In particular the following theorem appears there.

Theorem 1.2 (Gyárfás, Lehel, Schelp, Tuza) [18]**.**

(i) *For $m \ge 2n-1, (m,n) \ne (3,2)$, $r_{2-\ell oc}(K_m - K_n) = r_2(K_m - K_n)$. In particular $r_{2-\ell oc}(K_m) = r_2(K_m)$ for all $m \ge 1$.*

(ii) *There exist trees T for which $r_{2-\ell oc}(T) - r_2(T)$ is not bounded.*

(iii) *For all connected graphs G the ratio $r_{2-\ell oc}(G)/r_2(G) \leq 3/2$.*

Theorem 1.2 (iii) is generalized in [23] for all k and is given in the next theorem.

Theorem 1.3 (Truszczyński and Tuza [23])**.** *For each positive integer k there exists a constant $c = c(k)$ such that $r_{k-\ell oc}(G)/r_k(G) \leq c$ for all connected graphs G.*

In [12] Caro and Tuza investigate the relationship between $r_{k-\ell oc}(G)$ and $r_{k-mean}(G)$. They prove the following.

Theorem 1.4 (Caro and Tuza [12])**.**

(i) *There exists a constant $c(k) \leq 2(k-1)$ such that $r_{k-mean}(G) \leq c(k) r_{k-\ell oc}(G)$ for every graph G.*

(ii) *For every graph G, $r_{2-mean}(G) \leq r_2(G) + |G| - 2$.*

(iii) *For all $m \geq 3$, $r_{2-\ell oc}(K_m) = r_{2-mean}(K_m)$.*

Theorem 1.4 (iii) generalizes to the following theorem.

Theorem 1.5 (Schelp [21])**.** *$r_{k-\ell oc}(K_m) = r_{k-mean}(K_m)$ for all $m \geq 3$, $k \geq 2$.*

Caro and Tuza [12] mention that there are no known graphs G where $r_{k-\ell oc}(G) < r_{k-mean}(G)$ and ask whether $r_{k-\ell oc}(G) = r_{k-mean}(G)$ for all graphs G. This question is addressed for trees in [7] by Bollobás, Kostochka, and Schelp. They call a tree $T = (V, E)$ an ES-tree (Erdős-Sós tree), if every graph with average degree greater than $|V| - 2$ contains T. Many trees are known to be ES-trees.

Theorem 1.6 (Bollobás, Kostochka, Schelp [7])**.** *For every ES-tree T on d vertices and for sufficiently large k such that $k(k-1)/(d-1)$ is an integer, $r_{k-\ell oc}(T) = r_{k-mean}(T) = (d-2)k + 2$.*

Lemma 1.7. $r_3(K_3) = r_{3-mean}(K_3)$.

Proof. It is well known that $r_3(K_3) = 17 \leq r_{3-\ell oc}(K_3) = r_{3-mean}(K_3)$, the last equality following from Theorem 1.5. Thus the proof is completed by showing $r_{3-\ell oc}(K_3) \leq 17$.

Consider any 3-local coloring of K_{17}. For any fixed vertex v in the colored K_{17}, there are at least 6 incident edges colored with the same color, say color 1. Let $N_1(v)$ denote the end-vertices of those edges of color 1 incident to v so that $|N_1(v)| \geq 6$. If any pair of vertices in $N_1(v)$ are joined in color 1, K_{17} contains a monochromatic K_3 in color 1. Thus the 3-local coloring of K_{17} induces a 2-local coloring on $N_1(v)$. But by Theorem 1.2 (i) $r_{2-\ell oc}(K_3) = r_2(K_3)$. Since $|N_1(v)| \geq 6$ and $r_2(K_3) = 6$, there is a monochromatic K_3 in $N_1(v)$. ■

Observe that from the results given above Conjecture 1 holds when $k = 2$ and when $k = n = 3$. Also $r_{k-\ell oc}(K_n) = r_{k-mean}(K_n)$ with $r_{k-\ell oc}(K_n) \geq r_k(K_n)$. Thus proving Conjecture 1 amounts to showing that the last inequality is an equality.

If Conjecture 1 is true, it gives substantial insight to the classical Ramsey number. It says that if $r_k(K_n) = t$, then edge-coloring K_t by at most k colors is not what is important for it to contain monochromatic K_n. Rather K_t can be edge-colored by any number of colors as long as "on average" each vertex is incident to edges colored by at most k colors. Thus the number of colors used would not be important, only the average number of colors on edges incident to each vertex.

2. Vertex-Distinguishing Edge-Colorings

A proper edge-coloring of a graph is called *vertex-distinguishing* if every two distinct vertices are incident to different sets of colored edges. The mininum number of colors required for a vertex-distinguishing edge-coloring of a simple graph G is denoted by $\chi'_s(G)$ and called the *strong chromatic index* of G. A graph has a vertex-distinguishing edge-coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec-graph*. Vertex-distinguishing colorings have been considered in several papers (see [1, 3, 4, 5, 6, 7, 8, 9, 13, 16, 19, 20]). In [9] Burris and Schelp presented two conjectures the first of which was recently proved by Bazgan, Harket-Benhamdine, Li, and Woźniak and is given in the following theorem.

Theorem 2.1 (Bazgan, Harket-Benhamdine, Li, Woźniak [6]). *If G is a* *vdec* $-$ *graph, then* $\chi'_s(G) \leq |V(G)| + 1$.

Let $n_d = n_d(G)$ denote the number of vertices of degree d in the graph G. The other conjecture in [9] is the second conjecture of this paper.

Conjecture 2. Let G be a vdec-graph and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) = k$ or $k+1$.

Clearly if G has been given a vdec with k colors then $\binom{k}{d} \geq n_d$ for all d, so that in Conjecture 2 certainly $\chi'_s(G) \geq k$. Also there are many graphs where $\chi'_s(G) \geq k+1$. As an example consider any d-regular graph on $n = \binom{k}{d}$ vertices, where dn is even. If a vdec exists with k colors, then there must be exactly $\binom{k-1}{d-1}$ vertices incident to edges colored with any fixed color. However any edge of a given color is incident to two vertices, so that $\binom{k-1}{d-1}$ must be even. There are many pairs (k, d) where $d\binom{k}{d}$ is even and $\binom{k-1}{d-1}$ is odd (for example when k is a power of 2 and d is arbitrary). Therefore these graphs need $k+1$ colors.

Recently there has been significant progress on this conjecture. Balister, Bollobás, and Schelp [3] proved the conjecture holds when G is either a union of cycles or a union of paths. Specifically they proved the following result.

Theorem 2.2 (Balister, Bollobás, Schelp [3])**.**

(i) *Let G be a vertex-disjoint union of cycles, and let $n_2(G) = |V(G)| \leq \binom{k}{2}$, with k as small as possible. Then $\chi'_s(G) = k$ or $k+1$.*

(ii) *Let G be a vertex-disjoint union of paths with each path of length ≥ 2. Let $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$, with k as small as possible. Then $\chi'_s(G) = k$ or $k+1$.*

(iii) *Let G be any vdec-graph with $\Delta(G) = 2$. Let $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$, with k chosen as small as possible. Then $k \leq \chi'_s(G) \leq k+5$.*

This theorem, and in particular part (i) of the theorem, is a consequence of the following beautiful circuit packing theorem of Balister.

Theorem 2.3 (Balister [2])**.** *Let N be a positive integer and $\{m_i\}_{i=1}^t$ a sequence of integers, $m_i \geq 3$ for all i, such that $\binom{N}{2} = \sum_{i=1}^t m_i$ for N odd and $\binom{N}{2} - \frac{N}{2} = \sum_{i=1}^t m_i$ for N even. Then the edges of K_N for N odd (K_N minus a 1-factor for N even) can be written as an edge-disjoint union of circuits of length $m_1, m_2, \ldots, m_t$.*

It is easy to see that Theorem 2.2 (i) is almost an immediate corollary of Theorem 2.3. For let G be the vertex-disjoint union of cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$ with $\sum_{i=1}^{t} m_i = |E(G)| \leq \binom{k}{2}$, k as small as possible. If G is given a vdec by k colors, then the line graph $L(G)$ gives a collection of pairwise edge-disjoint circuits in K_k with vertices numbered $1, 2, \ldots, k$. Conversely if $C'_{m_1}, C'_{m_2}, \ldots, C'_{m_t}$ is a pairwise edge-disjoint collection of circuits in K_k, then G can be given a vdec by simultaneously transversing C'_{m_i}, and C_{m_i}, for each i, assigning the vertex number on the vertex passed on C'_{m_i} to the corresponding edge passed on C_{m_i}.

As difficult as Conjecture 2 has been for 2-regular graphs it is somewhat surprising that the conjecture has now been shown true for many graphs. The following has just recently been proved.

Theorem 2.4 (Balister, Kostochka, Li, Schelp [4]). *If G is a graph with n vertices, $\Delta(G) \geq \sqrt{2n} + 4, \delta(G) \geq 5$ and k is the smallest positive integer such that $n_d \leq \binom{k}{d}$ for all d, then $k \leq \chi'_s(G) \leq k + 1$.*

It is also possible, by use of the Lovász Local Lemma, to remove the restriction $\delta(G) \geq 5$ in the above theorem if one allows the bound on $\Delta(G)$ to become $\Delta(G) \geq C\sqrt{n}$ for some larger constant C.

The proof of Theorem 2.4 depends heavily on a useful balanced edge coloring result in [4]. Let a graph G be given a proper edge-coloring by k colors. Given a subset S of the set of k colors let $S(v)$ be the set of colors used to color the edges incident to v, let $V_S = \{v \in V(G) : S(v) = S\}$, and let $n_S = |V_S|$. Define an *optimal* k-edge-coloring with k colors to be a proper edge-coloring with minimal value of $\sum_S n_S^2$. Note that minimizing $\sum_S n_S^2$ amounts to bringing different values of n_S closer, keeping $\sum_S n_S = |V(G)|$ fixed.

Theorem 2.5 (Balister, Kostochka, Li, Schelp [4]). *In any optimal k-edge-coloring of $G, |n_S - n_{S'}| \leq 2$ for all color subsets S, S' of the color set with $|S| = |S'|$.*

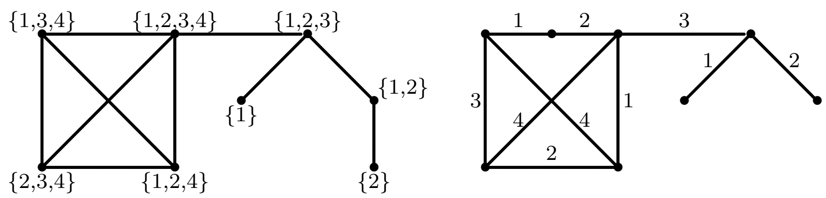
A corollary to this theorem is that any optimal k-edge-coloring with $n_d \leq \binom{k}{d} + 1$ for all $\delta(G) \leq d \leq \Delta(G)$ has $n_S \leq 2$ for all S. This follows since if $n_S > 2$ for some S, then $n_d = \sum_{|S'|=d} n_{S'} \geq n_s + \left(\binom{k}{d} - 1\right)(n_S - 2) = \binom{k}{d}(n_S - 2) + 2 \geq \binom{k}{d} + 2$, a contradiction. Thus for $k \geq \chi'(G)$ and $n_d \leq \binom{k}{d} + 1$ for all d, Theorem 2.5 can be applied so each color set is used at most twice at different vertices of G. The proof of Theorem 2.4 requires using the degree conditions of the theorem to reduce this use of any color set to at most one vertex.

In light of Theorems 2.2 and 2.4 it is expected that Conjecture 2 is true. The most difficult unresolved graphs are ones of small maximum degree. In particular does the conjecture hold for 3-regular graphs?

Very recently the following result has been proved for d-regular graphs with small components.

Theorem 2.6 (Balister, Riordan, Schelp [5]). *Let $d \geq 3$ and assume G is a d-regular graph which contains $d-2$ disjoint 1-factors. Let $G = \cup_{i=1}^{s} G_i$ where $G_1, G_2, \ldots, G_s$ are components of G. If $|V(G)| \leq \binom{k}{d}$ and $|V(G_i)| \leq \frac{3(k-1)}{3(d-1)}$ for all i, then $\chi'_s(G) \leq k+1$.*

For G a vdec-graph let $k = k(G)$ be the smallest positive integer such that $n_d \leq \binom{k}{d}$ for all $\delta(G) \leq d \leq \triangle(G)$. Recall for $v \in V(G)$, $S(v)$ denotes the set of colors used to color the edges incident to v. If a vdec with k colors exists for G, then each color meets an even number of vertices. Hence each color occurs in an even number of sets $S(v)$. Thus the symmetric difference of the sets $\bigoplus_{v \in V(G)} S(v) = \varnothing$. Let $k'(G)$ be the minimum k such that there exist distinct sets $S_v \subseteq \{1, 2, \cdots, k\}$ for all $v \in V(G)$ with $|S_v| = \deg v$ and $\bigoplus_{v \in V(G)} S_v = \varnothing$. Then it is clear that $\chi'_s(G) \geq k'(G) \geq k(G)$. There are many examples where $k'(G) > k(G)$, e.g., $G = K_4$, but there are no known regular graphs where $\chi'_s(G) > k'(G)$. The situation for non-regular graphs is different. Indeed, it is possible for two non-regular graphs with the same degree sequence to have different strong chromatic indices. An example is shown in the figure below.



G_1: $\chi'_s > k' = 4$ $\quad$ G_2: $\chi'_s = k' = 4$

This example motivates the following definition. Let $k''(G)$ be the smallest k such that for any set of vertices $\mathrm{X} \subseteq V(G)$ there exist distinct sets $S_v \subseteq \{1, 2, \cdots, k\}$, $v \in \mathrm{X}$, such that $|S_\nu| = \deg\ (v)$ and $|\bigoplus_{v \in \mathrm{X}} S_v| \leq |E(\mathrm{X}, \mathrm{X}^c)|$, where $E(\mathrm{X}, \mathrm{X}^c)$ is the set of edges between X and $\mathrm{X}^c = V(G) \backslash \mathrm{X}$.

If $S_v = S(v)$ for a vdec then $|\bigoplus_{v\in \mathrm{X}} S_v| \leq |E(\mathrm{X}, \mathrm{X}^c)|$ for all X, so that $\chi'_s(G) \geq k''(G)$. Also, if $\mathrm{X} = V(G)$ then $k''(G) \geq k'(G)$. Therefore, $\chi'_s(G) \geq k''(G) \geq k'(G) \geq k(G)$. Also, in [5] it is shown that $k''(G) \leq k(G) + 1$. Therefore, one has the following conjecture.

Conjecture (Balister, Riordan, Schelp [5])**.** For all vdec-graphs G, $\chi'_s(G) = k''(G)$.

Observe that this conjecture is consistent with the inequalities mentioned above. Also, by an exhaustive computer search this conjecture has been shown to hold for all vdec-graphs of order at most 11 and all regular graphs of order at most 20.

Note that this conjecture says that parity is the only reason that can make $\chi'_s(G) = k(G) + 1$ instead of $\chi'_s(G) = k(G)$. Also, clearly this conjecture implies the truth of Conjecture 2.

Finally, it is not even known whether there exists an absolute constant c such that $\chi'_s(G) \leq k(G) + c$ when G is a vdec-graph.

3. Local Edge-Colorings that are Global

M. Truszczyński [22] generalized the k-local colorings of Section 1 to what he called local (H, k)-colorings and considered a corresponding Ramsey number. Given a positive integer k and a graph H with at least $k + 1$ edges, a *local* (H, k)*-coloring* of the complete graph K_n is a coloring of its edges such that each of its subgraphs isomorphic to H has at most k edges colored differently. This leads to the following natural question considered by Clapsadle and Schelp in [15]. Does there exist necessary and sufficient conditions on H such that for $n \geq n_0$ (n_0 fixed) each local (H, k)-coloring of K_n is in fact a k-edge-coloring?

It is easy to see that a necessary condition is for H to contain all k-edge graphs as subgraphs. Suppose this is not the case and that H_0 is a k-edge graph which is not a subgraph of H. Select an isomorphic copy of H_0 in K_n. Color each edge of this copy of H_0 in K_n with a different color, say colors $1, 2, \ldots, k$. Color all remaining edges of K_n with a $(k + 1)$-st color. Clearly every isomorphic copy of H in K_n has at most k edges colored differently, since each such copy fails to contain at least one edge of the fixed copy of H_0 colored with colors $1, 2, \ldots, k$. Thus the described coloring of K_n is a local (H, k)-coloring with $k + 1$ colors. Hence the supposition is false and it is necessary for H to contain all k-edge graphs as subgraphs, for each local (H, k)-coloring of K_n to be a k-edge-coloring.

The sufficiency of the above condition is the unanswered portion of the third conjecture of this article.

Conjecture 3 (Clapsadle and Schelp [15]). Let k be a positive integer and H a graph with at least $k+1$ edges. A necessary and sufficient condition that each local (H, k)-coloring of K_n, $n \geq n_0$ (n_0 fixed), is a k-edge-coloring is that H contain each k-edge graph as a subgraph.

This conjecture has been established when $k = 2, 3, 4$ and when H is the graph formed by attaching a pendant edge to each vertex of K_k [15]. This graph H is of smallest order containing all k-edge graphs as subgraphs.
Clapsadle [14] has considered problems relating to the conjecture, but nothing more has been accomplished toward a complete solution. Since the conjecture holds for $k \leq 4$, it is likely to be true. If true, it would give a local edge-coloring criteria of K_n with a global consequence that K_n is k-edge-colored.

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Received 11 July 2000
Revised 3 September 2001