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EDGE COLORINGS AND TOTAL COLORINGS OF INTEGER DISTANCE GRAPHS

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Dedicated to Hansjoachim Walther on the occasion of his sixtieth birthday.

Abstract

An *integer distance graph* is a graph $G(D)$ with the set $\mathbb{Z}$ of integers as vertex set and two vertices $u, v \in \mathbb{Z}$ are adjacent if and only if $|u-v| \in D$ where the *distance set* D is a subset of the positive integers $\mathbb{N}$. In this note we determine the chromatic index, the choice index, the total chromatic number and the total choice number of all integer distance graphs, and the choice number of special integer distance graphs.

Keywords: integer distance graph, chromatic number, choice number, chromatic index, choice index, total chromatic number, total choice number.

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1. Introduction

If D is a subset of the positive integers, then the *integer distance graph* $G(\mathbb{Z}, D) = G(D)$ is defined as the graph with vertex set $V(G(D)) = \mathbb{Z}$, the set of integers, and two vertices u and v are adjacent if and only if their distance $|u-v|$ is an element of the so-called *distance set* D.

A (*vertex*) *coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are colored differently. The minimum number of colors necessary to color the vertices of G is the *chromatic number* $\chi(G)$ of G.

If $L = \{L(v) : v \in V(G)\}$ is a set of lists of colors then an *L-list* (*vertex*) *coloring* of a graph G is a coloring of the vertices of G such that each vertex obtains a color from its own list and adjacent vertices are colored differently. A graph G is called *k-choosable* if such a coloring exists for each choice of lists $L(v)$ of cardinality at least k. The minimum k such that G is k-choosable is the *choice number* $ch(G)$ of G.

For a distance set $D = \{d_1, d_2, \dots\} \subseteq \mathbb{N}$ we write $G(D) = G(d_1, d_2, \dots)$ and $\chi(G(D)) = \chi(D) = \chi(d_1, d_2, \dots)$, $ch(G(D)) = ch(D) = ch(d_1, d_2, \dots)$.

Integer distance graphs were introduced by Eggleton, Erdős, and Skilton [6]. There are a lot of papers in which the chromatic numbers of special integer distance graphs are determined (see, e.g., [4, 6, 7, 17, 22, 23, 24, 25]). We mention some of these results.

For example, $\chi(\mathbb{P}) = 4$ is proved in [6] where $\mathbb{P}$ is the set of primes. If D contains at most three elements, $\chi(D)$ is completely determined.

Obviously, $\chi(D) = 2$ for 1-element distance sets. If D contains only odd integers then $\chi(D) = 2$ (color all vertices alternately with two colors). Since $|D| + 1$ is a trivial upper bound for $\chi(D)$ if D is finite (see [4, 24]), we have for 2-element distance sets $\chi(D) = 2$ if D contains two odd elements and $\chi(D) = 3$ if D consists of two coprime elements of distinct parity.

If the greatest common divisor of D is 1, which is no loss of generality (see below), and $|D| = 3$ then $\chi(D) = 4$ if and only if $D = \{1, 2, 3n\}$ or $D = \{x, y, x + y\}$ and $x \not\equiv y \pmod 3$. For all other 3-element distance sets D it holds $\chi(D) \leq 3$ [22, 25].

If the greatest common divisor of D is 1 and $|D| = 4$ then $\chi(1, 2, 3, 4n)$ $= 5$ and $\chi(x, y, x + y, |y - x|) = 5$ if and only if x and y both are odd [18]. Possibly, these are all of the 4-element distance sets such that the chromatic number of the corresponding integer distance graph is 5.

As far as we know there are no specific results about the choice number of integer distance graphs and no published results about edge colorings and total colorings.

An *edge coloring* of a graph G is an assignment of colors to the edges of G in such a way that adjacent edges are colored differently. A *total coloring* is an assignment of colors to the vertices and edges such that adjacent vertices, adjacent edges as well as incident vertices and edges are colored differently, respectively. The minimum number of colors necessary to color the edges of G is the *chromatic index* $\chi'(G)$ and to color the vertices and edges the *total chromatic number* $\chi''(G)$.

Again, if the colors belong to specific lists assigned to the edges or to vertices and edges of G, respectively, and the cardinality of the lists is at least k then G is called *k-edge choosable* or *k-total choosable*, respectively, if such colorings exist for each choice of lists. The minimum k such that G is k-edge choosable is the *choice index* $ch'(G)$ of G, and the minimum k such that G is k-total choosable is the *total choice number* $ch''(G)$.

Obviously, for considerations of coloring properties of graphs one can restrict oneself without loss of generality to connected graphs. If d is an arbitrary divisor of the elements $d_1, d_2, \ldots$ of distance set D then the integer distance graph $G(D) = G(d_1, d_2, \ldots)$ is isomorphic to d disjoint copies of $G(\frac{d_1}{d}, \frac{d_2}{d}, \ldots)$. These copies are induced by the residue classes modulo d. Therefore, we will restrict ourselves throughout this paper to integer distance graphs whose distance set D has greatest common divisor 1, i.e., $G(D)$ is connected.

In this note we prove in Section 2 that $|D| + 1$ is an upper bound for the choice number $ch(D)$ of integer distance graphs $G(D)$. Moreover, we determine $ch(D)$ for distance sets of small cardinality, namely $D = \{x, y\}$ and $D = \{x, y, x+y\}$.

In Section 3 we determine the chromatic index and the choice index and in Section 4 the total chromatic number and the total choice number for an integer distance graphs $G(D)$.

2. Choice Number

It is proved in [24] that $|D|+1$ is an upper bound for the chromatic number $\chi(D)$ of an integer distance graph $G(D)$. The proof can be transferred to list colorings.

Theorem 1. *If $G(D)$ is an integer distance graph, then $ch(D) \leq |D| + 1$.*

Proof. Let $L = \{L(v) : v \in \mathbb{Z}\}$ be any set of lists of colors such that $L(v)$ is assigned to vertex v and all lists have length at least $|D| + 1$.

Color the vertices $0, 1, -1, 2, -2, \ldots$ successively such that vertex v obtains a color of its list $L(v)$ which is not used to color the previous colored neighbors of v. Such a color exists since the order of coloring implies that at most half of the neighbors, i.e., at most $|D|$, are previously colored. ■

Since $\chi(G) \leq ch(G)$ for all graphs G we obtain as a corollary of Theorem 1 that $ch(D) = |D| + 1$ for the integer distance graphs with $\chi(D) = |D| + 1$. Examples of such graphs are connected integer distance graphs $G(D)$ with distance set D with $|D| \leq 1$, or $D = \{x, y\}$, $x \not\equiv y \pmod 2$, or $D = \{x, y, x + y\}$, $x \not\equiv y \pmod 3$, or $D = \{x, y, x + y, |y - x|\}$, $x \equiv y \equiv 1 \pmod 2$, or $D = \{1, 2, \ldots, k - 1, nk\}$, $n, k \in \mathbb{N}$, $k \geq 2$.

For integer distance graphs $G(D)$ — as it is for graphs G — the difference between $ch(D)$ and $\chi(D)$ can be arbitrarily large.

For example, if $D_n = \{1, 3, 5, \ldots, 2n - 1\}$ then $\chi(D_n) = 2$, since $G(D_n)$ is bipartite. The subgraph of $G(D_n)$ induced by $2n$ consecutive vertices is isomorphic to the complete bipartite graph $K_{n,n}$. It is proved in [8] that $ch(K_{n,n}) = \log_2 n + o(\log n)$. Since $ch(D_n) \geq ch(K_{n,n})$, the difference $ch(D_n) - \chi(D_n)$ tends to infinity with n.

For distance sets of small cardinality the choice number and the chromatic number of the corresponding integer distance graph may differ.

Obviously, $ch(D) = 1$ if $D = \emptyset$ and $ch(D) = 2$ if $|D| = 1$. If $|D| = 2$, then $\chi(D) = 2$ or 3 (see above) but $ch(D) = 3$ for all distance sets $D = \{x, y\}$.

Theorem 2. *If $x \neq y$, then $ch(x, y) = 3$.*

Proof. Let $D = \{x, y\}$, $x < y$. $G(D)$ contains $P_2 \times P_3$ as subgraph (for example $V_{P_2 \times P_3} = \{-y, x - y, 0, x, y, x + y\}$, see Figure 1) which is not 2-choosable: Assume f is an L-list coloring of $G(D)$ with $L(0) = L(x) = \{a, b\}$, $L(x - y) = L(y) = \{b, c\}$, and $L(-y) = L(x + y) = \{a, c\}$.

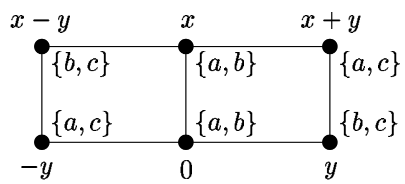

Figure 1. $P_2 \times P_3$ as subgraph of $G(x, y)$

If $f(0) = a$ then $f(-y) = c$, $f(x - y) = b$ but x cannot be colored by a color of its list. On the other hand, if $f(0) = b$ then $f(y) = c$, $f(x + y) = a$ and again x cannot be colored. Therefore, $ch(D) \geq ch(P_2 \times P_3) \geq 3$, and $ch(D) \leq |D| + 1 = 3$ by Theorem 1 which proves the assertion. ∎

Theorem 2 implies that $ch(D) \geq 3$ whenever $|D| \geq 2$.

The chromatic number $\chi(D)$ with $D = \{x, y, x+y\}$ is 3 or 4 as mentioned in the Introduction. We show in the next theorem that, however, the choice number $ch(D)$ is always 4.

Theorem 3. *If $x \neq y$, then $ch(x, y, x+y) = 4$.*

Proof. Let $D = \{x, y, x+y\}$, $x \neq y$, and assume that $ch(D) = 3$. Consider the subgraph of $G(D)$ (see Figure 2) which is induced by the vertex set $V = \{0, x+y, \ldots, 7(x+y), x, 2x+y, \ldots, 7x+6y, y, x+2y, \ldots, 6x+7y\}$.

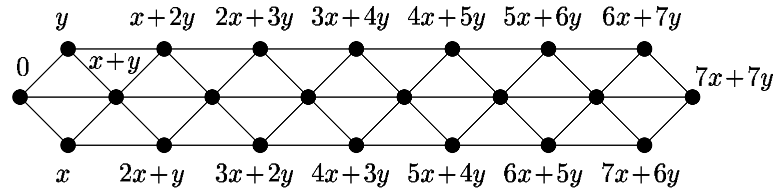

Figure 2. A subgraph of $G(x, y, x+y)$

Let L be a list of colors such that $L(i(x+y)) = L(j(x+y)+x) = \{a, b, c\}$ for $i = 0, \ldots, 7$, $j = 0, \ldots, 6$ and $L(k(x+y)+y) = \{a, b, d\}$ for $k = 0, 2, 4, 6$ and $\{b, c, d\}$ for $k = 1, 3, 5$.

Assume f is an L-list coloring of $G(D)$. The choice of the lists implies that $f(0) = f(3(x+y)) = f(6(x+y))$, $f(x+y) = f(4(x+y)) = f(7(x+y))$ and $f(2(x+y)) = f(5(x+y))$.

If $f(0) = a$, $f(x+y) = b$, $f(2(x+y)) = c$, then $f(y) = d$ and $x+2y$ can not be colored with a color from its list $\{b, c, d\}$, which is a contradiction to the assumption that f is an L-list coloring of $G(D)$. Any other permutation of the colors a, b, c for the vertices $0, x+y, 2(x+y)$ forces $f(k(x+y)+y) = f((k+1)(x+y)+y) = d$ for some k, $1 \leq k \leq 5$, which is a contradiction to the assumption, since $k(x+y)+y$ and $(k+1)(x+y)+y$ are neighbors. ∎

3. CHROMATIC INDEX AND CHOICE INDEX

Vizing [21] proved that the chromatic index $\chi'(G)$ of any simple graph G attains one of two values: $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of G. We show that all integer distance graphs $G(D)$ are class 1, i.e., $\chi'(G(D)) = \Delta(G(D)) = 2|D|$.

Theorem 4. *If $G(D)$ is an integer distance graph, then $\chi'(D) = 2\,|D|$.*

Proof. Each distance element $d \in D$ creates a subgraph $G(d)$ of $G(D)$ which consists of d disjoint infinite paths. The edges of these d paths can be colored with two colors. This implies that all edges of $G(D)$ can be colored with $2\,|D| = \Delta(G(D))$ colors. ■

It is conjectured that $\chi'(G) = ch'(G)$ for all graphs G (see, e.g., [2, 14]). This so-called *list edge coloring conjecture* is proved, for instance, for graphs G with $\Delta(G) = 2$ or with $\Delta(G) = 3$ and $\chi'(G) = 4$ [12], for bipartite graphs [9], for complete graphs of odd order [11], for planar graphs G with $\Delta(G) \geq 12$ [3], and for outerplanar graphs [10, 15]. We prove that this conjecture is also true for integer distance graphs.

Theorem 5. *If $G(D)$ is an integer distance graph, then $ch'(D) = \chi'(D) = 2\,|D|$.*

Proof. Let $D = \{d_1, d_2, d_3, \ldots, d_{|D|}\}$, $d_1 < d_2 < d_3 < \cdots < d_{|D|}$ be nonempty and finite (otherwise the statement is trivial). A list of $2\,|D|$ colors is assigned to each edge of $G(D)$. We color the edges of $G(D)$ successively such that each edge has at most $2\,|D| - 1$ previously colored adjacent edges, i.e., each edge can be colored with a color of its list.

We denote the edge from v to $v + d_i$ by $\{v, v + d_i\}$ and we start by coloring all edges incident with vertex 0. Since the number of these edges is $2\,|D|$ each edge can be colored with a color from its list. We proceed by coloring successively all edges incident with $1, -1, 2, -2, \ldots, k, -k$ and show that also the edges incident with $k + 1$ and $-k - 1$ can be colored.

First, we color those edges $\{k + 1, k + 1 - d_i\}$, $i = 1, \ldots, |D|$, which are uncolored so far step by step in an arbitrary order. These edges can be colored with a color from their list, since they are adjacent to at most $2\,|D| - 2$ previously colored edges, namely to at most $|D| - 1$ edges incident with $k + 1$ and at most $|D| - 1$ edges incident with $k + 1 - d_i$.

Next we color the edges $\{k + 1, k + 1 + d_i\}$ in the order $i = 1$, $i = 2$, and so on, up to $i = |D|$. The edge $\{k + 1, k + 1 + d_i\}$ is adjacent to at most $2\,|D| - 1$ previously colored edges, namely to $|D| + i - 1$ edges incident with vertex $k + 1$ and to at most $|D| - i$ edges which connect vertex $k + 1 + d_i$ with vertices $k + 1 + d_i - d_j$, $j = i + 1, \ldots, |D|$ (see Figure 3). Therefore, we can color the edges $\{k + 1, k + 1 + d_i\}$ with some color of their respective lists.

Next we color the edges incident with $-k - 1$ step by step. If we choose the order $\{-k - 1, -k - 1 + d_1\}$, $\{-k - 1, -k - 1 + d_2\}, \ldots,$

$\{-k-1, -k-1+d_{|D|}\}$, $\{-k-1, -k-1-d_1\}, \ldots, \{-k-1, -k-1-d_{|D|}\}$ for coloring the edges, then the same argument shows that there are for each of these edges at most $2\,|D|-1$ previously colored adjacent edges. Hence, these edges can also be colored with a color from their list.

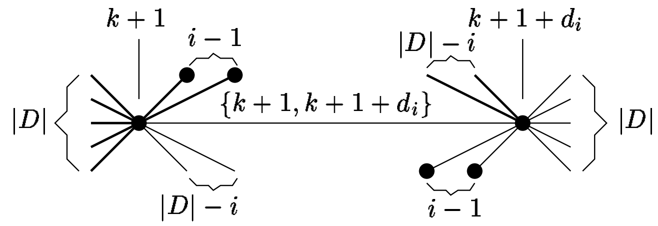

Figure 3. The edge $\{k+1, k+1+d_i\}$ and adjacent edges

Therefore, $ch'(D) \le 2\,|D|$. Since $ch'(G) \ge \chi'(G)$ for all graphs G, $ch'(D) = \chi'(D) = 2\,|D|$. ■

Obviously, the list edge coloring conjecture also holds for all subgraphs of integer distance graphs $G(D)$ whose maximum degree coincides with $\Delta(G(D))$.

4. TOTAL CHROMATIC NUMBER AND TOTAL CHOICE NUMBER

For total colorings it is conjectured that $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2$ (*total coloring conjecture*, see [1, 21]) and $ch''(G) = \chi''(G)$ for all graphs G (*list total coloring conjecture*, see [3]). The total coloring conjecture is proved, e.g., for complete graphs, for bipartite graphs, for graphs G with maximum degree $\Delta(G) \ge \frac{3}{4}|V(G)|$ [13] or $\Delta(G) \le 5$ [19], and for planar graphs G with $\Delta(G) \ne 6$ [20]. The list total coloring conjecture is verified, for example, for graphs G with $\Delta(G) \le 2$ [16], for planar graphs G with $\Delta(G) \ge 12$ [3], and for outerplanar graphs [15].

We prove the total coloring conjecture as well as the list total coloring conjecture for integer distance graphs using methods analogous to those of Theorem 5.

Theorem 6. *If $G(D)$ is an integer distance graph, then $ch''(D) = \chi''(D) = \Delta(G(D)) + 1 = 2\,|D| + 1$.*

Proof. Let $D = \{d_1, d_2, \ldots, d_{|D|}\}$ be nonempty and finite (if not, the assertion is obvious) and let L be an assignment of color lists of length $2|D| + 1$ to the vertices and edges of $G(D)$.

In a first step we color the vertex 0 and all $2|D|$ incident edges. Then we color successively vertices v and all edges incident with v in the order $v = 1, -1, 2, -2, \ldots, k, -k$.

Vertex $k+1$ is incident with at most $|D|$ edges and adjacent to at most $|D|$ respective end vertices which are previously colored. Therefore, vertex $k+1$ can be colored with a color from its list.

The so far uncolored edges $\{k+1, k+1-d_i\}$, $i = 1, 2, \ldots, |D|$ are adjacent to at most $2|D| - 2$ previously colored edges and hence can be colored step by step.

Each edge $\{k+1, k+1+d_i\}$, $i = 1, \ldots, |D|$ is adjacent to at most $2|D| - 1$ previously colored edges (see proof of Theorem 5) such that it can be colored with a color from its list, since vertex $k+1+d_i$ is so far uncolored.

Next we color vertex $-k-1$ and proceed by coloring all incident edges similar to that done in the proof of Theorem 5.

Therefore, $ch''(D) \leq 2|D|+1$. Since $ch''(D) \geq \chi''(D) \geq \Delta(G(D))+1 = 2|D| + 1$, the theorem is proved. ■

The total coloring conjecture and the list total coloring conjecture also hold for all subgraphs H of integer distance graphs $G(D)$ with $\Delta(H) = \Delta(G(D))$.

5. CONCLUDING REMARKS

More generally, one can define distance graphs instead of integer distance graphs (see [5, 6]). If S is a subset of the n-dimensional Euclidean space, $S \subseteq \mathbb{R}^n$, then the *distance graph* $G(S, D)$ has vertex set $V(G(S, D)) = S$, and two vertices are adjacent if and only if their Euclidean distance is an element of the distance set D which is a subset of the set of positive real numbers, $D \subseteq \mathbb{R}_+$.

Eggleton, Erdős, and Skilton [6] considered distance graphs on the real line, i.e., $n = 1$. They investigated, among others, the chromatic number $\chi(G(\mathbb{R}, D))$ for various intervals and unions of intervals D. If $n > 1$ and $D \neq \emptyset$, then obviously $ch(G(\mathbb{R}^n, D))$, $\chi'(G(\mathbb{R}^n, D))$, $ch'(G(\mathbb{R}^n, D))$, $\chi''(G(\mathbb{R}^n, D))$ as well as $ch''(G(\mathbb{R}^n, D))$ are infinite.

Some of the proofs of this paper can be transferred to distance graphs $G(\mathbb{R}, D)$, e.g., for the chromatic index it holds that $\chi'(G(\mathbb{R}, D)) = 2|D|$. Is it also possible to generalize the results for the other chromatic numbers?

As mentioned in the Introduction the chromatic numbers of all integer distance graphs with 3-element distance sets are completely determined. So far, there are only partial results known for the choice number of such graphs (see Section 2).

All the integer distance graphs $G(D)$ whose choice number was determined in Section 2 had the property that their clique number $\omega(G(D))$ is large with respect to the cardinality of D. It turned out for those graphs that $ch(D) = |D|+1$ whenever $\omega(G(D)) = |D|$. Is this true in general? For example, does $ch(x, y, x+y, |y-x|) = 5$ always hold? If x and y both are odd then this is true.

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