DECOMPOSITIONS OF MULTIGRAPHS INTO PARTS WITH TWO EDGES

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Abstract

Given a family $F$ of multigraphs without isolated vertices, a multigraph $M$ is called $F$-decomposable if $M$ is an edge disjoint union of multigraphs each of which is isomorphic to a member of $F$. We present necessary and sufficient conditions for the existence of such decompositions if $F$ comprises two multigraphs from the set consisting of a 2-cycle, a 2-matching and a path with two edges.

Keywords: edge decomposition, multigraph, line graph, 1-factor.

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1. Introduction

All multigraphs considered in what follows are loopless. Given a family $F$ of multigraphs without isolated vertices, an $F$-decomposition of a multigraph $M$ is a collection of submultigraphs which partition the edge set $E(M)$ of $M$...
and are all isomorphic to members of $\mathcal{F}$. If such a decomposition exists, $M$ is called $\mathcal{F}$-decomposable; and also $H$-decomposable if $H$ is the only member of $\mathcal{F}$. Let $\mathcal{F} = \{H_1, H_2, \ldots, H_t\}$. By an $H_i$-edge in an $\mathcal{F}$-decomposition of $M$ we mean an edge belonging to any decomposition part isomorphic to $H_i$ for some $i = 1, 2, \ldots, t$.

If $M$ is a multigraph, we write $M = (V, E)$ where $V = V(M)$ and $E = E(M)$ stand for the vertex set and edge set of $M$, respectively. Cardinalities of those sets, denoted by $v(M)$ and $e(M)$, are called the order and size of $M$, respectively. For $S \subset V(M)$, $M[S]$ denotes the submultigraph of $M$ induced by $S$. The number of edges incident to a vertex $x$ in $M$, denoted by $\text{val}_M(x)$, is called the valency of $x$, whilst the number of neighbours of $x$ in $M$, denoted by $\text{deg}_M(x)$, is called the degree of $x$. As usual $\Delta(M)$ stands for the maximum valency among vertices of $M$. For any two vertices $x, y$ of $M$, let $p_M(x, y)$ denote the number of edges joining $x$ and $y$. We call $p_M(x, y)$ the multiplicity of an edge $xy$ in $M$. Edges joining the same vertices are called parallel edges (if they are distinct).

The aim of our paper is to provide necessary and sufficient conditions for a multigraph $M$ to be $\{H_1, H_2\}$-decomposable, where $H_1, H_2$ are any two multigraphs out of $C_2$ (2-cycle), $P_3$ (path with two edges), and $2K_2$ (2-matching). Obviously, if $M$ is $H_i$-decomposable for some $i = 1, 2$, then $M$ is $\{H_1, H_2\}$-decomposable. Therefore the following known results are quoted.

**Theorem 1** (Skupień [7], see [4] for a proof). A multigraph $M$ is $2K_2$-decomposable iff its size $e(M)$ is even, $\Delta(M) \leq \frac{e(M)}{2}$ and $e(M[\{x, y, z\}]) \leq \frac{e(M)}{2}$ for all $\{x, y, z\} \subset V(M)$.

If $M$ is a simple graph then the very last condition in Theorem 1 means that $M \neq K_3 \cup K_2$, cf. Caro [2].

**Proposition 2.** A multigraph $M$ is $C_2$-decomposable iff $p_M(x, y) \equiv 0 \pmod{2}$ for all $x, y \in V(M)$.

**Theorem 3** [5, 3]. A simple graph $G$ is $P_3$-decomposable iff each component of $G$ is of even size.

**Corollary 4.** A graph $G$ is $\{P_3, 2K_2\}$-decomposable iff the size $e(G)$ of $G$ is even.
Given a multigraph $M$, define the $\ast$-line graph of $M$, denoted by $L^\ast(M)$, to be the graph with vertex set $V(L^\ast(M)) = E(M)$ and edge set $E(L^\ast(M)) = \{w_1w_2 : w_1, w_2 \in E(M), |w_1 \cap w_2| = 1\}$. Evidently, $L^\ast(M)$ is obtainable from the ordinary line graph $L(M)$ by removal of all edges which represent multiple adjacency of edges in the root multigraph $M$. In other words, the operator $L^\ast$ represents doubly adjacent edges in $M$ as if they were nonadjacent in $M$.

**Theorem 5** [4]. Given a multigraph $M$, the following statements are equivalent.

(i) $M$ is $P_3$-decomposable.

(ii) $L^\ast(M)$ has a 1-factor.

Therefore checking whether a multigraph $M$ is $P_3$-decomposable can be done in polynomial time $O(e(M)^{2.5})$, cf [4]. Some original sufficient conditions for $M$ to be $P_3$-decomposable may be found in [1, 4].

### 2. $\{C_2, P_3\}$-Decomposition

**Theorem 6.** Let $M$ be a multigraph and let $L(M)$ be the line graph of $M$. The following statements are equivalent.

(i) $M$ is $\{C_2, P_3\}$-decomposable.

(ii) Each component of $M$ has an even number of edges.

(iii) Each component of $L(M)$ has an even number of vertices.

(iv) $L(M)$ has a 1-factor.

**Proof.** Each of the implications in the cycle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) is obvious or well-known. Well-known is the implication (iii) $\Rightarrow$ (iv) following from the result of Sumner [8] and Las Vergnas [6] which says that every connected claw-free graph of even order has a 1-factor.

### 3. $\{P_3, 2K_2\}$-Decomposition

**Theorem 7.** Let $M$ be a multigraph. Let $L^\ast(M)$ and $\overline{L(M)}$ be the $\ast$-line graph and the complement of the line graph $L(M)$ of $M$, respectively. The following statements are mutually equivalent.
(i) $M$ is $\{P_3, 2K_2\}$-decomposable.

(ii) $M$ has an even number, $e(M)$, of edges and the multiplicity of any edge does not exceed $e(M)/2$.

(iii) The graph $\tilde{L} := L^*(M) \cup \overline{L(M)}$ has a 1-factor.

**Proof.** Implication (i) $\Rightarrow$ (ii) is true because $e(M)/2$ is the number of parts and parallel edges must be in different parts of a decomposition. Implication (ii) $\Rightarrow$ (iii) is true because the order $v(\tilde{L}) = e(M)$ is even and the minimum degree $\delta(\tilde{L}) \geq \frac{1}{2}v(\tilde{L})$, whence, by Dirac’s theorem, the graph $\tilde{L}$ has a Hamiltonian cycle. Implication (iii) $\Rightarrow$ (i) is obvious.

4. $\{C_2, 2K_2\}$-Decomposition

Given a multigraph $M$, let $G(M)$ denote the graph induced by the edge set $E(G(M)) := \{xy : p_M(x, y) \equiv 1 \pmod{2}\}$. Evidently, a graph isomorphic to $G(M)$ is obtainable from $M$ both by removing all edges of the maximal family of pairwise edge-disjoint copies of $C_2$ and by removing all resulting isolated vertices. Thus $2K_2$-edges in any $\{C_2, 2K_2\}$-decomposition of $M$ induce a multigraph $M'$ containing a subgraph isomorphic to $G(M)$ (in fact, $p_M'(x, y) \geq 1$ whenever $xy \in E(G(M))$).

If $E' \subset E(M)$, $f \in E(M)$, and $w \in V(M)$ then $M - E'$ (or $M - f$) is the spanning submultigraph of $M$ obtained by removing the edges only ($E'$ or $f$), while $M - w$ is obtained from $M$ by removing the vertex $w$ together with all edges incident to $w$.

![Figure 1. Eight families of multigraphs $M$.](image)

<table>
<thead>
<tr>
<th>edge</th>
<th>heavy</th>
<th>thin</th>
<th>doubled</th>
<th>dotted</th>
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<tbody>
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<td>even $\geq 2$</td>
<td>even $\geq 0$</td>
</tr>
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</table>
Theorem 8. Let $M$ be a multigraph and let $\overline{L^*(M)}$ be the complement of the $*$-line graph $L^*(M)$ of $M$. The following three statements are mutually equivalent.

(i) $M$ is $\{C_2, 2K_2\}$-decomposable.
(ii) $\overline{L^*(M)}$ has a 1-factor.
(iii) Each of the following five conditions holds:
   (0) $e(M)$ is even,
   (1) $\text{val}_M(x) + \deg_{G(M)}(x) \leq e(M)$ for every $x \in V(M)$,
   (2) if $xy \in E(G(M))$ then $\text{val}_M(x) + \text{val}_M(y) - p_M(x, y) < e(M)$,
   (3) if $yx, xz \in E(G(M))$ then $1 + \text{val}_M(x) + p_M(y, z) < e(M)$,
   (4) $M$ is different from each of the (forbidden) multigraphs shown in Figure 1.

A vertex $y$ is called an odd neighbour of a vertex $x$ if $M$ has an edge $xy$ whose multiplicity $p_M(x, y)$ is odd.

Proposition 9. The following condition $(i')$ is an equivalent of $(i)$ above for $i = 1, 2, 3$.

$(1')$ The number of odd neighbours of any vertex $x$ does not exceed the number of all edges nonincident to $x$;
$(2')$ There is no edge $xy$ adjacent to every other edge and with odd multiplicity $p_M(x, y)$;
$(3')$ There are no two adjacent edges $yx, xz$ both with odd multiplicities and such that among the remaining edges at most one is not a neighbour of both $yx$ and $xz$.

Proposition 10. Each multigraph depicted in Figure 1 satisfies all conditions $(0)$–$(3)$ and is not $\{C_2, 2K_2\}$-decomposable.

The following converse result is of importance.

Lemma 11. Every multigraph $M$ which satisfies conditions $(0)$–$(3)$, has $e(G(M)) \leq 4$, and is not $\{C_2, 2K_2\}$-decomposable is depicted in Figure 1.
Proof. Suppose that $M$ is a counterexample. Since $M$ is not $C_2$-decomposable, $e(G(M)) > 0$. Due to (0), $G(M)$ has two or four edges. Consider two main cases A and B.

A. $e(G(M)) = 4$. As $G(M)$ is not $2K_2$-decomposable, either $G(M)$ contains a triangle or otherwise $\Delta(G(M)) \geq 3$. Consider the following subcases.

A1. $\Delta(G(M)) = 4$. Then $G(M)$ is a star with a central vertex $w$ and $M - w$ is $C_2$-decomposable. Moreover, $e(M - w) \geq 4$ by (1). Since $M$ satisfies (2), not all edges of $M - w$ are incident to one vertex of $G(M)$. On the other hand, each edge of $M - w$ has both endvertices in $G(M)$ as well as there is no $2K_2$ in $M - w$ because otherwise $G(M)$ together with any two pairs of parallel edges of $M - w$ which do not intersect at $G(M)$ is $2K_2$-decomposable. Consequently, edges of $M - w$ induce a “multiple triangle” on three hanging vertices of $G(M)$. Therefore no parallel edges can join $w$ to a vertex off the “triangle”. Hence $M$ appears in Figure 1, a contradiction.

A2. $\Delta(G(M)) = 3$ and $G(M)$ contains no triangle. Let $w$ be the degree-3 central vertex of the star of $G(M)$, let $f$ and $wx_i$ with $i = 1, 2, 3$ be the four edges of $G(M)$ with notation such that the edge $f$ is incident to $x_3$ if $G(M)$ is connected. Then $e(M - w) > 2$ by (1). It is easily seen that each pair of parallel edges of $M - w$ has a vertex in $\{x_1, x_2, x_3\}$. Hence the multiplicity of $f$ is one if $f$ is not incident to $x_3$. The multiplicity of $f$ is one, too, otherwise. Namely, by (2), $M$ has a pair of parallel edges which are nonadjacent to the edge $wx_3$ of $G(M)$. These are $x_1 - x_2$ edges because otherwise the pair together with $G(M)$ is $2K_2$-decomposable (the edge $f$ being matched with $wx_i$ if $x_i$ is an endvertex of the pair, $i \neq 3$). Now, clearly, the multiplicity of $f$ is one. Consequently, by (3), each vertex $x_i$ is incident to parallel edges of $M - w$; moreover, one can see that all parallel edges of $M - w$ are of the form $x_ix_j$ only. Similarly, $\deg_M(w) = 3$ only, whence $M$ appears in Figure 1, a contradiction.

A3. $G(M)$ contains a triangle. Let the vertices of the triangle be denoted by $x_i, i = 1, 2, 3$. Let $f$ stand for the remaining edge of $G(M)$. Then each pair of parallel edges are incident to some $x_i$ because otherwise the pair and $G(M)$ make up a $2K_2$-decomposable submultigraph. Assume that the edge $f$ has no vertex in the triangle of $G(M)$. Hence the multiplicity of $f$ is one. Moreover, by (3), $M$ has two pairs of parallel edges of the form $x_iz$ and $x_j\tilde{z}$ where $x_i, x_j$ are distinct vertices of the triangle of $G(M)$ and $z, \tilde{z}$ are both off the triangle. Then $\tilde{z} = z$ because otherwise the two pairs
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and $G(M)$ would be $2K_2$-decomposable. Moreover, $f$ is either incident to $z$ or not; and in either case $M$ appears in Figure 1, a contradiction.

Assume that $f$ is incident to a vertex, say $x_1$, in the triangle of $G(M)$. Then, by (2), $M$ has parallel edges of the form $x_2z$ and $x_3\tilde{z}$ where $z, \tilde{z}$ are vertices off the triangle of $G(M)$. Hence $\tilde{z} = z$ can be seen. Moreover, the multiplicity of $f$ is one if $f$ is not incident to $z$. Then, as well as if $f = x_1z$, the multigraph $M$ appears in Figure 1, a contradiction.

**B.** $e(G(M)) = 2$. As $G(M)$ is not $2K_2$-decomposable, $\Delta(G(M)) = 2$, i.e., $E(G(M)) = \{wx_1, wx_2\}$. Each pair of parallel edges of $M - w$ has an endvertex in $\{x_1, x_2\}$ because otherwise $G(M)$ together with a nonincident pair is $2K_2$-decomposable. Then also two mutually nonadjacent pairs of parallel edges in $M - w$ taken together with $G(M)$ make up a $2K_2$-decomposable submultigraph of $M$. By (2), however, $M - w$ has parallel edges nonadjacent to either edge of $G(M)$. Hence, there is a vertex $y$ of $M$ which is adjacent to both $x_1$ and $x_2$ and $y \neq w$. Moreover, one can see that no other vertex can be a neighbour of $w$. Therefore $M$ appears in Figure 1, a contradiction.

**Proof of Theorem 8.** Note that the equivalence (i)$\Leftrightarrow$(ii) and implication (i)$\Rightarrow$(iii) are clear.

It remains to prove the converse implication (iii)$\Rightarrow$(i) for all $M$ with $e(G(M)) \geq 6$. To this end, let us assume to the contrary that $M$ is a multigraph with a minimum number of edges and $e(G(M)) \geq 6$, which satisfies (0)–(3) and still $M$ is not $\{C_2, 2K_2\}$-decomposable. Then $M$ contains parallel edges because otherwise $G(M) = M$ and, by (0), (1), (3) and Theorem 1, $M$ is $2K_2$-decomposable. By the minimality of $M$, for any pair of parallel edges $f_1, f_2$, at least one of the conditions (1)–(3) is false if $M \leftarrow M - \{f_1, f_2\}$. Moreover, $e(G(M))$ is even by (0) and the definition of $G(M)$. As the simple graph $G(M)$ is not $2K_2$-decomposable, $\Delta(G(M)) = \frac{e(G(M))}{2} \geq 3$ by Theorem 1. Let $w \in V(M)$ satisfy $\deg_{G(M)}(w) = \Delta(G(M))$. One can easily see that if we remove any pair of parallel edges incident to $w$, we get a multigraph satisfying (0)–(3), a contradiction to the minimality of $M$. Therefore $\deg_{G(M)}(w) = \text{val}_M(w)$. By Theorem 1, since $M$ is not $2K_2$-decomposable, $\Delta(M) = \frac{e(M)}{2}$ or $e(M[\{x, y, z\}]) > \frac{e(M)}{2}$ for some $\{x, y, z\} \subset V(M)$. Consider the following cases.

**A.** $\Delta(M) > \frac{e(M)}{2}$. Let $u \in V(M)$ satisfy $\text{val}_M(u) = \Delta(M)$. Then $u \neq w$ because otherwise (1) would be violated. Moreover, $\deg_{G(M)}(w) > \deg_{G(M)}(u)$ is clear. Therefore $u$ is incident to some parallel edges.

Let $t \in V(M)$ satisfy $p_M(u, t) \geq p_M(u, x)$ for any $x \in V(M)$. Then
p_M(u, t) ≥ 2 whence t ≠ w. Define M' to be a submultigraph of M obtained by removing two parallel u − t edges. By the minimality of M, one of the conditions (1)–(3) is false if M ↷ M'.

A1. Suppose that (1) is false for a vertex x of M'. Then x = w is the only possibility whence e(M) − 2 = e(M') < 2val_M(w) ≤ e(M), i.e., val_M(w) = \(\frac{e(M)}{2}\). Hence, since val_M(u) > val_M(w), the vertices u and w are adjacent and the edge uw is adjacent to all remaining edges of M. This contradicts (2) since clearly p_M(u, w) < 2 by the choice of w.

A2. Suppose that (2) is false for M'. Then there is a vertex y ∈ V(M) such that wy ∈ E(G(M)) and wy is adjacent to all remaining edges of M'. As M satisfies (2), y ∉ \{u, t\} whence p_M(u, t) = 2 (and moreover, p_M(u, x) ≤ 2 for any x ∈ V(M)). Thus \(4 ≤ \Delta(G(M)) < \Delta(M) ≤ \val_M(u) = p_M(u, t) + p_M(u, y) + p_M(u, w) ≤ 5\). Hence \(\Delta(M) = 5\) and p_M(u, y) = 2. Therefore \(10 = 2\Delta(M) > e(M) ≥ e(G(M)) + p_M(u, t) + p_M(u, y) ≥ 10\), a contradiction.

A3. Suppose that (3) is false for M'. As M satisfies (3) as well as val_M(w) = \deg_M(w) ≥ 4 and val_M(u) ≥ 5, there is a vertex y ∉ \{t, u, w\} such that uw, wy ∈ E(G(M)) and e(M) > 1 + val_M(w) + p_M(u, y) ≥ e(M') = e(M) − 2. Since M satisfies (2), M' has an edge different from nonadjacent to uw. Hence p_M(u, t) = 2 (and p_M(u, x) ≤ 2 for any x ∈ V(M)) whence \(5 ≥ p_M(u, t) + p_M(u, y) + p_M(u, w) = \val_M(u) ≥ 5\). Therefore \(\Delta(M) = 5\), p_M(u, y) = 2 and \(10 = 2\Delta(M) > e(M) ≥ e(G(M)) + p_M(u, t) + p_M(u, y) ≥ 10\), a contradiction.

B. \(\Delta(M) ≤ \frac{e(M)}{2}\). Then there are three vertices x, y, z ∈ V(M) such that \(e(M[\{x, y, z\}]) ≥ \frac{e(M)}{2}\) where the notation is chosen so that \(p_M(y, z) ≥ p_M(z, x) ≥ p_M(x, y) ≥ 1\). As \(e(M) ≥ 8\), \(p_M(y, z) ≥ 2\). Let \(M^+\) be a multigraph obtained from M by removing two y–z edges. Clearly, one of the conditions (1)–(3) is false if M ↷ M^+.

B1. Suppose that (1) is false for \(M^+\). Then \(e(M) − 2 = e(M^+) < 2\val_M(w) ≤ e(M), i.e., \val_M(w) = \frac{e(M)}{2}\). Since \(e(M[\{x, y, z\}]) > \frac{e(M)}{2}\), it follows that \(x = w\), \(p_M(y, z) ≥ \frac{e(M)}{2} − 1\) and \(wy, wz \in E(G(M))\), contrary to (3).

B2. Suppose that (2) is false for \(M^+\). As M satisfies (2), \(p_M(y, z) = 2\). Hence \(6 ≥ e(M[\{x, y, z\}]) ≥ \val_M(w) ≥ 4\), i.e., \(p_M(z, x) = 2 ≥ p_M(x, y)\). Therefore a contradiction arises since either \(p_M(x, y) = 1\) and \(10 = 2e(M[\{x, y, z\}]) > e(M) ≥ e(G(M)) + p_M(y, z) + p_M(x, z) ≥ 10\) or \(p_M(x, y) = 2\) and \(12 = 2e(M[\{x, y, z\}]) > e(M) ≥ e(G(M)) + p_M(y, z) + p_M(x, z) +\)
\[ p_M(x, y) \geq 12. \]

**B3.** Suppose that (3) is false for \( M^+ \). As \( M \) satisfies (3), \( w \notin \{x, y, z\} \) and \( p_M(y, z) = 2 \). Since \( e(M) \geq 8 \), \( e(M[\{x, y, z\}]) \geq 5 \) and therefore \( p_M(x, z) = 2 \). Thus \( wx, wz \in E(G(M)) \) and \( 1 + \text{val}_M(w) + p_M(x, z) \geq e(M^+) = e(M) - 2. \) Hence \( p_M(x, y) = 1. \) This implies \( 5 = e(M[\{x, y, z\}]) > \frac{e(M)}{2} \geq \text{val}_M(z) = p_M(y, z) + p_M(x, z) + p_M(w, z) = 5 \), a contradiction. ■

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**References**


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