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DECOMPOSITIONS OF MULTIGRAPHS INTO PARTS WITH TWO EDGES

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Abstract

Given a family $\mathcal{F}$ of multigraphs without isolated vertices, a multigraph M is called *$\mathcal{F}$-decomposable* if M is an edge disjoint union of multigraphs each of which is isomorphic to a member of $\mathcal{F}$. We present necessary and sufficient conditions for the existence of such decompositions if $\mathcal{F}$ comprises two multigraphs from the set consisting of a 2-cycle, a 2-matching and a path with two edges.

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1. Introduction

All multigraphs considered in what follows are loopless. Given a family $\mathcal{F}$ of multigraphs without isolated vertices, an *$\mathcal{F}$-decomposition* of a multigraph M is a collection of submultigraphs which partition the edge set $E(M)$ of M

and are all isomorphic to members of $\mathcal{F}$. If such a decomposition exists, M is called *$\mathcal{F}$-decomposable*; and also *H-decomposable* if H is the only member of $\mathcal{F}$. Let $\mathcal{F} = \{H_1, H_2, \dots, H_t\}$. By an *$H_i$-edge* in an $\mathcal{F}$-decomposition of M we mean an edge belonging to any decomposition part isomorphic to H_i for some $i = 1, 2, \dots, t$.

If M is a multigraph, we write $M = (V, E)$ where $V = V(M)$ and $E = E(M)$ stand for the vertex set and edge set of M, respectively. Cardinalities of those sets, denoted by $v(M)$ and $e(M)$, are called the *order* and *size* of M, respectively. For $S \subset V(M)$, $M[S]$ denotes the submultigraph of M induced by S. The number of edges incident to a vertex x in M, denoted by $\mathrm{val}_M(x)$, is called the *valency* of x, whilst the number of neighbours of x in M, denoted by $\deg_M(x)$, is called the *degree* of x. As usual $\Delta(M)$ stands for the maximum valency among vertices of M. For any two vertices x, y of M, let $p_M(x, y)$ denote the number of edges joining x and y. We call $p_M(x, y)$ the *multiplicity* of an edge xy in M. Edges joining the same vertices are called *parallel edges* (if they are distinct).

The aim of our paper is to provide necessary and sufficient conditions for a multigraph M to be $\{H_1, H_2\}$-decomposable, where H_1, H_2 are any two multigraphs out of C_2 (2-cycle), P_3 (path with two edges), and $2K_2$ (2-matching). Obviously, if M is H_i-decomposable for some $i = 1, 2$, then M is $\{H_1, H_2\}$-decomposable. Therefore the following known results are quoted.

Theorem 1 (Skupień [7], see [4] for a proof)**.** *A multigraph M is $2K_2$-decomposable iff its size $e(M)$ is even, $\Delta(M) \le \frac{e(M)}{2}$ and $e(M[\{x, y, z\}]) \le \frac{e(M)}{2}$ for all $\{x, y, z\} \subset V(M)$.*

If M is a simple graph then the very last condition in Theorem 1 means that $M \neq K_3 \dot\cup K_2$, cf. Caro [2].

Proposition 2. *A multigraph M is C_2-decomposable iff $p_M(x, y) \equiv 0 \pmod 2$ for all $x, y \in V(M)$.*

Theorem 3 [5, 3]**.** *A simple graph G is P_3-decomposable iff each component of G is of even size.*

Corollary 4. *A graph G is $\{P_3, 2K_2\}$-decomposable iff the size $e(G)$ of G is even.*

Given a multigraph M, define the $*$-*line graph* of M, denoted by $L^*(M)$, to be the graph with vertex set $V(L^*(M)) = E(M)$ and edge set $E(L^*(M)) = \{w_1w_2 :\ w_1, w_2 \in E(M),\ |w_1 \cap w_2| = 1\}$. Evidently, $L^*(M)$ is obtainable from the ordinary line graph $L(M)$ by removal of all edges which represent multiple adjacency of edges in the root multigraph M. In other words, the operator L^* represents doubly adjacent edges in M as if they were nonadjacent in M.

Theorem 5 [4]**.** *Given a multigraph M, the following statements are equivalent.*

(i) *M is P_3-decomposable.*

(ii) *$L^*(M)$ has a 1-factor.*

Therefore checking whether a multigraph M is P_3-decomposable can be done in polynomial time $O(e(M)^{2.5})$, cf [4]. Some original sufficient conditions for M to be P_3-decomposable may be found in [1, 4].

2. $\{C_2, P_3\}$-Decomposition

Theorem 6. *Let M be a multigraph and let $L(M)$ be the line graph of M. The following statements are equivalent.*

(i) *M is $\{C_2, P_3\}$-decomposable.*

(ii) *Each component of M has an even number of edges.*

(iii) *Each component of $L(M)$ has an even number of vertices.*

(iv) *$L(M)$ has a 1-factor.*

Proof. Each of the implications in the cycle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) is obvious or well-known. Well-known is the implication (iii) $\Rightarrow$ (iv) following from the result of Sumner [8] and Las Vergnas [6] which says that every connected claw-free graph of even order has a 1-factor. ■

3. $\{P_3, 2K_2\}$-Decomposition

Theorem 7. *Let M be a multigraph. Let $L^*(M)$ and $\overline{L(M)}$ be the $*$-line graph and the complement of the line graph $L(M)$ of M, respectively. The following statements are mutually equivalent.*

(i) *M is $\{P_3,\, 2K_2\}$-decomposable.*

(ii) *M has an even number, $e(M)$, of edges and the multiplicity of any edge does not exceed $e(M)/2$.*

(iii) *The graph $\tilde{L} := L^*(M) \cup \overline{L(M)}$ has a 1-factor.*

Proof. Implication (i) ⇒ (ii) is true because $e(M)/2$ is the number of parts and parallel edges must be in different parts of a decomposition. Implication (ii) ⇒ (iii) is true because the order $v(\tilde{L}) = e(M)$ is even and the minimum degree $\delta(\tilde{L}) \geq \frac{1}{2}v(\tilde{L})$, whence, by Dirac's theorem, the graph $\tilde{L}$ has a Hamiltonian cycle. Implication (iii) ⇒ (i) is obvious. ■

4. $\{C_2,\, 2K_2\}$-Decomposition

Given a multigraph M, let $G(M)$ denote the graph induced by the edge set $E(G(M)) := \{xy :\ p_M(x,y) \equiv 1 \pmod 2\}$. Evidently, a graph isomorphic to $G(M)$ is obtainable from M both by removing all edges of the maximal family of pairwise edge-disjoint copies of C_2 and by removing all resulting isolated vertices. Thus $2K_2$-edges in any $\{C_2,\, 2K_2\}$-decomposition of M induce a multigraph M' containing a subgraph isomorphic to $G(M)$ (in fact, $p_{M'}(x,y) \geq 1$ whenever $xy \in E(G(M))$).

If $E' \subset E(M)$, $f \in E(M)$, and $w \in V(M)$ then $M - E'$ (or $M - f$) is the spanning submultigraph of M obtained by removing the edges only (E' or f), while $M - w$ is obtained from M by removing the vertex w together with all edges incident to w.

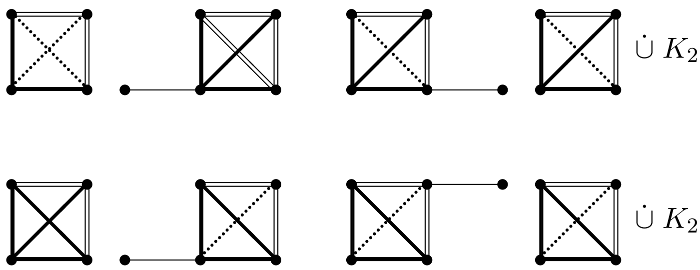

Figure 1. Eight families of multigraphs M

edge :	heavy	thin	doubled	dotted
multiplicity :	odd	1	even ≥ 2	even ≥ 0

Table 1. Codes in Figure 1

Theorem 8. *Let M be a multigraph and let $\overline{L^*(M)}$ be the complement of the $*$-line graph $L^*(M)$ of M. The following three statements are mutually equivalent.*

(i) *M is $\{C_2, 2K_2\}$-decomposable.*

(ii) *$\overline{L^*(M)}$ has a 1-factor.*

(iii) *Each of the following five conditions holds:*

(0) *$e(M)$ is even,*

(1) $\mathrm{val}_M(x) + \deg_{G(M)}(x) \leq e(M)$ *for every* $x \in V(M)$,

(2) *if* $xy \in E(G(M))$ *then* $\mathrm{val}_M(x) + \mathrm{val}_M(y) - p_M(x, y) < e(M)$,

(3) *if* $yx, xz \in E(G(M))$ *then* $1 + \mathrm{val}_M(x) + p_M(y, z) < e(M)$,

(4) *M is different from each of the (forbidden) multigraphs shown in Figure 1.*

A vertex y is called an *odd neighbour* of a vertex x if M has an edge xy whose multiplicity $p_M(x, y)$ is odd.

Proposition 9. *The following condition (i') is an equivalent of (i) above for $i = 1, 2, 3$.*

$(1')$ *The number of odd neighbours of any vertex x does not exceed the number of all edges nonincident to x;*

$(2')$ *There is no edge xy adjacent to every other edge and with odd multiplicity $p_M(x, y)$;*

$(3')$ *There are no two adjacent edges yx, xz both with odd multiplicities and such that among the remaining edges at most one is not a neighbour of both yx and xz.* ■

Proposition 10. *Each multigraph depicted in Figure 1 satisfies all conditions (0)–(3) and is not $\{C_2, 2K_2\}$-decomposable.* ■

The following converse result is of importance.

Lemma 11. *Every multigraph M which satisfies conditions (0)–(3), has $e(G(M)) \leq 4$, and is not $\{C_2, 2K_2\}$-decomposable is depicted in Figure 1.*

Proof. Suppose that M is a counterexample. Since M is not C_2-decomposable, $e(G(M)) > 0$. Due to (0), $G(M)$ has two or four edges. Consider two main cases A and B.

A. $e(G(M)) = 4$. As $G(M)$ is not $2K_2$-decomposable, either $G(M)$ contains a triangle or otherwise $\Delta(G(M)) \geq 3$. Consider the following subcases.

A1. $\Delta(G(M)) = 4$. Then $G(M)$ is a star with a central vertex w and $M - w$ is C_2-decomposable. Moreover, $e(M - w) \geq 4$ by (1). Since M satisfies (2), not all edges of $M - w$ are incident to one vertex of $G(M)$. On the other hand, each edge of $M - w$ has both endvertices in $G(M)$ as well as there is no $2K_2$ in $M - w$ because otherwise $G(M)$ together with any two pairs of parallel edges of $M - w$ which do not intersect at $G(M)$ is $2K_2$-decomposable. Consequently, edges of $M - w$ induce a "multiple triangle" on three hanging vertices of $G(M)$. Therefore no parallel edges can join w to a vertex off the "triangle". Hence M appears in Figure 1, a contradiction.

A2. $\Delta(G(M)) = 3$ and $G(M)$ contains no triangle. Let w be the degree-3 central vertex of the star of $G(M)$, let f and wx_i with $i = 1, 2, 3$ be the four edges of $G(M)$ with notation such that the edge f is incident to x_3 if $G(M)$ is connected. Then $e(M - w) > 2$ by (1). It is easily seen that each pair of parallel edges of $M - w$ has a vertex in $\{x_1, x_2, x_3\}$. Hence the multiplicity of f is one if f is not incident to x_3. The multiplicity of f is one, too, otherwise. Namely, by (2), M has a pair of parallel edges which are nonadjacent to the edge wx_3 of $G(M)$. These are x_1–x_2 edges because otherwise the pair together with $G(M)$ is $2K_2$-decomposable (the edge f being matched with wx_i if x_i is an endvertex of the pair, $i \neq 3$). Now, clearly, the multiplicity of f is one. Consequently, by (3), each vertex x_i is incident to parallel edges of $M - w$; moreover, one can see that all parallel edges of $M - w$ are of the form x_ix_j only. Similarly, $\deg_M(w) = 3$ only, whence M appears in Figure 1, a contradiction.

A3. $G(M)$ contains a triangle. Let the vertices of the triangle be denoted by x_i, $i = 1, 2, 3$. Let f stand for the remaining edge of $G(M)$. Then each pair of parallel edges are incident to some x_i because otherwise the pair and $G(M)$ make up a $2K_2$-decomposable submultigraph. Assume that the edge f has no vertex in the triangle of $G(M)$. Hence the multiplicity of f is one. Moreover, by (3), M has two pairs of parallel edges of the form x_iz and $x_j\tilde{z}$ where x_i, x_j are distinct vertices of the triangle of $G(M)$ and z, $\tilde{z}$ are both off the triangle. Then $\tilde{z} = z$ because otherwise the two pairs and $G(M)$ would be $2K_2$-decomposable. Moreover, f is either incident to z or not; and in either case M appears in Figure 1, a contradiction.

Assume that f is incident to a vertex, say x_1, in the triangle of $G(M)$. Then, by (2), M has parallel edges of the form x_2z and $x_3\tilde{z}$ where z, $\tilde{z}$ are vertices off the triangle of $G(M)$. Hence $\tilde{z} = z$ can be seen. Moreover, the multiplicity of f is one if f is not incident to z. Then, as well as if $f = x_1z$, the multigraph M appears in Figure 1, a contradiction.

B. $e(G(M)) = 2$. As $G(M)$ is not $2K_2$-decomposable, $\Delta(G(M)) = 2$, i.e., $E(G(M)) = \{wx_1, wx_2\}$. Each pair of parallel edges of $M - w$ has an endvertex in $\{x_1, x_2\}$ because otherwise $G(M)$ together with a nonincident pair is $2K_2$-decomposable. Then also two mutually nonadjacent pairs of parallel edges in $M - w$ taken together with $G(M)$ make up a $2K_2$-decomposable submultigraph of M. By (2), however, $M - w$ has parallel edges nonadjacent to either edge of $G(M)$. Hence, there is a vertex y of M which is adjacent to both x_1 and x_2 and $y \neq w$. Moreover, one can see that no other vertex can be a neighbour of w. Therefore M appears in Figure 1, a contradiction. ∎

Proof of Theorem **8.** Note that the equivalence (i)⇔(ii) and implication (i) ⇒ (iii) are clear.

It remains to prove the converse implication (iii) ⇒ (i) for all M with $e(G(M)) \geq 6$. To this end, let us assume to the contrary that M is a multigraph with a minimum number of edges and $e(G(M)) \geq 6$, which satisfies (0)–(3) and still M is not $\{C_2, 2K_2\}$-decomposable. Then M contains parallel edges because otherwise $G(M) = M$ and, by (0), (1), (3) and Theorem 1, M is $2K_2$-decomposable. By the minimality of M, for any pair of parallel edges f_1, f_2, at least one of the conditions (1)–(3) is false if $M \leftarrow M - \{f_1, f_2\}$. Moreover, $e(G(M))$ is even by (0) and the definition of $G(M)$. As the simple graph $G(M)$ is not $2K_2$-decomposable, $\Delta(G(M)) > \frac{e(G(M))}{2} \geq 3$ by Theorem 1. Let $w \in V(M)$ satisfy $\deg_{G(M)}(w) = \Delta(G(M))$. One can easily see that if we remove any pair of parallel edges incident to w, we get a multigraph satisfying (0)–(3), a contradiction to the minimality of M. Therefore $\deg_{G(M)}(w) = \mathrm{val}_M(w)$. By Theorem 1, since M is not $2K_2$-decomposable, $\Delta(M) > \frac{e(M)}{2}$ or $e(M[\{x, y, z\}]) > \frac{e(M)}{2}$ for some $\{x, y, z\} \subset V(M)$. Consider the following cases.

A. $\Delta(M) > \frac{e(M)}{2}$. Let $u \in V(M)$ satisfy $\mathrm{val}_M(u) = \Delta(M)$. Then $u \neq w$ because otherwise (1) would be violated. Moreover, $\deg_{G(M)}(w) > \deg_{G(M)}(u)$ is clear. Therefore u is incident to some parallel edges.

Let $t \in V(M)$ satisfy $p_M(u, t) \geq p_M(u, x)$ for any $x \in V(M)$. Then

$p_M(u,t) \geq 2$ whence $t \neq w$. Define M' to be a submultigraph of M obtained by removing two parallel $u-t$ edges. By the minimality of M, one of the conditions (1)–(3) is false if $M \leftarrow M'$.

A1. Suppose that (1) is false for a vertex x of M'. Then $x = w$ is the only possibility whence $e(M) - 2 = e(M') < 2\mathrm{val}_M(w) \leq e(M)$, i.e., $\mathrm{val}_M(w) = \frac{e(M)}{2}$. Hence, since $\mathrm{val}_M(u) > \mathrm{val}_M(w)$, the vertices u and w are adjacent and the edge wu is adjacent to all remaining edges of M. This contradicts (2) since clearly $p_M(u,w) < 2$ by the choice of w.

A2. Suppose that (2) is false for M'. Then there is a vertex $y \in V(M)$ such that $wy \in E(G(M))$ and wy is adjacent to all remaining edges of M'. As M satisfies (2), $y \notin \{u,t\}$ whence $p_M(u,t) = 2$ (and moreover, $p_M(u,x) \leq 2$ for any $x \in V(M)$). Thus $4 \leq \Delta(G(M)) < \Delta(M) = \mathrm{val}_M(u) = p_M(u,t) + p_M(u,y) + p_M(u,w) \leq 5$. Hence $\Delta(M) = 5$ and $p_M(u,y) = 2$. Therefore $10 = 2\Delta(M) > e(M) \geq e(G(M)) + p_M(u,t) + p_M(u,y) \geq 10$, a contradiction.

A3. Suppose that (3) is false for M'. As M satisfies (3) as well as $\mathrm{val}_M(w) = \deg_M(w) \geq 4$ and $\mathrm{val}_M(u) \geq 5$, there is a vertex $y \notin \{t,u,w\}$ such that $uw, wy \in E(G(M))$ and $e(M) > 1+\mathrm{val}_M(w)+p_M(u,y) \geq e(M') = e(M) - 2$. Since M satisfies (2), M' has an edge different from and nonadjacent to uw. Hence $p_M(u,t) = 2$ (and $p_M(u,x) \leq 2$ for any $x \in V(M)$) whence $5 \geq p_M(u,t) + p_M(u,y) + p_M(u,w) = \mathrm{val}_M(u) \geq 5$. Therefore $\Delta(M) = 5$, $p_M(u,y) = 2$ and $10 = 2\Delta(M) > e(M) \geq e(G(M)) + p_M(u,t) + p_M(u,y) \geq 10$, a contradiction.

B. $\Delta(M) \leq \frac{e(M)}{2}$. Then there are three vertices $x, y, z \in V(M)$ such that $e(M[\{x,y,z\}]) > \frac{e(M)}{2}$ where the notation is chosen so that $p_M(y,z) \geq p_M(z,x) \geq p_M(x,y) \geq 1$. As $e(M) \geq 8$, $p_M(y,z) \geq 2$. Let M^+ be a multigraph obtained from M by removing two y–z edges. Clearly, one of the conditions (1)–(3) is false if $M \leftarrow M^+$.

B1. Suppose that (1) is false for M^+. Then $e(M) - 2 = e(M^+) < 2\mathrm{val}_M(w) \leq e(M)$, i.e., $\mathrm{val}_M(w) = \frac{e(M)}{2}$. Since $e(M[\{x,y,z\}]) > \frac{e(M)}{2}$, it follows that $x = w$, $p_M(y,z) \geq \frac{e(M)}{2} - 1$ and $wy, wz \in E(G(M))$, contrary to (3).

B2. Suppose that (2) is false for M^+. As M satisfies (2), $p_M(y,z) = 2$. Hence $6 \geq e(M[\{x,y,z\}]) > \mathrm{val}_M(w) \geq 4$, i.e., $p_M(z,x) = 2 \geq p_M(x,y)$. Therefore a contradiction arises since either $p_M(x,y) = 1$ and $10 = 2e(M[\{x,y,z\}]) > e(M) \geq e(G(M)) + p_M(y,z) + p_M(x,z) \geq 10$ or $p_M(x,y) = 2$ and $12 = 2e(M[\{x,y,z\}]) > e(M) \geq e(G(M)) + p_M(y,z) + p_M(x,z) +$

$p_M(x, y) \geq 12$.

B3. Suppose that (3) is false for M^+. As M satisfies (3), $w \notin \{x, y, z\}$ and $p_M(y, z) = 2$. Since $e(M) \geq 8$, $e(M[\{x, y, z\}]) \geq 5$ and therefore $p_M(x, z) = 2$. Thus $wx, wz \in E(G(M))$ and $1 + \mathrm{val}_M(w) + p_M(x, z) \geq e(M^+) = e(M) - 2$. Hence $p_M(x, y) = 1$. This implies $5 = e(M[\{x, y, z\}]) > \frac{e(M)}{2} \geq \mathrm{val}_M(z) = p_M(y, z) + p_M(x, z) + p_M(w, z) = 5$, a contradiction. ■

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References

[1] K. Bryś, M. Kouider, Z. Lonc and M. Mahéo, *Decomposition of multigraphs*, Discuss. Math. Graph Theory **18** (1998) 225–232.

[2] Y. Caro, *The decomposition of graphs into graphs having two edges*, a manuscript.

[3] Y. Caro and J. Schönheim, *Decompositions of trees into isomorphic subtrees*, Ars Comb. **9** (1980) 119–130.

[4] J. Ivančo, M. Meszka and Z. Skupień, *Decomposition of multigraphs into isomorphic graphs with two edges*, Ars Comb. **51** (1999) 105–112.

[5] E.B. Yavorskiĭ, *Representations of oriented graphs and φ-transformations* [Russian], in: A. N. Šarkovskiĭ, ed., Theoretical and Applied Problems of Differential Equations and Algebra [Russian] (Nauk. Dumka, Kiev, 1978) 247–250.

[6] M. Las Vergnas, *A note on matchings in graphs*, Cahiers Centre Etudes Rech. Opér. **17** (1975) 257–260.

[7] Z. Skupień, Problem 270 [on 2-edge-decomposable multigraphs], Discrete Math. **164** (1997) 320–321.

[8] D.P. Sumner, *Graphs with* 1*-factors*, Proc. Amer. Math. Soc. **42** (1974) 8–12.

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