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DÜRER POLYHEDRA: THE DARK SIDE OF MELANCHOLIA

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Abstract

Dürer's engraving *Melencolia I* famously includes a perspective view of a solid polyhedral block of which the visible portion is an 8-circuit bounding a pentagon-triple+triangle patch. The polyhedron is usually taken to be a cube truncated on antipodal corners, but an infinity of others are compatible with the visible patch. Construction of all cubic polyhedra compatible with the visible portion (i.e., *Dürer Polyhedra*) is discussed, explicit graphs and symmetries are listed for small cases (≤ 18 vertices) and total counts are given for $10 \leq$ vertices ≤ 26.

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1. INTRODUCTION

How well can we reconstruct a simple geometrical object from a single snapshot in which one hemisphere is lit and the other is in darkness? In Albrecht Dürer's famous engraving of 1514, *Melencolia I*, a winged figure broods over a collection of objects that includes a large, strikingly irregular, polyhedral stone block described variously by art historians as a rhomboid [1], a truncated rhomboid or truncated rhombo-hedron [2] ('abgestumpfter Rhomboeder' [3]). Four faces only of the block are visible (a triangle and three pentagons). It is suggested [4] that the whole is a cube truncated on opposite vertices, a claim supported by a view of a truncated cube in Dürer's writings on perspective geometry [5], and a preliminary drawing in the Dresden Sketchbook [6] with hidden lines marked. The antipodally bi-truncated cube is sometimes called 'Dürer's octahedron' in the mathematical literature [7, 8], but this identification is clearly combinatorial rather than affine in nature, as the angles of the figure as drawn are famously irregular, even after allowance is made for perspective effects [9, 10, 11].

Here we ask another combinatorial question: what can we deduce with certainty about the hidden dark side of the mysterious polyhedron, simply from *what we can see* in the engraving? An infinite set of completions, each equally deserving the appellation of *Dürer Polyhedron*, is found. As the reasoning has applications to problems of blind-side reconstruction of polyhedra in other contexts, and uses only elementary graph theory, it is briefly summarised here along with the specific solutions for Dürer polyhedra.

2. METHOD

Assume that the object to be reconstructed is a *cubic polyhedron* P without holes or handles (i.e., that the graph of vertices and edges has three edges meeting at every vertex, is three-connected and can be drawn in the plane without edge crossings [12]). Note that in [12, p. 235] is also given a proof of the famous Steinitz theorem [13, 14]:

> A graph G is (isomorphic to) the graph $G(P)$ of a cubic polyhedron P if and only if it is planar and three-connected;

which is important for our investigation.

P^0 and P' are respectively the visible and invisible parts of P, separated by a 'terminator' cycle $C^0 = C(P^0)$. Let $P^\star$ denote the union of P' and C^0. The reconstruction problem is converted to one of finding suitable in-fillings of a Schlegel diagram of P^0 by using graphs embedded in the plane i.e., $G = G(P)$, $G^0 = G(P^0)$, $G^\star = G(P^\star)$. The terminator becomes the circuit $C^0 = C(P^0)$ bounding a finite face of G^0. Symbols v, e, f with appropriate superscripts denote the numbers of vertices, edges and faces of G, G^0 and $G^\star$. Clearly, $v^\star = v - v^0$, $e^\star = e - e^0$ and $f^\star = f - f^0$ and, as P is cubic,

$$e^\star = 3v^\star/2 + 3v^0/2 - e^0,$$

$$f^\star = v^\star/2 + v^0/2 + 2 - f^0.$$

In our case, P is a *Dürer polyhedron*, $P = D$ (i.e., one that contains the triangle + triple-pentagon patch seen in the engraving). $G(D^0)$ has $v^0 = 10$, $e^0 = 13$, $f^0 = 4$, with four divalent vertices in the circuit C^0, and hence $e^\star \geq 2$, $f^\star \geq 3$ for $v^\star \geq 0$, and, if $\hat{f}_r$ denotes the number of faces of size r, $\sum_r (6-r)\hat{f}_r = 12$, thus $\sum_r (6-r)\hat{f}_r^\star = 6$. The in-filling problem is sketched in Figure 1.

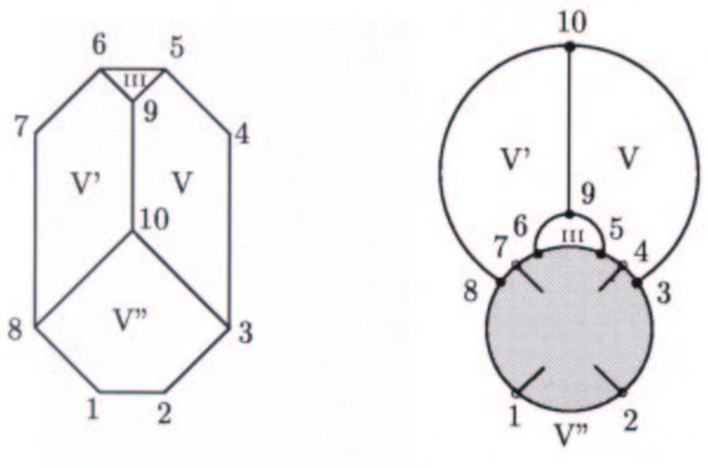


Figure 1. Dürer polyhedron and its Schlegel diagram. Regions III and V denote the triangular and pentagonal faces of the visible hemisphere of the polyhedron. • are saturated trivalent vertices; ∘ have two valencies in the visible portion, requiring incidence with an edge within the hidden hemisphere (the shaded circular disk).

A systematic procedure for generating all solutions is now described.

Let $G = G(P)$ with $C^0 = C(G^0)$ be a solution of our problem. Delete all edges that are incident with vertices of degree 3 in G^0, and after that, all isolated vertices. The remaining graph $B(G, C^0)$ has $c = c\big(B(G, C^0)\big)$ components B_i, where $1 \leq c \leq 2$. Each component B_i is an *internal bridge of G with respect to C^0*. Let $\mathbf{B}_j = \mathbf{B}_j(G, C^0)$ denote the set of all bridges B for which the number of vertices of degree 1 is $b^0 = b^0(B) = j$. Clearly, in our case is $j = 2$ or 4.

For $j = 2$ there exists exactly one solution (two bridges B_1, B_2 isomorphic to the complete graph K_2). See Figure 2.

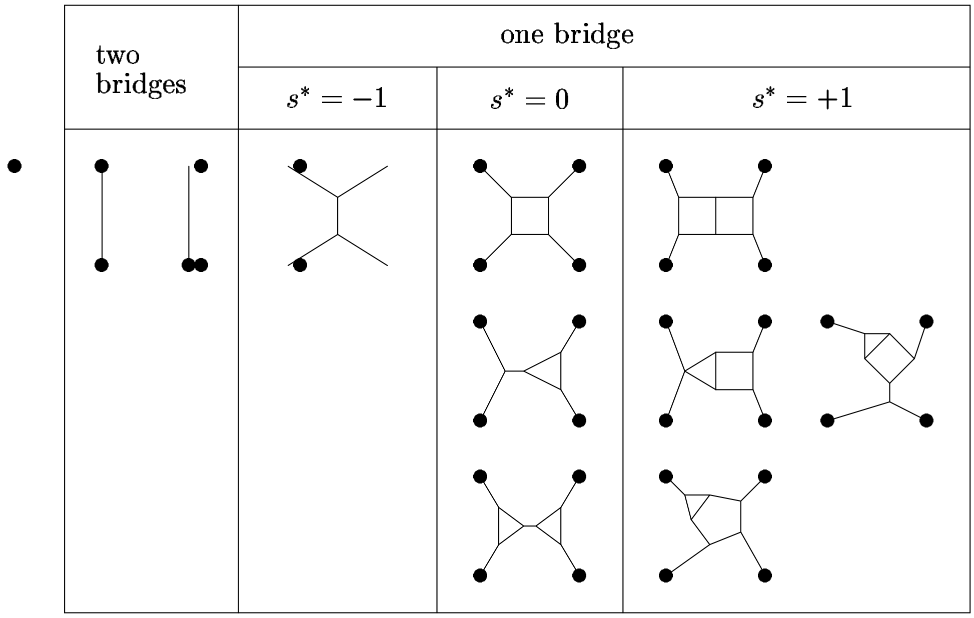


Figure 2. Possible bridges for $j = 4$. The complete set of possibilities with $s^\star \leq 1$.

For $j = 4$ there exist one-bridge solutions B, which have $v^\star = v^\star(B)$ vertices of degree 3, $e^\star = e^\star(B)$ edges, and the number $f^\star = f^\star(B)$ of faces of B is defined by $f^\star := f(G^\star) - 1$. Clearly, $3v^\star = 2e^\star - b^0$. The general solutions are therefore, for integer $s^\star \geq -b^0/2$:

$$\begin{aligned} e^\star &= 3s^\star + 2b^0, \\ v^\star &= 2s^\star + b^0, \\ f^\star &= s^\star + b^0 + 1. \end{aligned}$$

For $j = 4$ and small integers $s^\star$, the possible elements of $\mathbf{B}_4$ are listed in Figure 2.

Small cases are easily checked by working through published complete lists of graphs sorted by edge or vertex count [15] or generating new lists [16].

To solve our problem of finding polyhedra, we must take into account the sequence of vertices in C^0 in order to decide on the number and placing of the bridges within C^0.

The difficulty with the present approach is in assigning symmetry and checking isomorphism for the polyhedra generated by the distinct placements of the bridges within the Schegel diagram. The graph $G^0 = G(D^0)$ has C_s symmetry, which may be broken, enhanced or reduced on filling in the bridge(s). Hand enumeration of possibilities for bridges gives the set of the first 78 Dürer polyhedra shown in Figure 3, listed in order of increasing $v^\star$. Figure 4 shows the first 20 members of the series, drawn from the 'topological coordinates' [17], i.e., from scaled eigenvectors of their adjacency matrices, and assigns them to their maximal point group symmetries; affine realisations of any particular polyhedron may belong to subgroups of this maximal group.

3. DISCUSSION

The smallest Dürer polyhedron is the C_{3v} object obtained by truncation of a tetrahedron on three of its four vertices. Although it is a symmetrical, uniquely minimal solution without hidden vertices, it would be balanced on an edge when projected as in *Melencolia I*, and so can be rejected on physical grounds as a candidate for the polyhedron that Dürer had in mind when making the engraving, unless he was prepared to invoke hidden supports or ignore impending collapse.

The two distinct solutions for $v^\star = 2$ are related by a *Stone-Wales transformation* [18] i.e., a 90° rotation of the bridge B. One is the bi-truncated cube (D_{3d}) which in an unsymmetrical realisation is the traditional explanation for the solid in the engraving; the other has only C_s symmetry.

Clearly the class of Dürer polyhedra is infinite, as two distinct polyhedra at any even value of $v^\star$ can be obtained from the $v^\star = 2$ solutions by converting the central hidden edge to a ladder of $(v^\star - 2)/2$ fused squares. Each solution for $v^\star > 2$ outside this series also generates an infinite set of larger solutions by ladder extension. Some statistics on the initial growth of the series of Dürer polyhedra can be obtained by filtering the Dürer polyhedra from the lists of general cubic polyhedra available from the *plantri* program of Brinkmann and Mckay [19]. As Table 1 shows, the Dürer polyhedra show the expected combinatorial rise in absolute number as the number of vertices increases, but appear to form a decreasing fraction of the population of cubic polyhedra. *Post hoc* filtering is not an efficient way to generate subsets of the cubic polyhedra, but the program could be adapted to generate only Dürer polyhedra if it became important to have more complete lists.

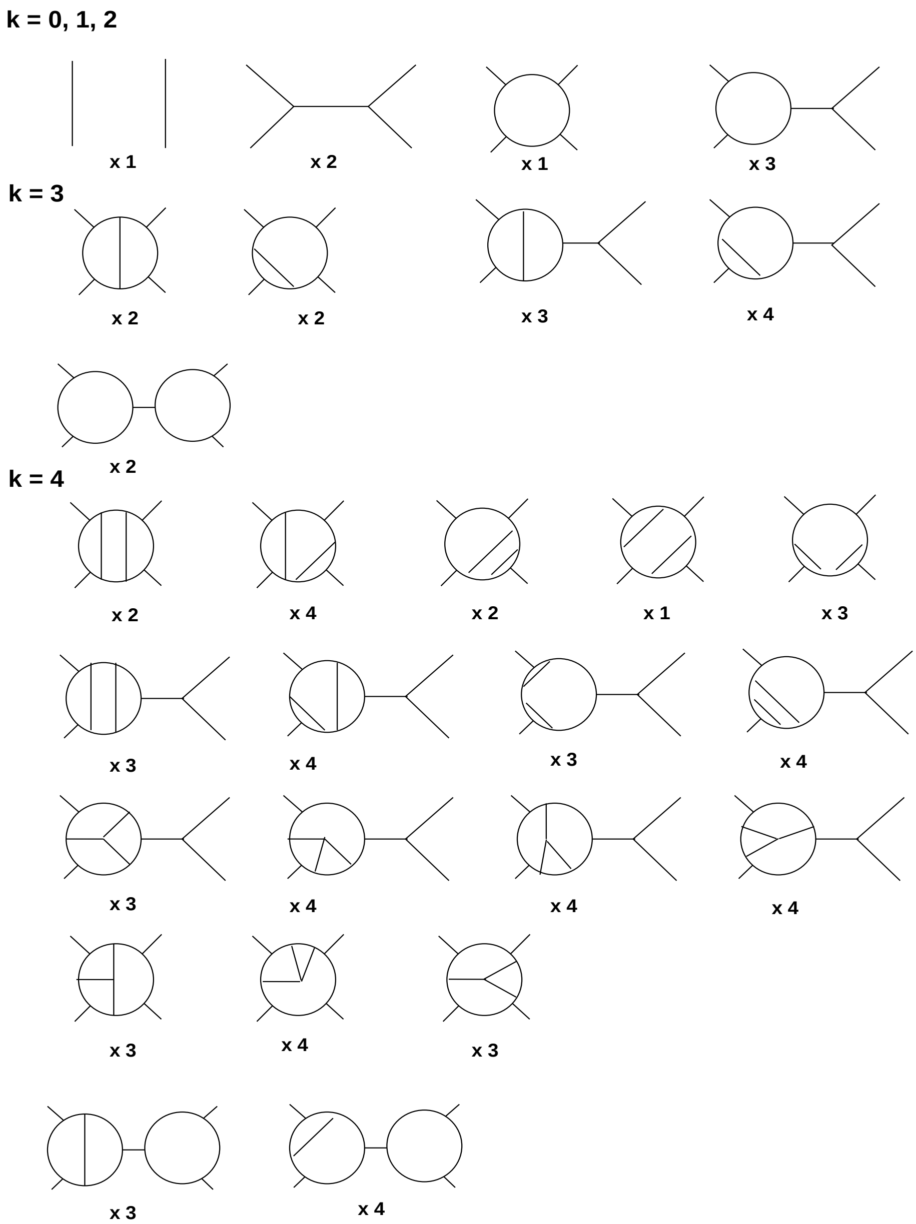
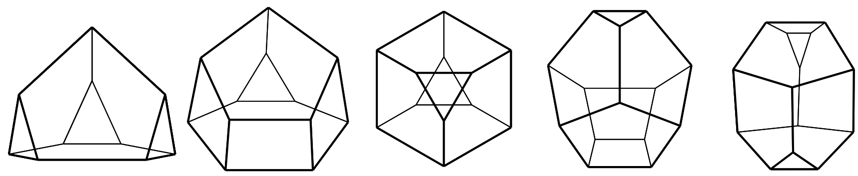
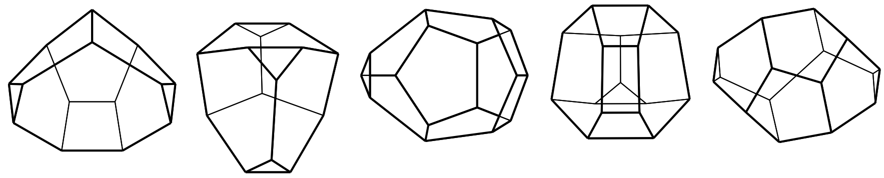


Figure 3. Dürer polyhedra on $v = 10 + 2k$ vertices, ($k = 0, 1, 2, 3, 4$), shown as bridges of the terminator circuit $C(D^0)$, each labelled with a multiplicity factor i.e., the number of non-isomorphic polyhedra derived by setting the given bridge in C.

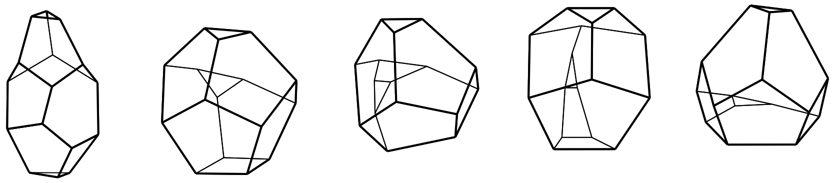
10:3031 C_{3v} 12:1331 C_s 12:206: D_{3d} 14:1251 C_s 14:2142 C_s



14:21501 C_s 14:3033 C_s 16:12601: C_s 16:1333 C_s 16:2062 C_2



16:21511 C_1 16:21511 C_1 16:22321: C_1 16:22402 C_s 16:224101 C_1



16:224101 C_1 16:305101 C_2 16:31312: C_s 16:31312 C_s 16:31312 C_1

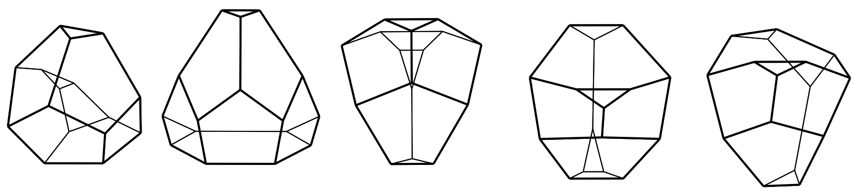


Figure 4. The first 20 Dürer polyhedra, drawn from eigenvectors of the adjacency matrix [17]. Polyhedra are labelled $v : \hat{f}_3\hat{f}_4\hat{f}_5 \ldots : S$, where v is the vertex count, $\hat{f}_r$ the number of faces of size r and S the maximal point group symmetry.

Table 1. Comparison of counts for Dürer and general cubic polyhedra. $N(D)$ and $N(c)$ are the respective numbers of Dürer and cubic polyhedra on v vertices, and $\%D$ is their ratio, expressed as a percentage. The counts are obtained by filtering the lists of distinct cubic polyhedra produced by the plantri program [19].

v	4	6	8	10	12	14	16	18	20	22	24	26
$N(D)$	0	0	0	1	2	4	13	58	308	1826	11810	80036
$N(c)$	1	1	2	5	14	50	233	1249	7595	49566	339722	2406841
$\%D$	0	0	0	20	14	8	6	5	4	3.7	3.5	3.3

All these polyhedra are combinatorially compatible with what can be seen in the engraving, though as the number of hidden vertices increases, the implied metric and physical constraints become more severe. The identity of the polyhedron in the engraving is a historical fact, recoverable or not, and external evidence seems to point to the second-smallest Dürer polyhedron, the bi-truncated cube, as the illustrated object. The purely internal evidence is open to wider interpretation. Without making the anachronistic claim that Dürer would have been able to find these polyhedra by the Euler theorem, we note that any one of the smaller members of the series could have been found empirically by truncating the cube.

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