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PARTIAL COVERS OF GRAPHS

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Abstract

Given graphs G and H, a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $(f(u), f(v))$ is an edge of H for every edge (u, v) of G. In this paper, we initiate the study of computational complexity of *locally injective homomorphisms* called *partial covers* of graphs. We motivate the study of partial covers by showing a correspondence to generalized (2,1)-colorings of graphs, the notion stemming from a practical problem of assigning frequencies to transmitters without interference. We compare the problems of deciding existence of partial covers and of full covers (locally bijective homomorphisms), which were previously studied.

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1. Introduction

Given graphs G and H, a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $(f(u), f(v))$ is an edge of H for every edge (u, v) of G. A homomorphism from G to H is also called an H*-coloring* of G (since homomorphisms to

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the complete graphs correspond to ordinary colorings). Hell and Nešetřil [14, 15] considered the complexity of deciding if an input graph G allows a homomorphism into a fixed parameter graph H. They proved that this question is polynomially solvable for graphs H which contain a loop or are bipartite, and NP-complete otherwise. Dyer and Greenhill [7] extended the research in this direction and characterized the graphs H for which counting homomorphisms from G into H is $\#P$-complete.

A homomorphism which is a local isomorphism (i.e., for every node u of G, the morphism f maps the neighborhood of u bijectively onto the neighborhood of $f(u)$) is called a *covering projection*, and if such a homomorphism exists, we say that G *covers* H. The motivation for the study of graph covers comes from the algebraic graph theory [4]. As special cases of covering spaces from algebraic topology [22], graph covers are used in many applications in topological graph theory [12].

Computational applications of graph covers are used by Angluin [2] to study "local knowledge" in distributed computing environments, and by Courcelle and Métivier [6] to show that nontrivial minor closed classes of graphs cannot be recognized by local computations.

Bodlaender in [3] raised the question of computational complexity of H-cover problems. The H-(partial) cover problem asks if a given input graph G (partially) covers H, the latter graph is considered a fixed parameter of the problem. In [1] Abello *et al.* it is shown that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. The complexity of the H-cover problem was further studied in [17, 18, 19]. Several infinite classes of both polynomial and NP-complete instances were recognized, however, currently no good conjecture about the characterization of graphs H, for which the H-cover problem is polynomially solvable, is at hand (assuming, of course, $P \neq$ NP).

In this paper, we initiate the study of computational complexity of *partial covers*. We say that G is a *partial cover* of H, if G is an induced subgraph of a (full) cover of H. In the algebraic setting, a partial covering projection is a *locally injective homomorphism*, i.e., a homomorphism which maps the neighborhood of every vertex injectively into the neighborhood of the image of this vertex. Partial covers of graphs are used in [17] as a tool in the gadget construction for NP-completeness reduction in the proof of Proposition 2.5.

We believe that the study of the computational complexity of partial covers is worthwhile for two reasons. First, as it is shown in Section 3, partial covers correspond to generalized $(2, 1)$-colorings of graphs, while the notion of a $(2, 1)$-coloring stems from a practical problem of assigning frequencies to transmitters to avoid interference. Secondly, as our results show, from the computational complexity point of view partial covers seem to be more difficult than full covers. This raises a hope that the number of polynomially solvable instances may be so restricted that it would allow a characterization theorem.

The paper is organized as follows. In Section 2 we compare the problems of deciding existence of full and partial covers. In Section 3 we reveal the connection to generalized $(2, 1)$-colorings.

2. Partial Covers Versus Full Covers

It is well known that the existence of a covering projection between two graphs implies that they have the same degree refinement matrix. The converse is not true, but it is shown in [21] that graphs with the same degree refinement matrix have a common finite cover. In general, partial covers do not preserve the degree refinement matrix. However, if two graphs happen to have the same degree refinement matrix, then any partial covering projection between them is necessarily a full covering (Proposition 2.3). This fact will be used in this subsection to derive our first complexity results on partial covers.

The *degree partition* of a graph G is the coarsest partition (i.e., the partition of the minimum number of classes) of the vertex set of G into classes (called blocks) $B_1, B_2, \ldots, B_k$, such that for every i, j and any two vertices $u, v \in B_i$, $|N(u) \cap B_j| = |N(v) \cap B_j|$. The symbol $N(u)$ denotes the neighborhood of the vertex u. Given an ordering of classes $B_1, \ldots, B_k$, the *weight* $t(u)$ of a vertex $u \in V(G)$ is the index $t(u) = j$, such that $u \in B_j$.

The degree partition is unique and can be obtained by the following procedure which recursively refines partitions by the numbers of neighbors of vertices in the blocks of the partition. The symbol k_i denotes the number of classes in the partition after i rounds of refining, and $B_{ij}, j = 1, 2, \ldots, k_i$ are the classes of this partition.

1. Define $k_0 = 1$ and $B_{01} = V(G)$.
2. Set $i = 0$.
3. Repeat until $k_i = k_{i+1}$:
  3.1. For every vertex $u \in V(G)$, compute the
    neighbor vector $n^i(u) = (n_1, n_2, \ldots, n_{k_i})$ where $n_j = |N(u) \cap B_{ij}|$.

3.2. Define a new partition of $V(G)$ so that u, v belong to the same class, if and only if $n^i(u) = n^i(v)$.
3.3. Let k_{i+1} be the number of the classes in the new partition.
3.4. Order the classes of the partition lexicographically according to the neighbor vectors, so that $u \in B_{i+1,j}$ and $v \in B_{i+1,h}$ with $j < h$ imply $n^i(u) >_{\text{lex}} n^i(v)$.
3.5. Set $i = i + 1$ and continue Step 3.

4. Set $k = k_i$ and $B_j = B_{ij}$ for $j = 1, 2, \ldots, k$.

Note that the first round distributes vertices into classes according to their degrees, and so $k_1 = 1$ and $B_{11} = V(G)$ for regular graphs, and the degree partition of a regular graph consists of a single block. Note also that this procedure does not only compute the degree partition, but also gives a unique ordering of the blocks of the degree partition (e.g., B_{11} contains the vertices of the maximum degree and B_{1,k_1} the vertices of the minimum degree).

Having the degree partition, let $r_{ij} = |N(u) \cap B_j|$ for any $u \in B_i$, for $1 \leq i, j \leq k$. The *degree refinement matrix* M_G of the graph G is the k by k square matrix

$$M_G = (r_{ij})_{i,j=1}^k.$$

The unique ordering of blocks of the degree partition implies that the degree refinement matrix is defined uniquely as well.

For purposes of the proof of Proposition 2.3, we introduce a lemma, which glues together the execution of the refining procedure on graphs G and H, which share a common degree refinement matrix.

Lemma 2.1. *Suppose graphs G and H have the same degree refinement matrix and let $B_j(G)$, $B_j(H)$, $j = 1, 2, \ldots, k$ be the degree partitions. Then for every $i \geq 0$ and $j, 1 \leq j \leq k_i$, there exists $X_{ij} \subseteq \{1, 2, \ldots, k\}$, such that $B_{ij}(G) = \bigcup_{h \in X_{ij}} B_h(G)$ and $B_{ij}(H) = \bigcup_{h \in X_{ij}} B_h(H)$. Moreover, $n^i_G(u) = n^i_H(v)$ for every $i \geq 0$, whenever $t_G(u) = t_H(v)$.*

Proof. We prove the statement by induction.

For $i = 0$, $X_{01} = \{1, 2, \ldots, k\}$. Let $u \in B_t(G)$ and $v \in B_t(H)$ be vertices of the same weight $t = t_G(u) = t_H(v)$. Then $\deg_G(u) = \sum_{j=1}^k r_{tj} = \deg_H(v)$ and hence $n^0(u) = (\deg_G(u)) = (\deg_H(v)) = n^0(v)$.

For $i > 0$ and vertices u, v of the same weight t, the assumption $n^{i-1}_G(u) = n^{i-1}_H(v)$ and the fact that the final neighbor vectors of u and v are equal (being the t^{th} row of the degree refinement matrix) imply $n^i_G(u) = n^i_H(v)$.

Then the existence of X_{ij} follows from the equality $(n^i_G(u))_j = (n^i_H(v))_j = \sum_{h \in X_{ij}} r_{th}$. ■

The following theorem is folklore:

Proposition 2.2. *If a graph G covers a connected graph H, then $M_G = M_H$.*

It is obvious that for regular graphs of the same valency, a locally injective homomorphism is a local isomorphism, i.e., every partial covering projection is a full cover. An analogous statement is true for graphs with the same degree refinement matrix:

Proposition 2.3 [8]**.** *If connected graphs G and H have the same degree refinement matrix, then every partial covering projection of G to H is also a full covering projection.*

Proof. Let f be a partial covering projection of G to H. For every vertex $u \in V(G)$, $\deg_G(u) \le \deg_H(f(u))$ follows from local injectivity of f.

We first prove that $t_G(u) \ge t_H(f(u))$ for every $u \in V(G)$. As in the proof of Lemma 2.1, run the refinement procedure simultaneously on G and H. Let $t^i_G(u)$ denote the weight of vertex u in the partition of G after the i-th run of Step 3, and similarly for H. We prove by induction on i that $t^i_G(u) \ge t^i_H(f(u))$ for every $u \in V(G)$.

For $i = 0$, $t^0_G(u) = t^0_H(f(u)) = 1$, since $k_0(G) = k_0(H) = 1$. Suppose $i > 0$. The weights of u and $f(u)$ depend on their neighbor vectors $n^{i-1}_G(u)$ and $n^{i-1}_H(f(u))$. For every neighbor z of u in G, $f(z)$ is a neighbor of $f(u)$ in H and, by induction hypothesis, $t^{i-1}_G(z) \ge t^{i-1}_H(f(z))$. Hence in the lexicographic ordering $n^{i-1}_G(u) \le_{\text{lex}} n^{i-1}_H(f(u))$. By Step 3.4, this means that $t^i_G(u) \ge t^i_H(f(u))$. Note that here we are implicitly using Lemma 2.1, as we are comparing neighbor vectors in G and H.

For $u \in B_1(G)$, $t_G(u) = 1 \ge t_H(f(u))$ and hence $t_H(f(u)) = 1$ as well. Thus vertices from the block $B_1(G)$ are mapped to vertices from $B_1(H)$. For the rest of the proof let v be an arbitrary vertex from $B_1(G)$.

Suppose there exists a vertex $u \in V(G)$, for which $t_G(u) > t_H(f(u))$. Consider a path from v to u in G. This path must contain an edge $(v', u') \in E(G)$, such that $t_G(v') = t_H(f(v'))$ and $t_G(u') > t_H(f(u'))$. Since v' and $f(v')$ have the same number of neighbors of each weight, this implies the existence of a neighbor w' of v' for which $t(w') < t(f(w'))$, a contradiction.

Now for all vertices $u \in V(G)$ we have $t_G(u) = t_H(f(u))$, and in particular $\deg_G(u) = \deg_H(f(u))$. Hence, f is a local epimorphism, and thus a full covering projection of G onto H. ∎

The consequence for the complexity of partial covers follows:

Corollary 2.4. *For any connected graph H, the H-cover problem is polynomially reducible to the H-partial cover problem.*

Proof. Given a graph G subject to the question if G covers H, determine (in polynomial time) if G has the same degree refinement matrix as H. If not, G cannot cover H. In the affirmative case, G partially covers H, if and only if it covers H fully. ∎

So all NP-completeness results about graph covers carry on to partial covers. We certainly do not wish to restate here all the results of [1, 17, 18, 19, 8], but let us mention two results which have consequence for the circular channel assignment problem of Leese [20], discussed in Section 3.

Proposition 2.5 [17]**.** *The H-cover problem is NP-complete when the graph H is k-regular $\lfloor \frac{k+1}{2} \rfloor$-edge connected or k-regular k-edge-colorable, for every $k \geq 3$.*

Moreover, Proposition 2.5 can be extended to the class of all k-regular graphs with $k \geq 3$:

Theorem 2.6 [8]**.** *The H-cover problem is NP-complete for all k-regular graphs H with $k \geq 3$.*

Proof. Without loss of generality we assume that H is connected and that H is not bipartite, since bipartite k-regular graphs by König-Hall marriage theorem are k-edge colorable and hence Proposition 2.5 would apply.

The Kronecker double cover $\tilde{H} = H \times K_2$ [1] is k-edge colorable k-regular connected graph and hence the $\tilde{H}$-cover problem is NP-complete due to Proposition 2.5.

We show a reduction of the $\tilde{H}$-cover problem to the H-cover problem. Consider a graph G whose covering projection $G \to \tilde{H}$ is questioned. We claim that G covers $\tilde{H}$ if and only if G is bipartite and G covers H.

[1] $V(\tilde{H}) = V(H) \times \{0,1\}, E(\tilde{H}) = \{((u,1),(v,0)),((u,0),(v,1))) : (u,v) \in E(H)\}$

The only if statement is trivial since $\tilde{H}$ is bipartite and only bipartite graphs can cover a bipartite graph (this holds even for a general graph homomorphism). Moreover any covering projection $G \to \tilde{H}$ can be extended to H by a composition with a covering projection $\tilde{H} \to H$.

In the other direction, assume that $f : G \to H$ is a covering projection and that G is bipartite, and a proper bicoloring using black and white colors is given. For each vertex u of H denote by $(u, 0)$ and $(u, 1)$ its two copies in $u \times K_2 \subset H \times K_2 = \tilde{H}$. We define a mapping $\tilde{f} : G \to \tilde{H}$ by $\tilde{f}(v) = (u, 0)$ if $f(v) = u$ and v is white, and $\tilde{f}(v) = (u, 1)$ if $f(v) = u$ and v is black.

Since each vertex has all neighbors colored by the complementary color, the above mentioned mapping satisfies all properties of a covering projection. ■

Corollary 2.7. *The H-partial cover problem is NP-complete for all k-regular graphs, $k \geq 3$.*

All graphs H with at most one cycle in each component of connectivity are polynomial instances both for the H-cover and H-partial cover problems, hence we get that the computational complexity for both these problems is completely classified for regular graphs H.

3. Generalized Distance Two Colorings of Graphs

Roberts proposed the following *distance two constrained labelings of graphs*, a notion stemming from the radio frequency assignment problem, where the task is to assign radio frequencies to transmitters at different locations without interference. Assuming that the distance function of the transmitters can be modelled by a graph distance, it is asked that transmitters which are close to each other receive different channels, and transmitters that are *very* close together receive channels that are at least two apart:

A $\lambda_{(2,1)}$-*labeling* of a graph G is an assignment of labels from the set $\{0, \ldots, \lambda\}$ to the vertices of G such that vertices at distance two are assigned different labels and adjacent vertices are assigned labels which differ by at least 2. The minimum value λ, for which G admits a $\lambda_{(2,1)}$-labeling, is denoted by $\lambda_{(2,1)}(G)$.

An upperbound for $\lambda_{(2,1)}(G)$ in terms of the maximum degree $\Delta(G)$ ($\lambda_{(2,1)}(G) \leq \Delta^2(G) + 2\Delta(G)$) was obtained in [11, 23] and this was improved in [5] to $\lambda_{(2,1)}(G) \leq \Delta^2(G) + \Delta(G)$. The conjecture if $\lambda_{(2,1)}(G) \leq \Delta^2(G)$ is still open, though it was proven true for some special graph classes (chordal graphs, graphs of diameter 2). From the complexity point of view, the problem to decide if a given graph allows a $\lambda_{(2,1)}$-labeling was proven NP-complete in [11, 23] if λ is part of the input, and in [10] for every fixed $\lambda \geq 4$.

A natural generalization of the distance two constrained labeling problem is considering channel (frequency) spaces with nonlinear metric. This would model the case when frequencies which are multiples can also interfere or when a fixed number of frequencies has to be assigned to every transmitter. The circular metric in the channel space was considered by Heuvel, Leese and Shepherd [16, 20]. Assuming that the distance function in the channel space can be modelled by a graph theoretical distance in a graph, whose vertices are the possible channels (frequencies), we arrive to a natural generalization of the concept of $\lambda_{(2,1)}$-labelings:

Definition 3.1. Let H be a graph. An $H_{(2,1)}$*-labeling* of a graph G is a mapping $f : V(G) \to V(H)$ which satisfies

1. $\mathrm{dist}_H(f(u), f(v)) \geq 2$ for every two adjacent vertices $u, v \in V(G)$;
2. $\mathrm{dist}_H(f(u), f(v)) \geq 1$ for every two vertices $u, v \in V(G)$, such that $\mathrm{dist}_G(u, v) = 2$,
   where the distance $\mathrm{dist}(u, v)$ is the number of edges of the shortest path connecting vertices u and v.

The notion of $H_{(2,1)}$-labelings straightforwardly relates to partial covers of graphs.

Observation 3.2. *A mapping $f : V(G) \to V(H)$ is an $H_{(2,1)}$-labeling, if and only if it is a partial covering projection of G to $\overline{H}$, the complement of the graph H.*

Proof. Recall that the local injectivity for a partial covering projection f from the graph G to the graph $\overline{H}$ can be expressed by the following conditions:

1. $(f(u), f(v)) \in E(\overline{H})$ for each edge $(u, v) \in E(G)$;
2. $f(u) \neq f(v)$ for any two distinct vertices $u, v \in V(G)$ which have a common neighbor in G.

Now, consider a mapping $f : V(G) \to V(H)$. For any two vertices $x, y \in V(H)$, the distance $\mathrm{dist}_H(x, y) \geq 1$ if and only if $x \neq y$, and so conditions

(2.) of Definition 3.1 and above are equivalent. The condition (1.) of Definition 3.1 states that the existence of an edge $(u, v) \in E(G)$ implies $(f(u), f(v)) \notin E(H)$, that is $(f(u), f(v)) \in E(\overline{H})$, which is exactly the condition (1.) presented here for the complement of H. ∎

Obviously, a graph G allows a $\lambda_{(2,1)}$-labeling if and only if it allows a $(P_\lambda)_{(2,1)}$-labeling (here P_k denotes the path with k edges). Similarly, G allows a circular $\lambda_{(2,1)}$-labeling, if and only if it allows a $(C_{\lambda+1})_{(2,1)}$-labeling (C_k denotes the cycle with k edges).

We have shown in [10] that deciding existence of $(P_\lambda)_{(2,1)}$-labeling is NP-complete for every fixed $\lambda \geq 4$. Here we have a similar result for the circular metric:

Theorem 3.3. *The $(C_{\lambda+1})_{(2,1)}$-labeling problem is NP-complete for every $\lambda \geq 5$.*

Proof. Since the complement of C_k is $(k-3)$-regular and $(k-3)$-edge-connected, the NP-completeness of $\overline{C_k}$-cover (for $k \geq 6$) follows from Proposition 2.5. By Corollary 2.4, the $\overline{C_k}$-partial cover problem is NP-complete, and Theorem 3.3 follows. ∎

In view of this observation, we may ask for the most detailed computational complexity characterization of partial covers of graphs, having in mind its impact on distance two constrained graph labelings.

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