Discussiones Mathematicae
Graph Theory 22 (2002) 39–50

SOME NEWS ABOUT OBLIQUE GRAPHS

Andrey A. Dobrynin and Leonid S. Melnikov*

Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Science
Novosibirsk 630090, Russia

e-mail: dobr@math.nsc.ru
e-mail: omeln@math.nsc.ru

Jens Schreyer and Hansjoachim Walther

Technische Universität Ilmenau
Germany

e-mail: jens.schreyer@tu-ilmenau.de
e-mail: hansjoachim.walther@tu-ilmenau.de

Abstract

A k-gon α of a polyhedral graph $G(V, E, F)$ is of *type* $\langle b_1, b_2, \ldots, b_k \rangle$ if the vertices incident with α in cyclic order have degrees $b_1, b_2, \ldots, b_k$ and $\langle b_1, b_2, \ldots, b_k \rangle$ is the lexicographic minimum of all such sequences available for α. A polyhedral graph G is *oblique* if it has no two faces of the same type. Among others it is shown that an oblique graph contains vertices of degree 3.

1. Introduction

In a polyhedral graph $G = G(V, E, F)$ with the vertex set $V = V(G)$, the edge set $E = E(G)$, and the face set $F = F(G)$ the number $d(x)$ of edges (= number of faces) incident with $x \in V$ is the *degree* of x. The number $d(\alpha)$ of edges (= number of vertices) incident with $\alpha \in F$ is the *degree* of α. α is a $d(\alpha)$-gon.

*The work of this author was supported by grants from Russian Foundation of Fundamental Research (project codes 99-01-00581 and 00-07-90296) and INTAS (project code 97-1001).

If $x \in V$ is incident with $\alpha \in F$ we write $x \in \alpha$.
$V_i := \{x \in V : d(x) = i\}$, $i = 3, 4, \ldots$ is the set of vertices of degree i and $v_i = |V_i|$ its cardinality. An edge $xy \in E : x, y \in V$ with $d(x) \leq d(y)$ is of *type* $\langle d(x), d(y) \rangle$. $\alpha \in F(G)$ is an $\langle a_1, a_2, \ldots, a_l \rangle$-face if α is an l-gon and the degree $d(x_i)$ of the vertex x_i incident with α is a_i, $i = 1, 2, \ldots, l$ in the cyclic order. Obviously, α is also an $\langle a_2, a_3, \ldots, a_l, a_1 \rangle$-face, an $\langle a_3, a_4, \ldots, a_l, a_1, a_2 \rangle$-face,..., and an $\langle a_l, a_{l-1}, \ldots, a_2, a_1 \rangle$-face, too.

The lexicographic minimum $\langle b_1, b_2, \ldots, b_l \rangle : \alpha$ is a $\langle b_1, b_2, \ldots, b_l \rangle$-face is called the *type* of α. For a triangle α of type $\langle a, b, c \rangle$ we have $a \leq b \leq c$. A polyhedral graph G is called *oblique* if all its faces are of different type. G is *superoblique* if both G and its dual G^* are oblique and they have no common face type. Let $z \geq 1$ be any given natural number. A polyhedral graph G is *z-oblique* if $F(G)$ contains at most z faces of the same type for any type of faces. Obviously, a 1-oblique graph is oblique and vice versa. A polyhedral graph is a *triangulation* if all its faces are triangles. B. Grünbaum and C.J. Shephard [2] listed all face transitive polyhedral graphs. Such graphs have only one type of faces. In [4] it has been shown that besides the face transitive polyhedral graphs there is exactly one polyhedral graph with only one type of faces (see Figure 1).

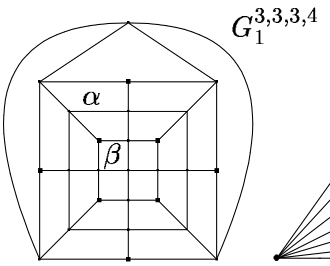

Figure 1

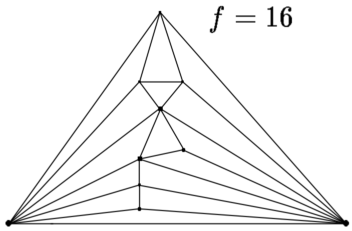

Figure 2

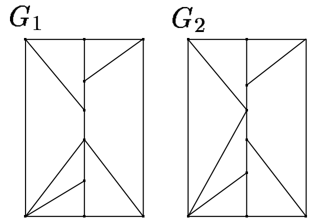

Figure 3

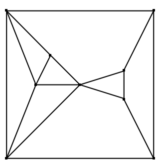
Figure 4

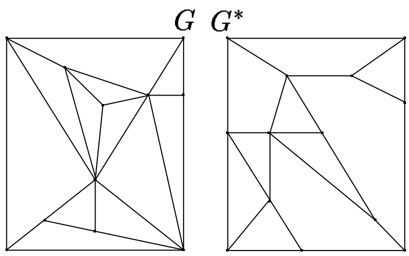

Figure 5

Moreover, it has been shown that the set of oblique triangulations is finite but not empty, and an oblique triangulation contains a vertex of degree 3.

An example of an oblique triangulation has been drawn in Figure 2. In [3] it has been proved that for any z the set of z-oblique graphs is finite. From a result by O. Borodin [1] it follows that an oblique graph contains a vertex of degree 3 or 4.

2. RESULTS

Theorem 1. *An oblique graph contains a vertex of degree* 3.

Theorem 2. *An oblique graph contains at least eight faces. There are exactly two non isomorphic oblique graphs with eight faces* (*see Figure* 3).

Theorem 3. *There are selfdual oblique graphs* (*see Figure* 4) *and super-oblique graphs as well* (*see Figure* 5).

Theorem 4. *An oblique triangulation G contains vertices with at least six different degrees, and G contains at least sixteen faces. There is exactly one such triangulation with sixteen faces.*

Proof of Theorem **1.** Let us suppose there is an oblique graph $G^\Delta = G^\Delta(V^\Delta, E^\Delta, F^\Delta)$ with maximum degree Δ without vertices of degree 3.

For any polyhedral graph G we define the *charge* $w(\alpha)$ of a face $\alpha \in F(G)$ in the following way:

$$w(\alpha) := 2(d(\alpha) - 3) + \sum_{x_i \in \alpha} \frac{d(x_i) - 6}{d(x_i)},$$

From Euler's polyhedral formula it is easy to see that

$$w(G) := \sum_{\alpha \in F(G)} w(\alpha) = -12. \tag{1}$$

In the following we will show the contradiction $w(G^\Delta) > -12$ for any oblique graph G^Δ without vertices of degree 3.

$$F_i := F_i(G^\Delta) := \{\alpha \in F^\Delta : \max_{x \in \alpha} d(x) = i\}, \quad i = 3, 4, \ldots, \Delta,$$

$$F^- := F^-(G^\Delta) := \{\alpha \in F^\Delta : w(\alpha) < 0\},$$

$$F_i^- := F_i^-(G^\Delta) := F^- \cap F_i, \quad i = 3, 4, \dots, \Delta,$$
$$w(F_i^-) := w(F_i^-(G^\Delta)) := \sum_{\alpha \in F_i^-} w(\alpha), \quad i = 3, 4, \dots, \Delta.$$

Remark. F^- consists only of triangles, because any l-gon $\beta \in F^\Delta : l \geq 4$ has a non negative charge. By $w\langle x, y, z\rangle$ we denote the charge $w(\alpha)$ of a face α of type $\langle x, y, z\rangle$.

Any type of faces is present in an oblique graph at most once, so we have:

$$F_3^- = \emptyset, \quad w(F_3^-) = 0, \text{ as there is no vertex of degree 3,}$$
$$F_4^- \subseteq \{\langle 4, 4, 4\rangle\}, \quad w(F_4^-) \geq -\frac{3}{2},$$
$$F_5^- \subseteq \{\langle 4, 4, 5\rangle, \langle 4, 5, 5\rangle, \langle 5, 5, 5\rangle\}, \quad w(F_5^-) \geq -\frac{27}{10}, \dots$$

(for $w(F_i^-)$: $4 \leq i \leq 15$ see table, 5 column).

Lemma 1. $\Delta \geq 9$.

Proof. For $\Delta \leq 8$ (see table, last column) we have

$$w(G^\Delta) \geq \sum_{j=4}^{\Delta} w(F_j^-) \geq \sum_{j=4}^{8} w(F_j^-) \geq -\frac{1541}{140} > -12,$$

a contradiction to (1). ∎

Remark. Let $\beta \in F_\Delta$ be an l-gon ($l \geq 4$) incident with a vertex of maximum degree Δ. Then $w(\beta) \geq w\langle 4, 4, 4, \Delta\rangle \geq 2 - \frac{3}{2} + \frac{\Delta-6}{\Delta} = \frac{3}{2} - \frac{6}{\Delta} \geq \frac{5}{6}$.

$$S^\Delta := S^\Delta(G^\Delta) := \sum_{\alpha \in F^\Delta} w(\alpha)$$

is the sum of the charges of all faces $\alpha \in F^\Delta$ incident with at least one vertex of maximum degree Δ.

In the third column of the table it can be found the charge contribution $w_i(a, b)$ of vertices x and y with degrees $a = d(x)$, $b = d(y)$, resp. to the charge $w(\alpha)$ in a triangle $\alpha = (x, y, z)$ of type $\langle a, b, c\rangle$ (increasing, up to

$i = 18$). Since there are at least Δ face types containing a vertex of degree Δ it is easy to see that

$$\begin{aligned}
&S^9 \geq w\langle 4,4,9\rangle + w\langle 4,5,9\rangle + w\langle 4,6,9\rangle + w\langle 5,5,9\rangle + w\langle 4,7,9\rangle \\
&+ w\langle 4,8,9\rangle + w\langle 5,6,9\rangle + w\langle 4,9,9\rangle + w\langle 5,7,9\rangle \\
&\geq 7\left(-\frac{1}{2}\right) + 5\left(-\frac{1}{5}\right) + 2(0) + 2\left(\frac{1}{7}\right) + 1\left(\frac{2}{8}\right) + 10\left(\frac{3}{9}\right) = -\frac{53}{84} > -1, \\
&S^{10} \geq \frac{113}{420} > 0,\ S^{11} \geq \frac{5653}{4620} > 1,\ S^{12} \geq \frac{10273}{4620} > 2,\ S^{13} \geq \frac{14893}{4620} > 3, \ldots
\end{aligned}$$

Lemma 2. $S^\Delta > \Delta - 10$.

Proof. For $\Delta \in \{9, \ldots, 13\}$ it is shown above. For $j \geq 12$ we have $w_j(a,b) \geq 0$ (see table, third column), consequently $\{\sum_{j=1}^{i} w_j(a,b)\}_{i=11,12,\ldots}$ is an increasing sequence. That means

$$\sum_{j=1}^{i} w_j(a,b) \geq -\frac{17447}{4620} > -4 \text{ for all } i = 1, 2, \ldots$$

In each of the Δ faces α incident with $z \in V^\Delta$ the vertex z contributes $\frac{\Delta-6}{\Delta}$ to the charge of α, therefore $S^\Delta \geq \sum_{j=1}^{\Delta} w_j(a,b) + \Delta\frac{\Delta-6}{\Delta} > -4 + (\Delta - 6) = \Delta - 10$. ■

Lemma 3. $\Delta \leq 12$.

Proof. For $\Delta \geq 13$ we have

$$w(G^\Delta) = \sum_{\alpha \in F^\Delta} w(\alpha) \geq S^\Delta + \sum_{j=4}^{\Delta-1} w(F_j^-).$$

The sequence $\{w(F_i^-)\}_{i=6,7,\ldots}$ is increasing, and together with Lemma 2 we obtain

$$\begin{aligned}
w(G^\Delta) &\geq S^\Delta + \sum_{j=4}^{12} w(F_j^-) + \sum_{j=13}^{\Delta-1} w(F_j^-) \\
&\geq \Delta - 10 - \frac{68533}{4620} + (\Delta - 13)\left(-\frac{81}{130}\right)
\end{aligned}$$

$$\geq \frac{49}{130}\Delta - \frac{19}{10} - \frac{68533}{4620}$$
$$\geq \frac{49}{130} \cdot 13 - \frac{19}{10} - \frac{68533}{4620} > -12,$$

a contradition to (1). ∎

Lemma 4. $\Delta \geq 10$.

Proof. Supposing $\Delta = 9$. The 9 faces α_μ incident with a vertex z of degree 9 have together a charge $\geq -\frac{53}{84}$ because the case $i = 9$ (see table) does not occur. So we have

$$w(G^9) \geq S^9 + \sum_{j=4}^{8} w(F_j^8) \geq -\frac{53}{84} - \frac{1541}{140} > -12,$$

a contradiction to (1). ∎

(i)	a b	$w_i(a,b)$	$\sum_{j=1}^{i} w_j(a,b)$	$w(F_i^-)$	$W_i^- = \sum_{j=4}^{i} w(F_j^-)$
(1)	4 4	-1	$-1 \quad = -1$		
(2)	4 5	$-\frac{7}{10}$	$-\frac{17}{10} \quad = -1.7$		
(3)	4 6	$-\frac{5}{10}$	$-\frac{22}{10} \quad = -2.2$		
(4)	5 5	$-\frac{2}{5}$	$-\frac{26}{10} \quad = -2.6$	$-\frac{3}{2}$	$-\frac{3}{2}$ = -1.5
(5)	4 7	$-\frac{5}{14}$	$-\frac{207}{70} \quad = -2.9571\ldots$	$-\frac{27}{10}$	$-\frac{42}{10}$ = -4.2
(6)	4 8	$-\frac{1}{4}$	$-\frac{449}{140} \quad = -3.2071\ldots$	$-\frac{28}{10}$	$-\frac{70}{10}$ = -7.0
(7)	5 6	$-\frac{1}{5}$	$-\frac{477}{140} \quad = -3.4071\ldots$	$-\frac{161}{70}$	$-\frac{651}{70}$ = -9.3
(8)	4 9	$-\frac{1}{6}$	$-\frac{1501}{420} \quad = -3.5738\ldots$	$-\frac{239}{140}$	$-\frac{1541}{140}$ = -11.007...
(9)	4 10	$-\frac{1}{10}$	$-\frac{1543}{420} \quad = -3.6738\ldots$	$-\frac{542}{420}$	$-\frac{5165}{420}$ = -12.297...
(10)	5 7	$-\frac{2}{35}$	$-\frac{1567}{420} \quad = -3.7309\ldots$	-1	$-\frac{5585}{420}$ = -13.297...
(11)	4 11	$-\frac{1}{22}$	$-\frac{17447}{4620} \quad = -3.7764\ldots$	$-\frac{46}{55}$	$-\frac{65299}{4620}$ = -14.133...
(12)	4 12	0	$-\frac{17447}{4620} \quad = -3.7764\ldots$	$-\frac{7}{10}$	$-\frac{68533}{4620}$ = -14.833...
(13)	6 6	0	$-\frac{17447}{4620} \quad = -3.7764\ldots$	$-\frac{81}{130}$	$-\frac{928351}{60060}$ = -15.457...
(14)	4 13	$+\frac{1}{26}$	$-\frac{224501}{60060} \quad = -3.7379\ldots$	$-\frac{39}{70}$	$-\frac{961813}{60060}$ = -16.014...
(15)	5 8	$+\frac{1}{20}$	$-\frac{110749}{30030} \quad = -3.6879\ldots$	$-\frac{1}{2}$	$-\frac{991843}{60060}$ = -16.514...
(16)	4 14	$+\frac{1}{14}$	$-\frac{108604}{30030} \quad = -3.6165\ldots$	$> -\frac{1}{2}$	... ...
(17)	4 15	$+\frac{1}{10}$	$-\frac{105601}{30030} \quad = -3.5165\ldots$	$> -\frac{1}{2}$	... ...
(18)	4 16	$+\frac{1}{8}$	$-\frac{407389}{120120} \quad = -3.3915\ldots$	$> -\frac{1}{2}$	... ...

The three cases $\Delta \in \{10, 11, 12\}$ remain.

Lemma 5. *For each $k \in \{9, 10, \ldots, \Delta - 1\}$ the graph G^{Δ} contains a vertex y^k of degree k.*

A simple calculation yields the proof, we will show it in case $\Delta = 12$:
If there is no vertex of a fixed degree $k \in \{9, 10, 11\}$ we have $F_k^- = \emptyset$ therefore $w(F_k^-) = 0$. Having a look into the table (fifth column) we obtain

$$\sum_{j=4}^{11} w(F_j^-) \geq -\frac{65299}{4620} + \frac{46}{55} = -\frac{5585}{420},$$

together with Lemma 2 we obtain

$$w(G^{12}) \geq S^{12} + \sum_{j=4}^{11} w(F_j^-) \geq 2 - \frac{5585}{420} > -12.$$

The cases $\Delta \in \{10, 11\}$ can be settled in an analogous way ∎

Now, let us complete the proof of Theorem 1:
In case of $\Delta = 12$ there are 12 faces incident with a fixed vertex z of degree 12, moreover there are at least 9 faces incident with a vertex u of degree 11 but not incident with z. With the help of our table we obtain

$$w(G^{12}) \geq S^{12} + \sum_{i=1}^{9} w_i(a,b) + 9\left(\frac{5}{11}\right) + \sum_{i=1}^{10} w(F_i^-)$$

$$\geq 2 - \frac{1543}{420} + \frac{45}{11} - \frac{5585}{420} > -12.$$

The cases $\Delta \in \{10, 11\}$ can be settled in a similar way using $S^{11} \geq \frac{5653}{4620}$ and $S^{10} \geq \frac{113}{420}$, resp. This completes the proof of Theorem 1. ∎

***Proof of Theorem* 2.** The correctness of Theorem 2 has been shown by computers. It is easy but a little bit costly to prove it without computers. ∎

***Proof of Theorem* 3.** It is easy to see that the graph of Figure 4 is a selfdual oblique one as well as the graph of Figure 5 is a superoblique one. ∎

***Proof of Theorem* 4.** Let $G = G(V, E, F)$ be an oblique triangulation with $p \leq 5$ different degrees. ■

Lemma 6. *There is no edge* $xy \in E : d(x) = d(y) = 3$.

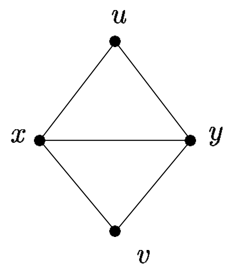

Figure 6

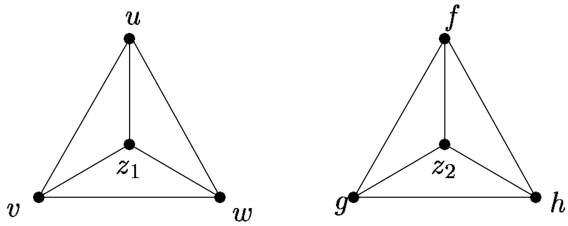

Figure 7

Proof. Otherwise, because of the triangulation, there are two edges $uv \in E$ (see Figure 6), but a polyhedral graph has no multiple edges. ■

Lemma 7. *G contains exactly one vertex z of degree* 3.

Proof. Because of *Theorem 1* there is at least one vertex of degree 3. Provided that there are at least two vertices z_1, z_2 of degree 3 (see Figure 7). The three vertices u, v, w incident with z_1 as well as the three vertices f, g, h incident with z_2 have pairwise different degrees because of the obliqueness of G.

Moreover, with the same argument the set $\{u, v, w, f, g, h\}$ consists of at least 5 vertices and in this set there are at least 5 different degrees. Together with the degree 3 of the vertices z_1, z_2 there are at least 6 different degrees in G. ■

Case 1. $p = 4$.
Let $3 < a < b < \Delta$ be the four different degrees in G with $\Delta \geq 6$. The following face types may occur:
$\langle 3, a, b\rangle, \langle 3, a, \Delta\rangle, \langle 3, b, \Delta\rangle, \langle a, a, a\rangle, \langle a, a, b\rangle, \langle a, a, \Delta\rangle,$
$\langle a, b, b\rangle, \langle a, b, \Delta\rangle, \langle a, \Delta, \Delta\rangle, \langle b, b, b\rangle, \langle b, b, \Delta\rangle, \langle b, \Delta, \Delta\rangle, \langle \Delta, \Delta, \Delta\rangle.$
There are only 5 face types containing exactly one vertex of degree Δ, but a vertex of degree Δ is incident with 6 faces. That means there are at least two vertices of degree Δ. Consequently there are at least 6+4=10 faces incident with a vertex of degree Δ, but there are only 8 face types containing Δ, a contradiction.

Case 2. $p = 5$.

Let $3 < a < b < c < \Delta$ be the five different degrees in G. The following face types containing a vertex of degree Δ may be present:
$\langle 3, a, \Delta\rangle, \langle 3, b, \Delta\rangle, \langle 3, c, \Delta\rangle, \langle a, a, \Delta\rangle, \langle a, b, \Delta\rangle, \langle a, c, \Delta\rangle, \langle a, \Delta, \Delta\rangle,$
$\langle b, b, \Delta\rangle, \langle b, c, \Delta\rangle, \langle b, \Delta, \Delta\rangle, \langle c, c, \Delta\rangle, \langle c, \Delta, \Delta\rangle, \langle \Delta, \Delta, \Delta\rangle.$
Because of Lemmata 6 and 7 there occur at most two of the three face types $\langle 3, a, \Delta\rangle, \langle 3, b, \Delta\rangle, \langle 3, c, \Delta\rangle$.

Case 2.1. $\Delta \geq 9$.

There are only 8 face types containing exactly one Δ, therefore there are at least two vertices of degree Δ in G. Then there are at least 9+7=16 faces incident with a vertex of degree Δ, but G contains at most 12 face types with a Δ-degree vertex, a contradiction.

Case 2.2. $\Delta = 8$.

With arguments like Case 2.1 there is exactly one vertex of degree $\Delta = 8$ in G, and all the 8 face types
$\langle 3, \phi, \Delta\rangle, \langle 3, \psi, \Delta\rangle, \langle a, a, \Delta\rangle, \langle a, b, \Delta\rangle, \langle a, c, \Delta\rangle, \langle b, b, \Delta\rangle, \langle b, c, \Delta\rangle, \langle c, c, \Delta\rangle,$
with $\phi \neq \psi$ and $\{\phi, \psi\} \subset \{a, b, c\}$ occur in G.
The edge (ϕ, Δ) occurs in the 8 face types (and therefore in G) exactly 5 times, but this number has to be an even one, a contradiction.

Case 2.3. $\Delta = 7$.

The $p = 5$ different degrees are $3 < 4 < 5 < 6 < 7 = \Delta$. The following nine face types with one 7-degree vertex are conceivable:
$\langle 3, 4, 7\rangle, \langle 3, 5, 7\rangle, \langle 3, 6, 7\rangle, \langle 4, 4, 7\rangle, \langle 4, 5, 7\rangle, \langle 4, 6, 7\rangle, \langle 5, 5, 7\rangle, \langle 5, 6, 7\rangle, \langle 6, 6, 7\rangle.$
Since the vertex of degree 3 in G can be incident to at most one vertex of degree 7 only the three face types $\langle 4, 7, 7\rangle, \langle 5, 7, 7\rangle, \langle 6, 7, 7\rangle$ with two 7-degree vertices are conceivable, and $\langle 7, 7, 7\rangle$ is the only face type with three 7-degree vertices. 0 or 2 of the three types $\langle 3, 4, 7\rangle, \langle 3, 5, 7\rangle, \langle 3, 6, 7\rangle$ may occur in G, because this number has to be even.

Let q be the number of vertices of degree 7 in G.
If $q = 2$ we have in G at least 7+5=12 faces incident with a vertex of degree 7, but there are only 11 face types with one or two 7-degree vertices.
If $q \geq 3$ we have at least 7+5+3 faces incident with a vertex of degree 7 but only twelve face types with a 7-degree vertex.

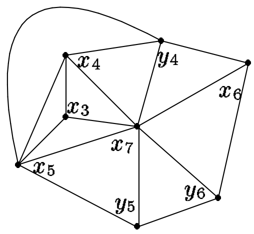

Figure 8a

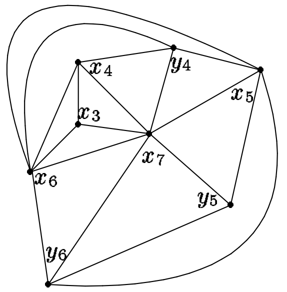

Figure 8b

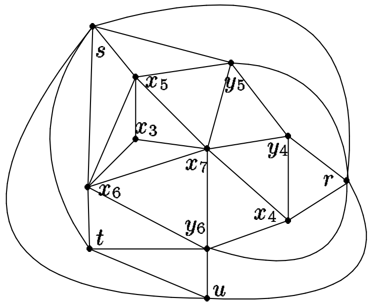

Figure 8c

If $q = 1$ exactly 7 of the 9 face types collected above are incident with the only vertex x_7 of degree 7.

Exactly two of the three types $\langle 3,4,7\rangle, \langle 3,5,7\rangle, \langle 3,6,7\rangle$ occur in G because if no of these face types occur in G we have at most 6 face types incident with the 7-degree vertex x_7.

We discuss the three possible cases (see Figure 8): The index of a vertex describes its degree.

Case 2.3.1. The neighbours x_4, x_5, x_7 of the 3-degree vertex x_3 have degrees 4,5,7, resp. (see Figure 8a).

Claim 1. $\langle 4,5,7\rangle$ does not occur in G. The edge type $\langle 4,7\rangle$ occurs 5 times in the following conceivable face types:
$\langle 3,4,7\rangle, \langle 4,4,7\rangle, \langle 4,5,7\rangle, \langle 4,6,7\rangle$. That means exactly one of the two face types $\langle 4,5,7\rangle, \langle 4,6,7\rangle$ does not occur.

With the same argument one of the two face types $\langle 4,5,7\rangle, \langle 5,6,7\rangle$ cannot occur. As we need seven face types with a 7-degree vertex in G the type $\langle 4,5,7\rangle$ does not occur.
The following seven face types occur:
$\langle 3,4,7\rangle, \langle 3,5,7\rangle, \langle 4,4,7\rangle, \langle 4,6,7\rangle, \langle 5,5,7\rangle, \langle 5,6,7\rangle, \langle 6,6,7\rangle$.
The only distribution of the face types around x_7 is drawn in Figure 8a.
$\Longrightarrow x_5 y_4 \in E(G)$, because x_4 is saturated. Now, y_4 and x_5 are saturated, in contradiction with the fact that G is a triangulation.

Case 2.3.2. The neighbours x_4, x_6, x_7 of the 3-degree vertex x_3 have degrees 4,6,7, resp. (see Figure 8b).

The only distribution of the face types around x_7 is drawn in Figure 8b. With arguments similar to the *Case* 2.3.1 there are edges $y_4x_6, x_5x_6, x_5y_6 \in E(G)$, and G is not 3-connected because there is a cut set x_5, y_6 of G consisting of less than three vertices, a contradiction.

Case 2.3.3. The neighbours x_5, x_6, x_7 of the 3-degree vertex x_3 have degrees 5,6,7, resp. (see Figure 8c). The only distribution of the face types around x_7 is drawn in Figure 8c.

1. There is a vertex r incident with x_4 and y_4. Now, x_4 and y_4 are saturated, consequently $(r, y_5), (r, y_6) \in E(G)$.

2. There is a vertex s with $(s, y_5), (s, x_5), (s, x_6) \in E(G)$. Now, y_5 is saturated, consequently $(r, s) \in E(G)$.

3. There is a vertex t with $(t, x_6), (t, y_6), (t, s) \in E(G)$.

4. There is a vertex u and $(u, t), (u, y_6), (u, r) \in E(G)$, because t must have a degree greater than 3. Now, r is saturated, because there is only one vertex of degree 7, consequently $(u, s) \in E(G)$. Now, s is saturated, G is constructed, otherwise G is only 1-connected. The arising graph has two faces of the same type $\langle 4, 4, 6 \rangle$, namely (u, t, y_6) and (r, x_4, y_4), a contradiction. This completes the proof of Theorem 4. ∎

Conclusions. With the help of computers it has been shown that there is an oblique triangulation with eighty four vertices and thirteen different degrees. Furthermore, the superoblique graph G of Figure 5 is the smallest one with regard to the number of vertices, and is unique in this sense.

Open problems.

1. What is the greatest number k such that there is an oblique graph with k different degrees?
2. The same question for oblique triangulations.
3. Is there an oblique graph without triangles?
4. Is there an oblique graph consisting of quadrangles only?
5. What is the greatest number $g(z)$ such that a z-oblique graph has $g(z)$ different degrees?

REFERENCES

[1] O. Borodin, *Structural properties of planar maps with the minimum degree* 5, Math. Nachr. **158** (1992) 109–117.

[2] B. Grünbaum and C.J. Shephard, *Spherical tilings with transitivity properties*, in: Geometrie (Springer-Verlag, 1982) 65–98.

[3] M. Voigt and H. Walther, *Polyhedral graphs with restricted number of faces of the same type*, Preprint No. M22/99, Technical University Ilmenau (submitted to Discr. Math.).

[4] H. Walther, *Polyhedral graphs with extreme numbers of types of faces*, Preprint No. M13/99, Technical University Ilmenau (submitted to Appl. Discr. Math.).

Received 29 January 2001
Revised 1 August 2001