Discussiones Mathematicae
Graph Theory 22 (2002) 31–37

CRITERIA FOR OF THE EXISTENCE OF UNIQUELY PARTITIONABLE GRAPHS WITH RESPECT TO ADDITIVE INDUCED-HEREDITARY PROPERTIES

Izak Broere

Department of Mathematics
Rand Afrikaans University
P.O. Box 524, Auckland Park, 2006 South Africa

e-mail: ib@na.rau.ac.za

Jozef Bucko

Department of Applied Mathematics
Faculty of Economics, Technical University
B. Němcovej, 040 01 Košice, Slovak Republic

e-mail: bucko@tuke.sk

and

Peter Mihók

Department of Applied Mathematics
Faculty of Economics, Technical University
B. Němcovej, 040 01 Košice, Slovak Republic
and
Mathematical Institute, Slovak Academy of Science
Grešákova 6, 040 01 Košice, Slovak Republic

e-mail: mihokp@tuke.sk

Abstract

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a graph G is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if there is exactly one (unordered) partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $G[V_i] \in \mathcal{P}_i$ for $i = 1, 2, \ldots, n$. We prove that for additive and induced-hereditary properties uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs exist if and only if $\mathcal{P}_i$ and $\mathcal{P}_j$ are either coprime or equal irreducible properties of graphs for every $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$.

Keywords: induced-hereditary properties, reducibility, divisibility, uniquely partitionable graphs.

2000 Mathematics Subject Classification: 05C15, 05C75.

1. Introduction

This paper is an extension of the paper [2] to induced-hereditary properties of graphs based on the Unique Factorization Theorem in the lattice of additive induced-hereditary properties (see [6]).

A *property* of graphs is any non-empty isomorphism-closed subclass of the class $\mathcal{I}$ of all simple finite graphs. A property $\mathcal{P}$ of graphs is said to be *induced-hereditary* (hereditary) if $H \leq G$ ($H \subseteq G$) and $G \in \mathcal{P}$ implies $H \in \mathcal{P}$ and *additive* if it is closed under disjoint union of graphs, i.e., if every component of G has property $\mathcal{P}$, then $G \in \mathcal{P}$, too. Obviously every hereditary property is induced-hereditary, but many important induced-hereditary properties ("perfect graphs", "claw-free graphs", "line-graphs", etc.) are not hereditary.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $G[V_i] \in \mathcal{P}_i$ for each $i = 1, 2, \ldots, n$. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ the class of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. A graph G of order at least n is said to be *uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable* if there is exactly one (unordered) $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. The class of all uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs will be denoted by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n)$.

The binary operation "$\circ$" of additive and hereditary properties have been considered in details in [2]. For technical reasons we consider also the *null-graph* $K_0 = (\emptyset, \emptyset)$, so that property $\Theta = \{K_0\}$ be the smallest property of graphs in the lattice $\mathbb{M}^a$ of all additive induced-hereditary properties partially ordered by set-inclusion (see [1]). The properties $\mathcal{I}$ and Θ are said to be *trivial*, since for every property $\mathcal{P} \in \mathbb{M}^a$, $\Theta \circ \mathcal{P} = \mathcal{P} \circ \Theta = \mathcal{P}$ and $\mathcal{I} \circ \mathcal{P} = \mathcal{P} \circ \mathcal{I} = \mathcal{I}$.

The notion of divisibility for the binary operation "$\circ$" on $\mathbb{M}^a$ is used in a natural way: Given any two graph properties $\mathcal{R}$ and $\mathcal{P}$, with $\mathcal{R}, \mathcal{P} \in \mathbb{M}^a$, we say that $\mathcal{P}$ is a *divisor* of $\mathcal{R}$, if $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ for some property $\mathcal{Q} \in \mathbb{M}^a$, we can also say that $\mathcal{P}$ *divides* $\mathcal{R}$ and that $\mathcal{R}$ is divisible by $\mathcal{P}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary properties we say that the additive induced-hereditary property $\mathcal{D} = gcd(\mathcal{P}, \mathcal{Q})$ is a *greatest common divisor* of $\mathcal{P}$ and $\mathcal{Q}$, if

(1) $\mathcal{D}$ divides $\mathcal{P}$ and $\mathcal{D}$ divides $\mathcal{Q}$;

(2) if $\mathcal{D}'$ divides $\mathcal{P}$ and $\mathcal{D}'$ divides $\mathcal{Q}$, then $\mathcal{D}'$ divides $\mathcal{D}$.

If $gcd(\mathcal{P}, \mathcal{Q}) = \Theta$, then we say that $\mathcal{P}$ and $\mathcal{Q}$ are *coprime*. A non-trivial additive induced-hereditary property $\mathcal{P}$ is said to be *irreducible*, if the only additive induced-hereditary properties which divide $\mathcal{P}$ are Θ and $\mathcal{P}$ itself and *reducible* otherwise.

These notions are well-defined since any additive induced-hereditary property can be expressed as a product of irreducible additive induced-hereditary properties:

Theorem 1 [6]**.** *Let $\mathcal{R} \in \mathbb{M}^a$ be a reducible property of graphs and suppose that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $n \geq 2$ is a factorization of $\mathcal{R}$ into irreducible factors. Then this factorization is unique* (*up to the order of the factors*)*.*

Hence any reducible property $\mathcal{R} \in \mathbb{M}^a$ can be written as $\mathcal{R} = {\mathcal{P}_1}^{e_1} \circ {\mathcal{P}_2}^{e_2} \circ \cdots \circ {\mathcal{P}_n}^{e_n}$, where $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are distinct irreducible properties and $e_1, e_2, \ldots, e_r$ are positive integers. Using the symbol $\mathcal{P}^0$ to denote the property Θ, one can clearly use this type of factorization to describe the greatest common divisor of any two properties similar to the way it is done in Number Theory.

The main result of this paper is:

Theorem 2. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be any non-trivial additive induced-hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph if and only if for each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ we have that $gcd(\mathcal{P}_i, \mathcal{P}_j) = \Theta$ or $\mathcal{P}_i = \mathcal{P}_j$ is an irreducible property.*

This result is an extension of the following Theorem of [6].

Theorem 3 [6]**.** *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ be a factorization of a reducible property $\mathcal{R} \in \mathbb{M}^a$ into irreducible factors. Then $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) \neq \emptyset$ and moreover if $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, then H is an induced subgraph of some uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph G.*

2. Preliminary Results

In the next sections we will consider non-trivial properties only. In the proof of our main result we will need some more notions and preliminary results.

Let $\mathcal{R}$ be an additive induced-hereditary property. For given graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, denote by

$$G_1 * G_2 * \cdots * G_n = \left\{ G : \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i \right\},$$

where $\bigcup_{i=1}^{n} G_i$ denotes the disjoint union and $\sum_{i=1}^{n} G_i$ the join of the graphs $G_1, G_2, \ldots, G_n$, respectively. A graph $G \in \mathcal{R}$ is said to be $\mathcal{R}$-*strict* if $G * K_1 \not\subseteq \mathcal{R}$.

The following Theorem describes the basic properties of the uniquely partitionable graphs (see [3, 4, 5]).

Theorem 4. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be any additive induced-hereditary properties of graphs, let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be the unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(G)$, $n \geq 2$. Then*

1. *$G \notin \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_{j-1} \circ \mathcal{P}_{j+1} \circ \cdots \circ \mathcal{P}_n$, for every $j = 1, 2, \ldots, n$,*
2. *the subgraphs $G[V_i]$ are $\mathcal{P}_i$-strict, $i = 1, 2, \ldots, n$,*
3. *for $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ the set $V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k}$ induces a uniquely $(\mathcal{P}_{i_1} \circ \mathcal{P}_{i_2} \circ \cdots \circ \mathcal{P}_{i_k})$-partitionable subgraph of G,*
4. *$\delta(G) \geq \max_{j} \sum_{i=1, i \neq j}^{n} \delta(\mathcal{P}_i)$.*

The next Lemmata will be used in the proof of our main result. They are based on the analogous result for uniquely colourable graphs (see [4]).

Lemma 1. *Let $\mathcal{P}_i \in \mathbb{M}^a$, for $i = 1, 2, \ldots n$. Let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and suppose that $\{V_1, V_2, \ldots, V_n\}$ is its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Then for every $j = 1, 2, \ldots, n$ the graph G_w^j obtained from G by adding a vertex w and edges joining w to vertices of the set $V_i, i \neq j$, such that $G[V_i \cup \{w\}] \notin \mathcal{P}_i$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$, is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1, \ldots, V_j \cup \{w\}, \ldots, V_n\}$ is its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition.*

Proof. Let G be any uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. By Theorem 4 it follows that the subgraphs $G[V_i]$ are $\mathcal{P}_i$-strict, $i = 1, 2, \ldots, n$, thus since $G[V_i] * K_1 \not\subseteq \mathcal{P}_i$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$ we can add edges between w and vertices of V_i so that $\{V_1, \ldots, V_j \cup \{w\}, \ldots, V_n\}$ be the only vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of G_w^j. ■

Using Lemma 1 we immediately have the following:

Lemma 2. *Let $\mathcal{P}_i \in \mathbb{M}^a$, for $i = 1, 2, \ldots n$. Let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Let $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $V(H) \cap V(G) = \emptyset$ and $\{W_1, W_2, \ldots, W_n\}$ be a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(H)$. Let the graph $G_H = (V(G) \cup V(H), E(G) \cup E(H) \cup E^*)$ be obtained from $G \cup H$ by adding edges so that for every $j = 1, 2, \ldots, n$ and for each $w \in W_j$ $G_H[V(G) \cup \{w\}] = G_w^j$, then G_H is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_n \cup W_n\}$ is its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition.*

Using the Unique Factorization Theorem 1, we have the following useful technical Lemma:

Lemma 3. *Let additive induced-hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ satisfy $\mathcal{P}_i \subseteq \mathcal{Q}_i$ for each $i = 1, 2, \ldots, n$, then $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$. If furthermore at least one of the inclusions $\mathcal{P}_i \subseteq \mathcal{Q}_i$ is strict (i.e., $\mathcal{P}_j \subset \mathcal{Q}_j$ for some j), then $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \subset \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$.*

Proof. The inclusion $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ follows easily by the definition of the product of properties. Let us suppose that for some $j \in \{1, 2, \ldots, n\}, \mathcal{P}_j \neq \mathcal{Q}_j$ and $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$. Then there are two different factorizations of the property $\mathcal{R}$ into irreducible factors, which contradicts Theorem 1. ■

3. Proof of the Main Result

We shall prove the Theorem 2 for $n = 2$ only, since the presented arguments can be repeated in the case $n \geq 3$, analogously.

Suppose there exists an irreducible property $\mathcal{Q}$ such that $\mathcal{P}_1 = \mathcal{Q} \circ \mathcal{P}_1'$ and $\mathcal{P}_2 = \mathcal{Q} \circ \mathcal{P}_2'$ and at least $\mathcal{P}_1' \neq \Theta$. Let $\{V_1, V_2\}$ be any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of a graph G. Since $\mathcal{P}_1 = \mathcal{Q} \circ \mathcal{P}_1'$ and $\mathcal{P}_2 = \mathcal{Q} \circ \mathcal{P}_2'$ let $\{V_{11}, V_{12}\}$ $(\{V_{21}, V_{22}\})$ be the

$(\mathcal{Q}, \mathcal{P}_1^{'})$-partition ($(\mathcal{Q}, \mathcal{P}_2^{'})$-partition) of $G[V_1]$ ($G[V_2]$). Then for every graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2$ we have at least the following two different $(\mathcal{P}_1, \mathcal{P}_2)$-partitions: $\{V_{11} \cup V_{12}, V_{21} \cup V_{22}\}$ and $\{V_{21} \cup V_{12}, V_{11} \cup V_{22}\}$ because we can assume that V_{11}, V_{21} and V_{12} are not empty.

To prove the converse, Theorem 3 can be applied if $\mathcal{P}_1 = \mathcal{P}_2$ is an irreducible property. Hence suppose that $gcd(\mathcal{P}_1, \mathcal{P}_2) = \Theta$. By Theorem 1 let $\mathcal{P}_1 = \mathcal{P}_{11} \circ \mathcal{P}_{12} \circ \cdots \circ \mathcal{P}_{1n}$ and $\mathcal{P}_2 = \mathcal{P}_{21} \circ \mathcal{P}_{22} \circ \cdots \circ \mathcal{P}_{2m}$ be the unique factorizations of $\mathcal{P}_1$ and $\mathcal{P}_2$ into irreducible factors. From our assumption that $gcd(\mathcal{P}_1, \mathcal{P}_2) = \Theta$ it follows that for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ $\mathcal{P}_{1i} \neq \mathcal{P}_{2j}$.

Let us take a uniquely $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partitionable graph G, which exists by Theorem 3. Let $\{V_{11}, \ldots, V_{1n}, V_{22}, \ldots, V_{2m}\}$ be the unique vertex $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partition of G. We shall now construct a $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partitionable graph H with an appropriate vertex partition $\{W_{11}, W_{12} \ldots, W_{1n}, W_{21}, W_{22}, \ldots, W_{2m}\}$ with $H[W_{ki}] \in \mathcal{P}_{ki}$ and prove that the graph G_H constructed by Lemma 2 is uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable.

To construct the graph H we need some convenient notation. Let us denote by $\prod_{i \in J} \mathcal{P}_i$ the product $\mathcal{P}_{j_1} \circ \mathcal{P}_{j_2} \circ \cdots \circ \mathcal{P}_{j_r}$ for $J = \{j_1, j_2, \ldots, j_r\}$. For any proper nonempty subsets $X \subset \{1, 2, \ldots, n\}$ and $Y \subset \{1, 2, \ldots, m\}$ we define a property $R_{XY} = \prod_{i \in X} \mathcal{P}_{1i} \circ \prod_{j \in Y} \mathcal{P}_{2j}$. Then we have for all partitions $N_1 \cup N_2$ and $M_1 \cup M_2$ of $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$, respectively, that $R_{N_1 M_1} \circ R_{N_2 M_2} = \mathcal{P}_1 \circ \mathcal{P}_2$. Note that this equation describes a decomposition of $\mathcal{P}_1 \circ \mathcal{P}_2$ with different order of the same irreducible properties and that $R_{N_1 M_1} = \mathcal{P}_1$ and $R_{N_2 M_2} = \mathcal{P}_2$ if and only if $M_1 = N_2 = \emptyset$, in a situation described we call this partition trivial.

Now, according to Lemma 3, for any nontrivial partition $N_1 \cup N_2, M_1 \cup M_2$ we cannot have $R_{N_1 M_1} \subseteq \mathcal{P}_1$ and $R_{N_2 M_2} \subseteq \mathcal{P}_2$, therefore for all nontrivial partitions $N_1 \cup N_2$ and $M_1 \cup M_2$ there is a graph $F_{N_1 M_1} \in R_{N_1 M_1}$ with $F_{N_1 M_1} \notin \mathcal{P}_1$ or there is a graph $F^*_{N_2 M_2} \in R_{N_2 M_2}$ with $F^*_{N_2 M_2} \notin \mathcal{P}_2$ (in cases where only one such graph exists, we take the other to be K_0). Let us define $H = \bigcup (F_{N_1 M_1} \cup F^*_{N_2 M_2})$ where the disjoint union of graphs is taken over all nontrivial partitions $N_1 \cup N_2$ and $M_1 \cup M_2$. To construct the graph G_H by Lemma 2 we consider the vertex $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partition of H formed with respect to the fact that $F_{N_1 M_1} \in R_{N_1 M_1}$ and $F^*_{N_2 M_2} \in R_{N_2 M_2}$.

Let us consider now any vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G_H, say $\{U_1, U_2\}$ and let $\{V^*_{11}, \ldots, V^*_{1n}, V^*_{21}, \ldots, V^*_{2m}\}$ be the unique $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partition of G_H. Since $G[U_1] \in \mathcal{P}_1$ the set U_1 can be further partitioned into n sets inducing subgraphs which are in the properties $\mathcal{P}_{1i}$. Similarly, the set U_2 can be partitioned into m sets inducing subgraphs which are in the properties $\mathcal{P}_{2j}$. By the above these $n+m$ sets are exactly the sets $\{V_{11}^*, \ldots, V_{1n}^*, V_{21}^*, \ldots, V_{2m}^*\}$, but not necessarily in this order. However the partition of U_1 is exactly $\{V_{11}^*, V_{12}^*, \ldots, V_{1n}^*\}$ since otherwise there is a non-trivial partition $N_1 \cup N_2, M_1 \cup M_2$ such that $F_{N_1 M_1} \in \mathcal{P}_1$ and $F_{N_2 M_2}^* \in \mathcal{P}_2$, a contradiction. Since the partition of U_1 is exactly $\{V_{11}^*, V_{12}^*, \ldots, V_{1n}^*\}$ it follows that the partition of U_2 must be $\{V_{21}^*, V_{22}^*, \ldots, V_{2m}^*\}$. ■

Acknowledgement

We are grateful to the referees for their suggestions and comments in preparing the revised version of the paper.

References

[1] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *Survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[2] I. Broere and J. Bucko, *Divisibility in additive hereditary graph properties and uniquely partitionable graphs*, Tatra Mt. Math. Publ. **18** (1999) 79–87.

[3] J. Bucko, M. Frick, P. Mihók and R. Vasky, *Uniquely partitionable graphs*, Discuss. Math. Graph Theory **17** (1997) 103–114.

[4] F. Harary, S.T. Hedetniemi and R.W. Robinson, *Uniquely colourable graphs*, J. Combin. Theory **6** (1969) 264–270.

[5] P. Mihók, *Additive hereditary properties and uniquely partitionable graphs*, in: M. Borowiecki and Z. Skupien, eds., Graphs, hypergraphs and matroids (Zielona Góra, 1985) 49–58.

[6] P. Mihók, *Unique factorization theorem*, Discuss. Math. Graph Theory **20** (2000) 143–154.

Received 8 August 2000
Revised 2 July 2001