WEAKLY $\mathcal{P}$-SATURATED GRAPHS

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Abstract

For a hereditary property $\mathcal{P}$ let $\kappa_\mathcal{P}(G)$ denote the number of forbidden subgraphs contained in $G$. A graph $G$ is said to be weakly $\mathcal{P}$-saturated, if $G$ has the property $\mathcal{P}$ and there is a sequence of edges of $G$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 \subset G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{l-1} + e_l = G_l = K_n$ ($G_{i+1} = G_i + e_{i+1}$) has the following property: $\kappa_\mathcal{P}(G_{i+1}) > \kappa_\mathcal{P}(G_i)$, $0 \leq i \leq l-1$.

In this paper we shall investigate some properties of weakly saturated graphs. We will find upper bound for the minimum number of edges of weakly $\mathcal{P}_k$-saturated graphs of order $n$. We shall determine the number $\text{wsat}(n, \mathcal{P})$ for some hereditary properties.

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1. Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph $G$ has a vertex set $V(G)$ and an edge set $E(G)$. Let $v(G)$, $e(G)$ denote the number of vertices and the number of edges of $G$, respectively. We say that $G$ contains $H$ whenever $G$ contains a subgraph isomorphic to $H$. 

The degree of $v \in V(G)$ is denoted by $d_G(v)$. The number of edges of a path is called the length of the path.

Let $\mathcal{I}$ denote the class of all graphs with isomorphic graphs being regarded as equal. If $\mathcal{P}$ is a proper nonempty subclass of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property of being in $\mathcal{P}$. We shall use the terms class of graphs and property of graphs interchangeably.

A property $\mathcal{P}$ is called hereditary if every subgraph of a graph $G$ with property $\mathcal{P}$ also has property $\mathcal{P}$.

We list some properties to introduce the necessary notation which will be used in the paper. Let $k$ be a non-negative integer.

\[
\begin{align*}
\mathcal{O} &= \{ G \in \mathcal{I} : G \text{ is totally disconnected} \}, \\
\mathcal{O}_k &= \{ G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k + 1 \text{ vertices} \}, \\
\mathcal{I}_k &= \{ G \in \mathcal{I} : G \text{ contains no subgraph isomorphic to } K_{k+2} \}, \\
\mathcal{S}_k &= \{ G \in \mathcal{I} : \Delta(G) \leq k \}, \\
\mathcal{D}_k &= \{ G \in \mathcal{I} : G \text{ is } k\text{-degenerated, i.e., } \delta(H) \leq k \text{ for any } H \leq G \}, \\
\mathcal{W}_k &= \{ G \in \mathcal{I} : \text{ the length of the longest path in } G \text{ is at most } k \}.
\end{align*}
\]

Let $\mathcal{P}$ be a nontrivial hereditary property. Then there is a nonnegative integer $c(\mathcal{P})$, called the completeness of $\mathcal{P}$, such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$. Obviously

\[
c(\mathcal{O}_k) = c(\mathcal{I}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{W}_k) = k.
\]

For a hereditary property $\mathcal{P}$ the set of all minimal forbidden subgraphs of $\mathcal{P}$ is defined by

\[
\mathcal{F}(\mathcal{P}) = \{ G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}.
\]

A graph is called $\mathcal{P}$-maximal if it does not contain any forbidden subgraph but it will contain a forbidden subgraph when any new edge is added to the graph. Let $\mathcal{M}(\mathcal{P})$ be the set of all $\mathcal{P}$-maximal graphs. The set of $\mathcal{P}$-maximal graphs of order $n$ is denoted by $\mathcal{M}(n, \mathcal{P})$.

Many problems of extremal graph theory can be formulated as follows: What is the maximum (minimum) number of edges in a $\mathcal{P}$-maximal graph
of order $n$? For a given hereditary property $\mathcal{P}$ we define those two numbers in the following manner:

$$
\text{ex}(n, \mathcal{P}) = \max \{ e(G) : G \in M(n, \mathcal{P}) \},
$$

$$
\text{sat}(n, \mathcal{P}) = \min \{ e(G) : G \in M(n, \mathcal{P}) \}.
$$

The set of all $\mathcal{P}$-maximal graphs of order $n$ with exactly $\text{ex}(n, \mathcal{P})$ edges is denoted by $\text{Ex}(n, \mathcal{P})$. The members of $\text{Ex}(n, \mathcal{P})$ are called $\mathcal{P}$-extremal graphs. By the symbol $\text{Sat}(n, \mathcal{P})$ is denoted the set of all $\mathcal{P}$-maximal graphs of order $n$ with $\text{sat}(n, \mathcal{P})$ edges. These graphs are called $\mathcal{P}$-saturated.

The most famous Turán’s Theorem [6] establishes the number of edges of $I_k$-extremal graphs. On the other hand, Erdős, Hajnal and Moon [2] calculated the number $\text{sat}(n, I_k)$.

Bollobás [1] introduced the concept of a weakly $k$-saturated graph. Consider a graph of order $n$ and add all those edges which are the only missing edge of complete graph of order $k$ (i.e., we add the edge $e$ if there are $k$ such vertices of the graph, that the graph contains all the edges spanned by these $k$ vertices, saving $e$). If by repeating this process a sufficient number of times the complete graph of order $n$ is obtained, the original graph will be called weakly $k$-saturated.

Bollobás showed that if a graph $G$ of order $n$ is weakly $k$-saturated (for $3 \leq k \leq 7$) with the minimum number of edges then $e(G) = (k - 2)n - \binom{k-1}{2}$. In the general case (i.e., for $k \geq 3$) the equality has been proved by Kalai [5].

Let $\mathcal{P}$ be a hereditary property and let $k_{\mathcal{P}}(G)$ denote the number of forbidden subgraphs contained in $G$. A graph $G$ is said to be weakly $\mathcal{P}$-saturated, if $G$ has the property $\mathcal{P}$ and there is a sequence of edges of $\mathcal{G}$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 \subset G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{l-1} + e_l = G_l = K_n$ ($G_{i+1} = G_i + e_{i+1}$) has the following property: $k_{\mathcal{P}}(G_{i+1}) > k_{\mathcal{P}}(G_i)$, $0 \leq i \leq l - 1$. This sequence of edges will be called the complementary sequence of $G$ with respect to $\mathcal{P}$ or briefly the complementary sequence if it does not lead us to misunderstanding.

According to our terminology a weakly $k$-saturated graph is called weakly $I_{k-2}$-saturated.

Let us denote a set of all weakly $\mathcal{P}$-saturated graphs of order $n$ by $\text{WSat}(n, \mathcal{P})$. Let the minimum and the maximum number of edges in a
graph of $WSat(n, \mathcal{P})$ be denoted by
\[
wsat(n, \mathcal{P}) = \min\{e(G) : G \in WSat(n, \mathcal{P})\},
\]
and
\[
wex(n, \mathcal{P}) = \max\{e(G) : G \in WSat(n, \mathcal{P})\}.
\]

From Theorem of Kalai and Theorem of Erdős, Hajnal, Moon it follows that $wsat(n, \mathcal{I}_k) = sat(n, \mathcal{I}_k)$.
In Section 2 we shall describe a hereditary property $\mathcal{P}$ such that $wsat(n, \mathcal{P}) < sat(n, \mathcal{P})$. We will also investigate some properties of weakly saturated graphs.
In Section 3 examples of weakly $\mathcal{D}_k$-saturated graphs and an upper bound for the number $wsat(n, \mathcal{D}_k)$ will be given. In Section 4 we shall determine the number $wsat(n, \mathcal{P})$ for some hereditary properties.

2. SOME PROPERTIES OF WEAKLY $\mathcal{P}$-SATURATED GRAPHS

From the definition of weakly $\mathcal{P}$-saturated graphs it follows that any $\mathcal{P}$-maximal graph is weakly $\mathcal{P}$-saturated. First we prove that the maximum number of edges of weakly $\mathcal{P}$-saturated graphs is equal to the maximum number of edges of $\mathcal{P}$-maximal graphs.

**Theorem 1.** Let $n \geq 1$. If $\mathcal{P}$ is a hereditary property, then $\text{wex}(n, \mathcal{P}) = \text{ex}(n, \mathcal{P})$.

**Proof.** Every $\mathcal{P}$-maximal graph is weakly $\mathcal{P}$-saturated. Thus $\text{wex}(n, \mathcal{P}) \geq \text{ex}(n, \mathcal{P})$. On the other hand, if a graph of order $n$ has more than $\text{ex}(n, \mathcal{P})$ edges then it contains a forbidden subgraph. Hence $\text{wex}(n, \mathcal{P}) \leq \text{ex}(n, \mathcal{P})$.

Any non-negative integer valued function $f : \mathcal{I} \to \mathbb{N}$ is called the graph invariant (invariant, for short). For a hereditary property $\mathcal{P}$ let us define the number
\[
f(\mathcal{P}) = \min\{f(H) : H \in \text{F}(\mathcal{P})\}.
\]

**Theorem 2.** Let $f(G)$ be an invariant satisfying:

1. $f(H) \leq f(G)$ for $H \subseteq G$,
2. $f(G + e) \leq f(G) + 1$ for $e \in E(G)$.

Then for any graph $G \in WSat(n, \mathcal{P})$ with $n \geq c(\mathcal{P}) + 2$, we have
\[
f(G) \geq f(\mathcal{P}) - 1.
\]
Proof. From the definition of weakly $\mathcal{P}$-saturated graphs, it follows that there is an edge $e \in E(G)$ and a graph $F \in \mathcal{F}(\mathcal{P})$ such that $F \subseteq G + e$. Thus $f(\mathcal{P}) \leq f(F) \leq f(G + e) \leq f(G) + 1$.

The chromatic number and the clique number are examples of invariant satisfying assumptions of Theorem 2. The edge connectivity $\lambda(G)$ does not satisfy the assumption (1) of Theorem 2, but we shall prove that for $G \in \text{WSat}(n, \mathcal{P})$ the inequality $\lambda(G) \geq \lambda(\mathcal{P}) - 1$ also holds.

**Theorem 3.** Let $\lambda(\mathcal{P}) > 0$ and $G \in \text{WSat}(n, \mathcal{P})$. Then

$$\lambda(G) \geq \lambda - 1.$$  

**Proof.** Let $S$ be an edge cutset of $G$ such that $\lambda(G) = |S|$. Let $G'$, $G''$ be two components of $G - S$. Since $G$ is weakly $\mathcal{P}$-saturated, it follows that there is a complementary sequence $e_1, e_2, \ldots, e_l$ of $G$. Let $e_i$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$, which joins a vertex of $G'$ with a vertex of $G''$. Let $F$ denote a subgraph of $G_{i-1} + e_i$, which contains the edge $e_i$ and is isomorphic with some graph of $\mathcal{F}(\mathcal{P})$. Then the set $S \cup \{e_i\}$ is an edge cutset of $F$. Thus $\lambda \leq \lambda(F) \leq |S| + 1 = \lambda(G) + 1$.

From the next theorem it follows that the behaviour of $\text{wsat}(n, \mathcal{P})$ is not monotone in general.

**Theorem 4.** Let $\mathcal{P}$ be the hereditary property such that $\mathcal{F}(\mathcal{P}) = \{2K_2\}$. Then

$$\text{wsat}(n, \mathcal{P}) = \begin{cases} 3, & \text{for } n = 4, \\ 1, & \text{for } n \geq 5. \end{cases}$$

**Proof.** It is easy to see that there is no weakly $\mathcal{P}$-saturated graph of order 4 with two edges. Since the graphs $K_{1,3}$ and $K_3 \cup K_1$ are weakly $\mathcal{P}$-saturated, we have $\text{wsat}(4, \mathcal{P}) = 3$.

If $n \geq 5$ then $K_2 \cup (n-2)K_1$ is a weakly $\mathcal{P}$-saturated graph. By adding (as long as possible) an edge joining two vertices of $(n - 2)K_1$ we obtain two independent edges, i.e., $2K_2$, and results in $K_{n-2}$. Since $n - 2 \geq 3$, it follows that every vertex of $K_2$ (in the original graph), we can join with every vertex of just obtained $K_{n-2}$.

From Theorem of Kalai and Theorem of Erdős, Hajnal and Moon, it follows that $\text{wsat}(n, \mathcal{I}_k) = \text{sat}(n, \mathcal{I}_k)$. Such equality also holds for the property $\mathcal{D}_1$. 

Weakly $\mathcal{P}$-Saturated Graphs

Proof. From the definition of weakly $\mathcal{P}$-saturated graphs, it follows that there is an edge $e \in E(G)$ and a graph $F \in \mathcal{F}(\mathcal{P})$ such that $F \subseteq G + e$. Thus $f(\mathcal{P}) \leq f(F) \leq f(G + e) \leq f(G) + 1$.

The chromatic number and the clique number are examples of invariant satisfying assumptions of Theorem 2. The edge connectivity $\lambda(G)$ does not satisfy the assumption (1) of Theorem 2, but we shall prove that for $G \in \text{WSat}(n, \mathcal{P})$ the inequality $\lambda(G) \geq \lambda(\mathcal{P}) - 1$ also holds.

**Theorem 3.** Let $\lambda(\mathcal{P}) > 0$ and $G \in \text{WSat}(n, \mathcal{P})$. Then

$$\lambda(G) \geq \lambda - 1.$$  

**Proof.** Let $S$ be an edge cutset of $G$ such that $\lambda(G) = |S|$. Let $G'$, $G''$ be two components of $G - S$. Since $G$ is weakly $\mathcal{P}$-saturated, it follows that there is a complementary sequence $e_1, e_2, \ldots, e_l$ of $G$. Let $e_i$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$, which joins a vertex of $G'$ with a vertex of $G''$. Let $F$ denote a subgraph of $G_{i-1} + e_i$, which contains the edge $e_i$ and is isomorphic with some graph of $\mathcal{F}(\mathcal{P})$. Then the set $S \cup \{e_i\}$ is an edge cutset of $F$. Thus $\lambda \leq \lambda(F) \leq |S| + 1 = \lambda(G) + 1$.

From the next theorem it follows that the behaviour of $\text{wsat}(n, \mathcal{P})$ is not monotone in general.

**Theorem 4.** Let $\mathcal{P}$ be the hereditary property such that $\mathcal{F}(\mathcal{P}) = \{2K_2\}$. Then

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**Proof.** It is easy to see that there is no weakly $\mathcal{P}$-saturated graph of order 4 with two edges. Since the graphs $K_{1,3}$ and $K_3 \cup K_1$ are weakly $\mathcal{P}$-saturated, we have $\text{wsat}(4, \mathcal{P}) = 3$.

If $n \geq 5$ then $K_2 \cup (n-2)K_1$ is a weakly $\mathcal{P}$-saturated graph. By adding (as long as possible) an edge joining two vertices of $(n - 2)K_1$ we obtain two independent edges, i.e., $2K_2$, and results in $K_{n-2}$. Since $n - 2 \geq 3$, it follows that every vertex of $K_2$ (in the original graph), we can join with every vertex of just obtained $K_{n-2}$.

From Theorem of Kalai and Theorem of Erdős, Hajnal and Moon, it follows that $\text{wsat}(n, \mathcal{I}_k) = \text{sat}(n, \mathcal{I}_k)$. Such equality also holds for the property $\mathcal{D}_1$. 

Weakly $\mathcal{P}$-Saturated Graphs
Theorem 5. Let $n \geq 1$. Then

$$\text{sat}(n, D_1) = \text{wsat}(n, D_1) = n - 1.$$ 

Proof. Since $F(D_1) = \{C_p : p \geq 3\}$, $\lambda(D_1) = 2$ and every tree is weakly $D_1$-saturated, it follows that $\text{wsat}(n, D_1) \leq n - 1$. From Theorem 3 we have $\lambda(G) \geq 1$ for $G \in \text{WSat}(n, D_1)$ then $\text{wsat}(n, D_1) \geq n - 1$. Thus $\text{wsat}(n, D_1) = n - 1$. Since the only $D_1$-maximal graphs are trees, we have $\text{sat}(n, D_1) = n - 1$. 

The next theorem describes a hereditary property $P$ for which the minimum number of edges of weakly $P$-saturated graphs of order $n$ is less than the number of edges of $P$-saturated graphs of order $n$.

Theorem 6. Let $P$ be the hereditary property such that $\text{ex}(n, P) = \text{sat}(n, P)$, $\lambda(P) = \lambda(H_0) = 1$, $H_0 \in F(P)$ and every $P$-maximal graph is connected. Then $\text{wsat}(n, P) < \text{sat}(n, P)$, $n \geq v(H_0)$.

Proof. Let $H_0 \in F(P)$ with $\lambda(H_0) = 1$ and let $e$ be a cutedge of $H_0$. Denote by $H_1, H_2$ components of $H_0 - e$. Let $v(H_1) = n_1$, $v(H_2) = n_2$. We define the graph $G = G_1 \cup G_2$ of order $n$ assuming that $v(G_1) = n_1$, $v(G_2) = n - n_1$ and for $i = 1, 2$, $G_i$ is $P$-maximal. Obviously $n - n_1 \geq n_2$. Since all forbidden subgraphs are connected it follows that the graph $G$ has property $P$. Defined graph $G$ is not connected, then by the assumption of the theorem, $G$ is not $P$-maximal. Thus $e(G) < \text{ex}(n, P) = \text{sat}(n, P)$.

On the other hand, we will show that the graph $G$ is weakly $P$-saturated. Since each component of $G$ is a $P$-maximal graph, it follows that if we add any edge of $G$ which joins two vertices of the same component we obtain a new forbidden subgraph containing the edge $e$. After adding all missing edges of each component we obtain the graph being a sum of complete graphs. Then each edge, which joins a vertex of the component of order $n_1$ with a vertex of the component of order $n - n_1$, belongs to a subgraph isomorphic to $H_0$. Thus the graph $G$ is weakly $P$-saturated and $e(G) \geq \text{wsat}(n, P)$. Hence $\text{wsat}(n, P) < \text{sat}(n, P)$.

In the next section we will show that the assumptions of Theorem 6 for the property $D_k$ ($k \geq 2$) holds.
3. Weakly $D_k$-Saturated Graphs

The set of minimal forbidden subgraphs for property $D_k$ was characterized by Mihók [4]. To describe the set $F(D_k)$ we need some more notations. For a nonnegative integer $k$ and a graph $G$, we denote the set of all vertices of $G$ of degree $k+1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of $G$ and $G_1, \ldots, G_s$, $s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by $H_i$, $i = 1, \ldots, s$.

**Theorem 7.** [4] A graph $G$ belongs to $F(D_k)$ if and only if $G$ is connected, $\delta(G) \geq k + 1$, $V(G) - M(G)$ is an independent set of vertices of $G$ and for each cutset $S \subset V(G) - M(G)$ we have that $\delta(H_i) \leq k$ for each $i = 1, \ldots, s$.

Let us present some useful examples of $F(D_k)$.

**Example 1.** Let $H_k$, $k \geq 2$, be the graph such that $V(H_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, w_1, w_2\}$ with the following properties: vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two complete graphs and $v_i w_i, v_i x_j, w_i y_j \in E(H_k)$ for $i = 1, 2, j = 1, \ldots, k$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph_example1}
\caption{The graph $H_k$ for $k = 2$}
\end{figure}

**Example 2.** Let $H'_k$, $k \geq 2$, be the graph such that $V(H'_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, v_3, w_1, w_2, w_3\}$ with the following properties: vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two graphs obtained from $K_k$ by removing $\left\lfloor \frac{k}{2} \right\rfloor$ independent edges and $v_i x_j, w_i y_j \in E(H'_k)$ for $i = 1, 2, 3, j = 1, \ldots, k$, and $v_1 w_1, v_2 v_3, w_2 w_3 \in E(H'_k)$.
By Example 2 we have that $\lambda(D_k) = 1$ for $k \geq 2$. Since $D_k$-maximal graphs are connected and $s(n, D_k) = ex(n, D_k)$ (see e.g. [3]), it follows that the assumptions of Theorem 6 holds. Then we immediately have

**Corollary 8.** $wsat(n, D_k)$ is weakly $D_k$-saturated.

**Theorem 10.** Let $k \geq 2$ and $n = 2(k+q)+r$, where $q \geq 1$, $0 \leq r \leq 2k+3$. Then

$$wsat(n, D_k) \leq \begin{cases} 
\frac{(k+2)(k+1)-1}{2(k+2)}n, & \text{for } r = 0, \\
\frac{(k+2)(k+1)-1}{2(k+2)}(n - r - (k + 2)) + (r + k + 2)k - \binom{k+1}{2}, & \text{for } 0 < r < k + 3, \\
\frac{(k+2)(k+1)-1}{2(k+2)}(n - r) + rk - \binom{k+1}{2}, & \text{for } r \geq k + 3.
\end{cases}$$
Proof. To prove the theorem it is enough to show that there is a weakly $D_k$-saturated graph $G$ of order $n$ with such number of edges. Let $k \geq 2$ and $n = 2(k + 2)q + r$, where $q \geq 1$, $0 \leq r \leq 2k + 3$. Put $G' = H_k - v_2w_2$. If $r \geq k + 3$, then $G = qG' \cup H$, where $H \in M(r, D_k)$. If $0 \leq r < k + 3$, then $G = (q - 1)G' \cup H$, where $H \in M(2(k + 2) + r, D_k)$. If $r = 0$, then $G = qG'$. By Lemma 9 it follows that each component of $G$ is a weakly $D_k$-saturated graph. Then we can add edges in each component of $G$ to obtain a complete graph. After having added those edges we can join any vertices of two different components.

4. The Number $wsat(n, \mathcal{P})$ for Some Hereditary Properties

In this section we will calculate the minimum number of edges of weakly saturated graphs for some hereditary properties.

Theorem 11. Let $k \geq 1$ and $n \geq k + 2$. Then

$$WSat(n, \mathcal{O}_k) \supseteq \{T_r \cup T_s \cup tT_1 : r + s = k + 2, \ r + s + t = n \text{ and } T_i \text{ is an arbitrary tree of order } i\}$$

and

$$wsat(n, \mathcal{O}_k) = k.$$

Proof. First we prove that the graph $G = T_r \cup T_s \cup tT_1$, where $r + s = k + 2$, $r + s + t = n$ is weakly $\mathcal{O}_k$-saturated. If we add an edge of $G'$, which joins a vertex of $T_r$ and a vertex of $T_s$ then we obtain a tree of order $k + 2$, i.e., we obtain a forbidden subgraph for property $\mathcal{O}_k$. If we join a vertex of the subgraph $tT_1$ with a vertex of the obtained tree of order $k + 2$ we have a connected graph of order $k + 3$. Thus new edge belongs to a tree of order $k + 2$. Repeating this process we obtain a connected graph of order $n$ in which each vertex of $tT_1$ is adjacent with any vertex of the tree of order $k + 2$. Since for each edge of the complement of a connected graph there is a spanning tree which contains this edge, it follows that $G$ is weakly $\mathcal{O}_k$-saturated. Hence $wsat(n, \mathcal{O}_k) \leq e(G) = k$.

On the other hand, let $G$ be a graph such that $G \in WSat(n, \mathcal{O}_k)$ and $e(G) = wsat(n, \mathcal{O}_k)$. Let $e_1$ be the first edge such that $G + e_1$ contains a forbidden subgraph, i.e., the graph $G + e_1$ contains a tree of order $k + 2$. Thus $wsat(n, \mathcal{O}_k) = e(G) \geq k$. ■
The proof of the next theorem is very similar to the proof of Theorem 11, then it is omitted.

**Theorem 12.** Let \( k \geq 1 \) and \( n \geq k + 2 \). Then

\[
\text{WSat}(n, W_k) \supseteq \{ P_r \cup P_s \cup tP_1 : r + s = k + 2, \ r + s + t = n \}
\]

and

\[
\text{wsat}(n, W_k) = k.
\]

It is easy to see that the graphs \( K_{k+1} + tK_1 \), where \( k+1+t = n \) are weakly \( S_k \)-saturated. There are some other weakly \( S_k \)-saturated graphs of order \( n \). For example the graph \( G_1 \) (Figure 4.1) is weakly \( S_2 \)-saturated and the graph \( G_2 \) (Figure 4.1) is weakly \( S_3 \)-saturated.

![](image)

**Figure 4.1.** The graphs \( G_1 \) and \( G_2 \)

**Theorem 13.** Let \( n \geq k + 2 \geq 4 \). Then

\[
\text{wsat}(n, S_k) = \binom{k+1}{2}.
\]

**Proof.** Let \( G \) be a weakly \( S_k \)-saturated graph of order \( n \) with the minimum number of edges. Then there is a complementary sequence \( e_1, e_2, \ldots, e_l \) of \( G \). Let \( e_1 = u_1v_1 \) and \( d_G(u_1) = k \). Let \( e_{f(1)}, \ldots, e_{f(t_1)} \) be the subsequence of \( e_1, e_2, \ldots, e_l \) such that every edge \( e_{f(i)} \), \( 1 \leq i \leq t_1 \) is adjacent with the vertex \( u_1 \). If in the graph \( G' = (G + e_{f(1)} + e_{f(2)} + \ldots + e_{f(t_1)}) \) there is no vertex of degree less than \( k \) then let \( e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)} \) be the new sequence of edges of \( E(G) \) with the following property: \( e_{f(1)}, \ldots, e_{f(t_1)} \) is the subsequence of \( e_1, e_2, \ldots, e_l \) such that every edge \( e_{f(i)} \), \( 1 \leq i \leq t_1 \) is adjacent with the vertex \( u_1 \) and \( e_{f(t_1)+1}, \ldots, e_{f(l)} \) is the subsequence of \( e_1, e_2, \ldots, e_l \) such that any edge \( e_{f(i)} \), \( t_1 \leq i \leq l \) is not adjacent with the vertex \( u_1 \). If in the graph \( G'' \) there is a vertex of degree less than \( k \)
then let $e_{f(t_1+1)}$ be the first edge of $e_1, e_2, \ldots, e_l$, which is not adjacent with the vertex $u_1$. Let $e_{f(t_1+1)} = u_2v_2$ and $u_2$ be a vertex of $G'$ such that $d_{G'}(u_2) \geq k$ and $u_1 \neq u_2$. Let $e_{f(t_1+1)}, \ldots, e_{f(t_2)}$ denote edges of $\{e_1, e_2, \ldots, e_l\} - \{e_{f(t_1)}, \ldots, e_{f(t_1)+1}\}$ which are adjacent with the vertex $u_2$. If in the graph $G'' = (G' + e_{f(t_1+1)} + e_{f(t_1)+2} + \ldots + e_{f(t_2)}$, there is no vertex of degree less than $k$ we form a new sequence of edges of $E(G)$, $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ with the following property: $e_{f(1)}, \ldots, e_{f(t_1)}$ is a subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex $u_1$ and $e_{f(t_1+1)}, \ldots, e_{f(t_2)}$ is a subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(t_1 < i \leq t_2)$ is adjacent with the vertex $u_2$ and $e_{f(t_2)+1}, \ldots, e_{f(l)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that any edge $e_{f(i)}$, $(t_2 < i \leq l)$ is not adjacent with the vertex $u_1$ and $u_2$. If in the graph $G''$ there is a vertex of degree less than $k$, we will repeat this steps until we will obtain a new sequence $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ of edges of $G$. With this sequence of edges $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ is related a sequence of vertices $u_1, u_2, \ldots, u_r$. It is easy to see that $r \leq k$, because after $k$ steps there is no vertex of degree less than $k$. Then for the vertex $u_t \in \{u_1, \ldots, u_r\}$ we have

\[(1)\quad d_G(u_t) + t - 1 - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}| \geq k,\]

for the vertex $x \in V(G) - \{u_1, \ldots, u_r\}$ we have

\[(2)\quad d_G(x) + r - |N_G(x) \cap \{u_1, \ldots, u_r\}| \geq k.\]

Thus

\[e(G) \geq \sum_{t \leq t \leq r} (d_G(u_t) - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}|) + \frac{1}{2} \sum_{x \in V(G) - \{u_1, \ldots, u_r\}} (d_G(x) - |N_G(x) \cap \{u_1, \ldots, u_r\}|) \geq \sum_{1 \leq t \leq r} (k + 1 - t) + \frac{1}{2} (n - r)(k - r).\]

The right side of inequality achieves the minimum for $r = k$. Thus

\[e(G) \geq \sum_{1 \leq t \leq r} (k + 1 - t) = \frac{1}{2} (k + 1)(k + 1).\]

On the other hand, the graph $K_{k+1} \cup (n - k - 1)K_1$ is weakly $S_k$-saturated. Thus $\text{wsat}(n, S_k) \leq \frac{(k+1)}{2}$.

In the next theorem we determine the number $\text{wsat}(n, \mathcal{P})$ for a hereditary property with one forbidden subgraph which is a cycle of odd length.

**Theorem 14.** Let $k \geq 1$ and $n \geq 2k + 2$. If $\mathcal{P}$ is the hereditary property such that $F(\mathcal{P}) = \{C_{2k+1}\}$, then $\text{wsat}(n, \mathcal{P}) = n - 1$. 
Proof. Since $\lambda(P) = 2$, by Theorem 3 it follows that every weakly $P$-saturated graph is connected. Then $\text{wsat}(n, P) \geq n - 1$. To prove that the inequality $\text{wsat}(n, P) \leq n - 1$ holds it is sufficient to show that there is a weakly $P$-saturated graph of order $n$ with $n - 1$ edges.

Let us show first that $P_{2k+2}$ is a weakly $P$-saturated graph. Let $V(P_{2k+2}) = \{v_1, \ldots, v_{2k+2}\}$ and $d(v_1) = d(v_{2k+2}) = 1$. It is easy to see that if we add the edge $v_1v_{2k+1}$ then we obtain a cycle of order $2k + 1$. Similarly if we add the edge $v_2v_{2k+2}$ a new cycle of order $2k + 1$ appears. Now we can add the edge $v_1v_4$. The edge $v_1v_4$ belongs to the cycle $v_1, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_4, v_1$. To prove that if we add any edge $v_1v_{2t}$ then a new cycles of order $2k + 1$ appears we will use induction on $t$. This is true for $t = 1, 2$. When the edges $v_1v_{2i}$ for $i < t$ are added the vertices $v_1, v_2, v_{2t-2}, v_{2t-3}, \ldots, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_2, v_1$ induce a cycle of order $2k + 1$ which contains the edge $v_1v_{2t}$. In the same manner, after having added edges $v_1v_{2i+1}$ for $k \geq i > t$ we can add the edge $v_1v_{2t+1}$. A new cycle $v_1, v_{2t+3}, \ldots, v_{2k+2}, v_2, v_3, \ldots, v_{2t+1}, v_1$ of order $2k + 1$ appears. Finally the vertex $v_1$ with all vertices of $P_{2k+2}$ is joined. Similarly we can join each vertex $v_t$ ($2 \leq t \leq 2k + 2$) with all vertices of $P_{2k+2}$. Thus we obtain a graph $K_{2k+2}$. Hence $P_{2k+2}$ is a weakly $P$-saturated graph.

Let $G$ be the graph of order $n \geq 2k + 2$ with the following properties: $G$ contains an induced path of order $2k + 2$, the remaining vertices of $G$ form an independent set and each vertex of this set is adjacent with exactly one vertex of the path. Since the path of order $2k + 2$ is weakly $P$-saturated, it follows that the graph $G$ is weakly $P$-saturated. Hence $\text{wsat}(n, P) \leq n - 1$.

In order to determine the number $\text{wsat}(n, P)$ for hereditary property such that $F(P) = \{C_{2k}\}$ we need the following lemma.

Lemma 15. Let $k \geq 2$ and $P$ be the hereditary property such that $F(P) = \{C_{2k}\}$, and $G$ be a bipartite graph of order $n \geq 2k + 1$. Then $G \not\in \text{WSat}(n, P)$.

Proof. On the contrary, suppose that there is a weakly $P$-saturated bipartite graph $G$ of order $n$. Let $e_1, e_2, \ldots, e_l$ be a complementary sequence of $G$. Let $e_i = xy$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$ such that its ends $x, y$ belong to the same colour class of $G$. (Notice, that the colour classes of $G$ are uniquely determined because of connectivity of $G$.) Since the edge $e_l$ belongs to an even cycle $C_{2k}$ then there is an edge $e_j$, $j < i$ of this cycle (and the sequence given above) with both ends in one colour class which is impossible.
Theorem 16. Let $k \geq 2$ and $n \geq 2k + 1$. Let $\mathcal{P}$ be the hereditary property such that $F(\mathcal{P}) = \{C_{2k}\}$. Then

$$\text{wsat}(n, \mathcal{P}) = n.$$ 

Proof. Let $G \in \text{WSat}(n, \mathcal{P})$. By Theorem 3 and Lemma 15 it follows that $G$ is connected and contains an odd cycle. Thus $\text{wsat}(n, \mathcal{P}) \geq n$.

To prove that the inequality $\text{wsat}(n, \mathcal{P}) \leq n$ holds it is sufficient to show that there is a weakly $\mathcal{P}$-saturated graph of order $n$ with $n$ edges.

First we prove that $C_{2k+1}$ is a weakly $\mathcal{P}$-saturated graph. Let $V(C_{2k+1}) = \{v_1, v_2, \ldots, v_{2k+1}\}$. It is easy to see that if we add the edge $v_1v_3$ or the edge $v_{2k+1}$, a cycle (containing this edge) of order $2k$ appears. To prove that if we add any edge $v_1v_t$ ($3 \leq t \leq 2k$) then we obtain a new cycle of order $2k$ we use induction on $t$. This is true for $t = 3$. After adding edges $v_1v_i$ for $3 \leq i < t$ the vertices $v_1, v_{t-2}, v_{t-3}, \ldots, v_2, v_{2k+1}, v_{2k}, \ldots, v_t, v_1$ induce a cycle of order $2k$ which contains the edge $v_1v_t$. Then the vertex $v_1$ can be joined with all vertices of $C_{2k+1}$. In the similar manner we can show that we can join any vertex $v_t \in V(C_{2k+1})$ with all vertices of $C_{2k+1}$. Hence $C_{2k+1}$ is weakly $\mathcal{P}$-saturated.

Let $G$ be the graph with the following properties: $G$ contains an induced cycle of order $2k + 1$, remaining vertices of $G$ form an independent set and each vertex of this set is adjacent with exactly one vertex of the cycle. Since the cycle of order $2k + 1$ is weakly $\mathcal{P}$-saturated (can be extended to $K_{2k+1}$), it follows that the graph $G$ also has this property, i.e., $G$ is weakly $\mathcal{P}$-saturated. Hence $\text{wsat}(n, \mathcal{P}) \leq n$.

References


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