WEAKLY $\mathcal{P}$-SATURATED GRAPHS

Mieczysław Borowiecki

and

Elżbieta Sidorowicz

Institute of Mathematics
University of Zielona Góra
65-246 Zielona Góra, Podgórna 50, Poland

e-mail: M.Borowiecki@im.uz.zgora.pl

e-mail: E.Sidorowicz@im.uz.zgora.pl

Abstract

For a hereditary property $\mathcal{P}$ let $k_\mathcal{P}(G)$ denote the number of forbidden subgraphs contained in $G$. A graph $G$ is said to be weakly $\mathcal{P}$-saturated, if $G$ has the property $\mathcal{P}$ and there is a sequence of edges of $G$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 = G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{i-1} + e_i = G_i = K_n$ ($G_{i+1} = G_i + e_{i+1}$) has the following property: $k_\mathcal{P}(G_{i+1}) > k_\mathcal{P}(G_i)$, $0 \leq i \leq l - 1$.

In this paper we shall investigate some properties of weakly saturated graphs. We will find upper bound for the minimum number of edges of weakly $D_k$-saturated graphs of order $n$. We shall determine the number $\text{wsat}(n, \mathcal{P})$ for some hereditary properties.

Keywords: graph, extremal problems, hereditary property, weakly saturated graphs.

1991 Mathematics Subject Classification: 05C35.

1. Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph $G$ has a vertex set $V(G)$ and an edge set $E(G)$. Let $v(G)$, $e(G)$ denote the number of vertices and the number of edges of $G$, respectively. We say that $G$ contains $H$ whenever $G$ contains a subgraph isomorphic to $H$. 
M. Borowiecki and E. Sidorowicz

The degree of \( v \in V(G) \) is denoted by \( d_G(v) \). The number of edges of a path is called the length of the path.

Let \( \mathcal{I} \) denote the class of all graphs with isomorphic graphs being regarded as equal. If \( \mathcal{P} \) is a proper nonempty subclass of \( \mathcal{I} \), then \( \mathcal{P} \) will also denote the property of being in \( \mathcal{P} \). We shall use the terms class of graphs and property of graphs interchangeably.

A property \( \mathcal{P} \) is called hereditary if every subgraph of a graph \( G \) with property \( \mathcal{P} \) also has property \( \mathcal{P} \).

We list some properties to introduce the necessary notation which will be used in the paper. Let \( k \) be a non-negative integer.

\[
\mathcal{O} = \{ G \in \mathcal{I} : G \text{ is totally disconnected} \},
\]

\[
\mathcal{O}_k = \{ G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k + 1 \text{ vertices} \},
\]

\[
\mathcal{I}_k = \{ G \in \mathcal{I} : \text{G contains no subgraph isomorphic to } K_{k+2} \},
\]

\[
\mathcal{S}_k = \{ G \in \mathcal{I} : \Delta(G) \leq k \},
\]

\[
\mathcal{D}_k = \{ G \in \mathcal{I} : G \text{ is } k\text{-degenerated, i.e., } \delta(H) \leq k \text{ for any } H \leq G \},
\]

\[
\mathcal{W}_k = \{ G \in \mathcal{I} : \text{the length of the longest path in } G \text{ is at most } k \}.
\]

Let \( \mathcal{P} \) be a nontrivial hereditary property. Then there is a nonnegative integer \( c(\mathcal{P}) \), called the completeness of \( \mathcal{P} \), such that \( K_{c(\mathcal{P})+1} \in \mathcal{P} \) but \( K_{c(\mathcal{P})+2} \notin \mathcal{P} \). Obviously

\[
c(\mathcal{O}_k) = c(\mathcal{I}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{W}_k) = k.
\]

For a hereditary property \( \mathcal{P} \) the set of all minimal forbidden subgraphs of \( \mathcal{P} \) is defined by

\[
F(\mathcal{P}) = \{ G \in I : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}. 
\]

A graph is called \( \mathcal{P} \)-maximal if it does not contain any forbidden subgraph but it will contain a forbidden subgraph when any new edge is added to the graph. Let \( M(\mathcal{P}) \) be the set of all \( \mathcal{P} \)-maximal graphs. The set of \( \mathcal{P} \)-maximal graphs of order \( n \) is denoted by \( M(n, \mathcal{P}) \).

Many problems of extremal graph theory can be formulated as follows: What is the maximum (minimum) number of edges in a \( \mathcal{P} \)-maximal graph
Weakly $\mathcal{P}$-Saturated Graphs

of order $n$? For a given hereditary property $\mathcal{P}$ we define those two numbers in the following manner:

\[
\begin{align*}
\text{ex}(n, \mathcal{P}) &= \max \{ e(G) : G \in \text{M}(n, \mathcal{P}) \}, \\
\text{sat}(n, \mathcal{P}) &= \min \{ e(G) : G \in \text{M}(n, \mathcal{P}) \}.
\end{align*}
\]

The set of all $\mathcal{P}$-maximal graphs of order $n$ with exactly $\text{ex}(n, \mathcal{P})$ edges is denoted by $\text{Ex}(n, \mathcal{P})$. The members of $\text{Ex}(n, \mathcal{P})$ are called $\mathcal{P}$-extremal graphs. By the symbol $\text{Sat}(n, \mathcal{P})$ is denoted the set of all $\mathcal{P}$-maximal graphs of order $n$ with $\text{sat}(n, \mathcal{P})$ edges. These graphs are called $\mathcal{P}$-saturated.

The most famous Turán’s Theorem [6] establishes the number of edges of $I_k$-extremal graphs. On the other hand, Erdős, Hajnal and Moon [2] calculated the number $\text{sat}(n, I_k)$.

Bollobás [1] introduced the concept of a weakly $k$-saturated graph. Consider a graph of order $n$ and add all those edges which are the only missing edge of complete graph of order $k$ (i.e., we add the edge $e$ if there are $k$ such vertices of the graph, that the graph contains all the edges spanned by these $k$ vertices, saving $e$). If by repeating this process a sufficient number of times the complete graph of order $n$ is obtained, the original graph will be called weakly $k$-saturated.

Bollobás showed that if a graph $G$ of order $n$ is weakly $k$-saturated (for $3 \leq k \leq 7$) with the minimum number of edges then $e(G) = (k - 2)n - \binom{k-1}{2}$. In the general case (i.e., for $k \geq 3$) the equality has been proved by Kalai [5].

Let $\mathcal{P}$ be a hereditary property and let $k_P(G)$ denote the number of forbidden subgraphs contained in $G$. A graph $G$ is said to be weakly $\mathcal{P}$-saturated, if $G$ has the property $\mathcal{P}$ and there is a sequence of edges of $G$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 \subset G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{i-1} + e_i = G_i = K_n \ (G_{i+1} = G_i + e_{i+1})$ has the following property: $k_P(G_{i+1}) > k_P(G_i), 0 \leq i \leq l - 1$. This sequence of edges will be called the complementary sequence of $G$ with respect to $\mathcal{P}$ or briefly the complementary sequence if it does not lead us to misunderstanding.

According to our terminology a weakly $k$-saturated graph is called weakly $I_{k-2}$-saturated.

Let us denote a set of all weakly $\mathcal{P}$-saturated graphs of order $n$ by $\text{WSat}(n, \mathcal{P})$. Let the minimum and the maximum number of edges in a
graph of WSat\((n, P)\) be denoted by
\[
\text{wsat}(n, P) = \min\{e(G) : G \in \text{WSat}(n, P)\},
\]
\[
\text{wex}(n, P) = \max\{e(G) : G \in \text{WSat}(n, P)\}.
\]

From Theorem of Kalai and Theorem of Erdős, Hajnal, Moon it follows
that \(\text{wsat}(n, I_k) = \text{sat}(n, I_k)\). In Section 2 we shall describe a hereditary
property \(P\) such that \(\text{wsat}(n, P) < \text{sat}(n, P)\). We will also investigate some
properties of weakly saturated graphs. In Section 3 examples of weakly
\(D_k\)-saturated graphs and an upper bound for the number \(\text{wsat}(n, D_k)\) will
be given. In Section 4 we shall determine the number \(\text{wsat}(n, P)\) for some
hereditary properties.

2. SOME PROPERTIES OF WEAKLY \(P\)-SATURATED GRAPHS

From the definition of weakly \(P\)-saturated graphs it follows that any
\(P\)-maximal graph is weakly \(P\)-saturated. First we prove that the maximum
number of edges of weakly \(P\)-saturated graphs is equal to the maximum
number of edges of \(P\)-maximal graphs.

Theorem 1. Let \(n \geq 1\). If \(P\) is a hereditary property, then \(\text{wex}(n, P) = \text{ex}(n, P)\).

Proof. Every \(P\)-maximal graph is weakly \(P\)-saturated. Thus \(\text{wex}(n, P) \geq \text{ex}(n, P)\). On the other hand, if a graph of order \(n\) has more than \(\text{ex}(n, P)\)
edges then it contains a forbidden subgraph. Hence \(\text{wex}(n, P) \leq \text{ex}(n, P)\).

Any non-negative integer valued function \(f : \mathcal{I} \to \mathbb{N}\) is called the graph
invariant (invariant, for short). For a hereditary property \(P\) let us define
the number
\[
f(P) = \min\{f(H) : H \in \mathcal{F}(P)\}.
\]

Theorem 2. Let \(f(G)\) be an invariant satisfying:

1. \(f(H) \leq f(G)\) for \(H \subseteq G\),
2. \(f(G + e) \leq f(G) + 1\) for \(e \in E(G)\).

Then for any graph \(G \in \text{WSat}(n, P)\) with \(n \geq c(P) + 2\), we have
\[
f(G) \geq f(P) - 1.
\]
Proof. From the definition of weakly $\mathcal{P}$-saturated graphs, it follows that there is an edge $e \in E(G)$ and a graph $F \in F(\mathcal{P})$ such that $F \subseteq G + e$. Thus $f(\mathcal{P}) \leq f(F) \leq f(G + e) \leq f(G) + 1$.

The chromatic number and the clique number are examples of invariant satisfying assumptions of Theorem 2. The edge connectivity $\lambda(G)$ does not satisfy the assumption (1) of Theorem 2, but we shall prove that for $G \in \text{WSat}(n, \mathcal{P})$ the inequality $\lambda(G) \geq \lambda(\mathcal{P}) - 1$ also holds.

Theorem 3. Let $\lambda(\mathcal{P}) = \lambda > 0$ and $G \in \text{WSat}(n, \mathcal{P})$. Then

$$\lambda(G) \geq \lambda - 1.$$  

Proof. Let $S$ be an edge cutset of $G$ such that $\lambda(G) = |S|$. Let $G'$, $G''$ be two components of $G - S$. Since $G$ is weakly $\mathcal{P}$-saturated, it follows that there is a complementary sequence $e_1, e_2, \ldots, e_l$ of $G$. Let $e_i$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$, which joins a vertex of $G'$ with a vertex of $G''$. Let $F$ denote a subgraph of $G - e_i$, which contains the edge $e_i$ and is isomorphic with some graph of $F(\mathcal{P})$. Then the set $S \cup \{e_i\}$ is an edge cutset of $F$. Thus $\lambda \leq \lambda(F) \leq |S| + 1 = \lambda(G) + 1$.

From the next theorem it follows that the behaviour of $\text{wsat}(n, \mathcal{P})$ is not monotone in general.

Theorem 4. Let $\mathcal{P}$ be the hereditary property such that $F(\mathcal{P}) = \{2K_2\}$. Then

$$\text{wsat}(n, \mathcal{P}) = \begin{cases} 3, & \text{for } n = 4, \\ 1, & \text{for } n \geq 5. \end{cases}$$

Proof. It is easy to see that there is no weakly $\mathcal{P}$-saturated graph of order 4 with two edges. Since the graphs $K_{1,3}$ and $K_3 \cup K_1$ are weakly $\mathcal{P}$-saturated, we have $\text{wsat}(4, \mathcal{P}) = 3$.

If $n \geq 5$ then $K_2 \cup (n - 2)K_1$ is a weakly $\mathcal{P}$-saturated graph. By adding (as long as possible) an edge joining two vertices of $(n - 2)K_1$ we obtain two independent edges, i.e., $2K_2$, and results in $K_{n-2}$. Since $n - 2 \geq 3$, it follows that every vertex of $K_2$ (in the original graph), we can join with every vertex of just obtained $K_{n-2}$.

From Theorem of Kalai and Theorem of Erdős, Hajnal and Moon, it follows that $\text{wsat}(n, \mathcal{I}_k) = \text{sat}(n, \mathcal{I}_k)$. Such equality also holds for the property $\mathcal{D}_1$.  

\textbf{Weakly }$\mathcal{P}$-\textbf{Saturated Graphs} 21
Theorem 5. Let $n \geq 1$. Then

$$\text{sat}(n, D_1) = \text{wsat}(n, D_1) = n - 1.$$ 

Proof. Since $F(D_1) = \{C_p : p \geq 3\}$, $\lambda(D_1) = 2$ and every tree is weakly $D_1$-saturated, it follows that $\text{wsat}(n, D_1) \leq n - 1$. From Theorem 3 we have $\lambda(G) \geq 1$ for $G \in \text{WSat}(n, D_1)$ then $\text{wsat}(n, D_1) \geq n - 1$. Thus $\text{wsat}(n, D_1) = n - 1$. Since the only $D_1$-maximal graphs are trees, we have $\text{sat}(n, D_1) = n - 1$.■

The next theorem describes a hereditary property $\mathcal{P}$ for which the minimum number of edges of weakly $\mathcal{P}$-saturated graphs of order $n$ is less than the number of edges of $\mathcal{P}$-saturated graphs of order $n$.

Theorem 6. Let $\mathcal{P}$ be the hereditary property such that $\text{ex}(n, \mathcal{P}) = \text{sat}(n, \mathcal{P})$, $\lambda(\mathcal{P}) = \lambda(H_0) = 1$, $H_0 \in F(\mathcal{P})$ and every $\mathcal{P}$-maximal graph is connected. Then $\text{wsat}(n, \mathcal{P}) < \text{sat}(n, \mathcal{P})$, $n \geq v(H_0)$.

Proof. Let $H_0 \in F(\mathcal{P})$ with $\lambda(H_0) = 1$ and let $e$ be a cutedge of $H_0$. Denote by $H_1, H_2$ components of $H_0 - e$. Let $v(H_1) = n_1$, $v(H_2) = n_2$. We define the graph $G = G_1 \cup G_2$ of order $n$ assuming that $v(G_1) = n_1$, $v(G_2) = n - n_1$ and for $i = 1, 2$, $G_i$ is $\mathcal{P}$-maximal. Obviously $n - n_1 \geq n_2$. Since all forbidden subgraphs are connected it follows that the graph $G$ has property $\mathcal{P}$. Defined graph $G$ is not connected, then by the assumption of the theorem, $G$ is not $\mathcal{P}$-maximal. Thus $e(G) < \text{ex}(n, \mathcal{P}) = \text{sat}(n, \mathcal{P})$.

On the other hand, we will show that the graph $G$ is weakly $\mathcal{P}$-saturated. Since each component of $G$ is a $\mathcal{P}$-maximal graph, it follows that if we add any edge of $G$ which joins two vertices of the same component we obtain a new forbidden subgraph containing the edge $e$. After adding all missing edges of each component we obtain the graph being a sum of complete graphs. Then each edge, which joins a vertex of the component of order $n_1$ with a vertex of the component of order $n - n_1$, belongs to a subgraph isomorphic to $H_0$. Thus the graph $G$ is weakly $\mathcal{P}$-saturated and $e(G) \geq \text{wsat}(n, \mathcal{P})$. Hence $\text{wsat}(n, \mathcal{P}) < \text{sat}(n, \mathcal{P})$. ■

In the next section we will show that the assumptions of Theorem 6 for the property $D_k$ ($k \geq 2$) holds.
3. Weakly $D_k$-Saturated Graphs

The set of minimal forbidden subgraphs for property $D_k$ was characterized by Mihók [4]. To describe the set $F(D_k)$ we need some more notations. For a nonnegative integer $k$ and a graph $G$, we denote the set of all vertices of $G$ of degree $k + 1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of $G$ and $G_1, \ldots, G_s$, $s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by $H_i$, $i = 1, \ldots, s$.

**Theorem 7.** [4] A graph $G$ belongs to $F(D_k)$ if and only if $G$ is connected, $\delta(G) \geq k + 1$, $V(G) - M(G)$ is an independent set of vertices of $G$ and for each cutset $S \subset V(G) - M(G)$ we have that $\delta(H_i) \leq k$ for each $i = 1, \ldots, s$.

Let us present some useful examples of $F(D_k)$.

**Example 1.** Let $H_k$, $k \geq 2$, be the graph such that $V(H_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, w_1, w_2\}$ with the following properties: vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two complete graphs and $v_ix_j, w_iy_j \in E(H_k)$ for $i = 1, 2, j = 1, \ldots, k$.

- $x_1$ - $v_1$ - $w_1$ - $y_1$
- $x_2$ - $v_2$ - $w_2$ - $y_2$

Figure 3.1. The graph $H_k$ for $k = 2$

**Example 2.** Let $H'_k$, $k \geq 2$, be the graph such that $V(H'_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, v_3, w_1, w_2, w_3\}$ with the following properties: vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two graphs obtained from $K_k$ by removing $\lfloor \frac{k}{2} \rfloor$ independent edges and $v_ix_j, w_iy_j \in E(H'_k)$ for $i = 1, 2, 3, j = 1, \ldots, k$, and $v_1w_1, v_2v_3, w_2w_3 \in E(H'_k)$.
By Example 2 we have that $\lambda(D_k) = 1$ for $k \geq 2$. Since $D_k$-maximal graphs are connected and $\text{sat}(n, D_k) = \text{ex}(n, D_k)$ (see e.g. [3]), it follows that the assumptions of Theorem 6 holds. Then we immediately have

**Corollary 8.** $\text{wsat}(n, D_k) < \text{sat}(n, D_k)$ for $n \geq 2(k + 3)$, $k \geq 2$.

To determine upper bound for the number $\text{wsat}(n, D_k)$ we need the following lemma.

**Lemma 9.** Let $k \geq 2$. Then the graph $H_k - v_2w_2$ is weakly $D_k$-saturated.

**Proof.** Put $G = H_k - v_2w_2$. If the edge $v_2w_2$ is added to $G$ then $G = H_k \in F(D_k)$ is obtained. If we add $v_1v_2$ or $w_1w_2$ to $H_k$ then we obtain the graph $K_{k+2}$ which belongs to $F(D_k)$. After adding the edge $x_iy_j$, $(1 \leq i, j \leq k)$, edges $(E(G) \cup \{v_1v_2, w_1w_2, x_iy_j\}) - \{v_1x_i, w_1y_j\}$ induce $H_k$. Now we can add the edge $v_1y_j$, $1 \leq j \leq k$ since edges $(E(G) \cup \{v_2w_2, w_1w_2, v_1y_j\}) - \{w_2y_j, v_1w_1\}$ induce $H_k$. If we add the edge $v_2w_j$ $(1 \leq j \leq k)$, we obtain the graph $H_k$ induced by $(E(G) \cup \{w_1w_2, v_2y_j\}) - \{w_1y_j\}$. In a similar manner we can show that if we add edges $x_iw_1$ and $x_iw_2$ $(1 \leq i \leq k)$, a new forbidden subgraph appears. The last two edges $v_1v_2$, $v_2w_1$ we can add because edges $(E(G) \cup \{x_1y_1, v_1w_2, v_1v_2, w_1w_2\}) - \{x_1v_1, w_2y_1, v_1w_1\}$ and $(E(G) \cup \{x_1y_1, v_2w_1, v_1v_2, w_1w_2\}) - \{x_1v_1, w_2y_1, v_1w_1\}$ induce $H_k$. \n
**Theorem 10.** Let $k \geq 2$ and $n = 2(k+2)q + r$, where $q \geq 1$, $0 \leq r \leq 2k+3$. Then

$$\text{wsat}(n, D_k) \leq \begin{cases} \frac{(k+2)(k+1)}{2(k+2)} - n, & \text{for } r = 0, \\ \frac{(k+2)(k+1)}{2(k+2)}(n - r - (k + 2)) + (r + k + 2)k - \binom{k+1}{2}, & \text{for } 0 < r < k + 3, \\
\frac{(k+2)(k+1)}{2(k+2)}(n - r) + rk - \binom{k+1}{2}, & \text{for } r \geq k + 3. \end{cases}$$
Proof. To prove the theorem it is enough to show that there is a weakly $D_k$-saturated graph $G$ of order $n$ with such number of edges. Let $k \geq 2$ and $n = 2(k + 2)q + r$, where $q \geq 1$, $0 \leq r \leq 2k + 3$. Put $G' = H_k - v_2w_2$. If $r \geq k + 3$, then $G = qG' \cup H$, where $H \in M(r, D_k)$. If $0 \leq r < k + 3$, then $G = (q - 1)G' \cup H$, where $H \in M(2(k + 2) + r, D_k)$. If $r = 0$, then $G = qG'$. By Lemma 9 it follows that each component of $G$ is a weakly $D_k$-saturated graph. Then we can add edges in each component of $G$ to obtain a complete graph. After having added those edges we can join any vertices of two different components.

4. The Number $\text{wsat}(n, \mathcal{P})$ for Some Hereditary Properties

In this section we will calculate the minimum number of edges of weakly saturated graphs for some hereditary properties.

Theorem 11. Let $k \geq 1$ and $n \geq k + 2$. Then

$$\text{WSat}(n, \mathcal{O}_k) \supseteq \{T_r \cup T_s \cup iT_1 : r + s = k + 2, r + s + t = n \text{ and } T_i \text{ is an arbitrary tree of order } i\}$$

and

$$\text{wsat}(n, \mathcal{O}_k) = k.$$

Proof. First we prove that the graph $G = T_r \cup T_s \cup iT_1$, where $r + s = k + 2$, $r + s + t = n$ is weakly $\mathcal{O}_k$-saturated. If we add an edge of $G$, which joins a vertex of $T_r$ and a vertex of $T_s$ then we obtain a tree of order $k + 2$, i.e., we obtain a forbidden subgraph for property $\mathcal{O}_k$. If we join a vertex of the subgraph $iT_1$ with a vertex of the obtained tree of order $k + 2$ we have a connected graph of order $k + 3$. Thus new edge belongs to a tree of order $k + 2$. Repeating this process we obtain a connected graph of order $n$ in which each vertex of $iT_1$ is adjacent with any vertex of the tree of order $k + 2$. Since for each edge of the complement of a connected graph there is a spanning tree which contains this edge, it follows that $G$ is weakly $\mathcal{O}_k$-saturated. Hence $\text{wsat}(n, \mathcal{O}_k) \leq e(G) = k$.

On the other hand, let $G$ be a graph such that $G \in \text{WSat}(n, \mathcal{O}_k)$ and $e(G) = \text{wsat}(n, \mathcal{O}_k)$. Let $e_1$ be the first edge such that $G + e_1$ contains a forbidden subgraph, i.e., the graph $G + e_1$ contains a tree of order $k + 2$. Thus $\text{wsat}(n, \mathcal{O}_k) = e(G) \geq k$. \hfill \blacksquare
The proof of the next theorem is very similar to the proof of Theorem 11, then it is omitted.

**Theorem 12.** Let $k \geq 1$ and $n \geq k + 2$. Then

$$\text{WSat}(n, W_k) \supseteq \{P_r \cup P_s \cup tP_1 : r + s = k + 2, r + s + t = n\}$$

and

$$\text{wsat}(n, W_k) = k.$$

It is easy to see that the graphs $K_{k+1} + tK_{1}$, where $k + 1 + t = n$ are weakly $S_k$-saturated. There are some other weakly $S_k$-saturated graphs of order $n$. For example the graph $G_1$ (Figure 4.1) is weakly $S_2$-saturated and the graph $G_2$ (Figure 4.1) is weakly $S_3$-saturated.

![Figure 4.1. The graphs $G_1$ and $G_2$](image)

**Theorem 13.** Let $n \geq k + 2 \geq 4$. Then

$$\text{wsat}(n, S_k) = \binom{k + 1}{2}.$$

**Proof.** Let $G$ be a weakly $S_k$-saturated graph of order $n$ with the minimum number of edges. Then there is a complementary sequence $e_1, e_2, \ldots, e_l$ of $G$. Let $e_1 = u_1v_1$ and $d_G(u_1) = k$. Let $e_{f(1)}, \ldots, e_{f(t_1)}$ be the subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex $u_1$. If in the graph $G' = ((G + e_{f(1)}) + e_{f(2)}) + \ldots + e_{f(t_1)}$ there is no vertex of degree less than $k$ then let $e_{f(1)}, e_{f(2)}, \ldots, e_{f(t_l)}$ be the new sequence of edges of $E(G)$ with the following property: $e_{f(1)}, \ldots, e_{f(t_1)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex $u_1$ and $e_{f(t_1)+1}, \ldots, e_{f(l)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that any edge $e_{f(i)}$, $(t_1 \leq i \leq l)$ is not adjacent with the vertex $u_1$. If in the graph $G''$ there is a vertex of degree less than $k$
then let \( e_{f(t_1+1)} \) be the first edge of \( e_1, e_2, \ldots, e_l \), which is not adjacent with the vertex \( u_1 \). Let \( e_{f(t_1+1)} = u_2v_2 \) and \( u_2 \) be a vertex of \( G' \) such that \( d_{G'}(u_2) \geq k \) and \( u_1 \neq u_2 \). Let \( e_{f(t_1+1)}, \ldots, e_{f(t_2)} \) denote edges of \( \{e_1, e_2, \ldots, e_l\} - \{e_{f(t_1)}, \ldots, e_{f(t_1+1)}\} \) which are adjacent with the vertex \( u_2 \). If in the graph \( G'' = (G' + e_{f(t_1+1)} + e_{f(t_1)+2} + \ldots + e_{f(t_2)} \) there is no vertex of degree less than \( k \) we form a new sequence of edges of \( E(G) \), \( e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)} \) with the following property: \( e_{f(1)}, \ldots, e_{f(t_1)} \) is a subsequence of \( e_1, e_2, \ldots, e_l \) such that every edge \( e_{f(i)}, \ (1 \leq i \leq t_1) \) is adjacent with the vertex \( u_1 \) and \( e_{f(t_1)+1}, \ldots, e_{f(t_2)} \) is a subsequence of \( e_1, e_2, \ldots, e_l \) such that every edge \( e_{f(i)}, \ (t_1 < i \leq t_2) \) is adjacent with the vertex \( u_2 \) and \( e_{f(t_2)+1}, \ldots, e_{f(l)} \) is the subsequence of \( e_1, e_2, \ldots, e_l \) such that any edge \( e_{f(i)}, \ (t_2 < i \leq l) \) is not adjacent with the vertex \( u_1 \) and \( u_2 \). If in the graph \( G'' \) there is a vertex of degree less than \( k \), we will repeat this steps until we will obtain a new sequence \( e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)} \) of edges of \( G \). With this sequence of edges \( e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)} \) is related a sequence of vertices \( u_1, u_2, \ldots, u_r \). It is easy to see that \( r \leq k \), because after \( k \) steps there is no vertex of degree less than \( k \). Then for the vertex \( u_t \in \{u_1, \ldots, u_r\} \) we have

\[
(1) \quad d_G(u_t) + t - 1 - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}| \geq k,
\]

for the vertex \( x \in V(G) - \{u_1, \ldots, u_r\} \) we have

\[
(2) \quad d_G(x) + r - |N_G(x) \cap \{u_1, \ldots, u_r\}| \geq k.
\]

Thus

\[
e(G) \geq \sum_{1 \leq t \leq r} (d_G(u_t) - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}|) + \frac{1}{2} \sum_{x \in V(G) - \{u_1, \ldots, u_r\}} (d_G(x) - |N_G(x) \cap \{u_1, \ldots, u_r\}|) \\
\geq \sum_{1 \leq t \leq r} (k + 1 - t) + \frac{1}{2}(n - r)(k - r).
\]

The right side of inequality achieves the minimum for \( r = k \). Thus

\[
e(G) \geq \sum_{1 \leq t \leq r} (k + 1 - t) = \frac{1}{2}(k + 1)k.
\]

On the other hand, the graph \( K_{k+1} \cup (n - k - 1)K_1 \) is weakly \( S_k \)-saturated. Thus \( \text{wsat}(n, S_k) \leq \binom{k+1}{2} \).

In the next theorem we determine the number \( \text{wsat}(n, \mathcal{P}) \) for a hereditary property with one forbidden subgraph which is a cycle of odd length.

**Theorem 14.** Let \( k \geq 1 \) and \( n \geq 2k+2 \). If \( \mathcal{P} \) is the hereditary property such that \( F(\mathcal{P}) = \{C_{2k+1}\} \), then \( \text{wsat}(n, \mathcal{P}) = n - 1 \).
Proof. Since $\lambda(P) = 2$, by Theorem 3 it follows that every weakly $P$-saturated graph is connected. Then $\text{wsat}(n, P) \geq n - 1$. To prove that the inequality $\text{wsat}(n, P) \leq n - 1$ holds it is sufficient to show that there is a weakly $P$-saturated graph of order $n$ with $n - 1$ edges.

Let us show first that $P_{2k+2}$ is a weakly $P$-saturated graph. Let $V(P_{2k+2}) = \{v_1, \ldots, v_{2k+2}\}$ and $d(v_1) = d(v_{2k+2}) = 1$. It is easy to see that if we add the edge $v_1v_{2k+1}$ then we obtain a cycle of order $2k + 1$. Similarly if we add the edge $v_2v_{2k+2}$ a new cycle of order $2k + 1$ appears. Now we can add the edge $v_1v_4$. The edge $v_1v_4$ belongs to the cycle $v_1, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_4, v_1$. To prove that if we add any edge $v_1v_{2l}$ then a new cycles of order $2k + 1$ appears we will use induction on $t$. This is true for $t = 1, 2$. When the edges $v_1v_{2l}$ for $i < t$ are added the vertices $v_1, v_{2t-2}, v_{2t-3}, \ldots, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_{2t}, v_1$ induce a cycle of order $2k + 1$ which contains the edge $v_1v_{2t}$. In the same manner, after having added edges $v_1v_{2l+1}$ for $k \geq i > t$ we can add the edge $v_1v_{2l+1}$. A new cycle $v_1, v_{2t+3}, \ldots, v_{2k+2}, v_2, v_3, \ldots, v_{2t+1}, v_1$ of order $2k + 1$ appears. Finally the vertex $v_1$ with all vertices of $P_{2k+2}$ is joined. Similarly we can join each vertex $v_t$ ($2 \leq t \leq 2k + 2$) with all vertices of $P_{2k+2}$. Thus we obtain a graph $K_{2k+2}$. Hence $P_{2k+2}$ is a weakly $P$-saturated graph.

Let $G$ be the graph of order $n \geq 2k + 2$ with the following properties: $G$ contains an induced path of order $2k + 2$, the remaining vertices of $G$ form an independent set and each vertex of this set is adjacent with exactly one vertex of the path. Since the path of order $2k + 2$ is weakly $P$-saturated, it follows that the graph $G$ is weakly $P$-saturated. Hence $\text{wsat}(n, P) \leq n - 1$.

In order to determine the number $\text{wsat}(n, P)$ for hereditary property such that $F(P) = \{C_{2k}\}$ we need the following lemma.

Lemma 15. Let $k \geq 2$ and $P$ be the hereditary property such that $F(P) = \{C_{2k}\}$, and $G$ be a bipartite graph of order $n \geq 2k + 1$. Then $G \notin \text{WSat}(n, P)$.

Proof. On the contrary, suppose that there is a weakly $P$-saturated bipartite graph $G$ of order $n$. Let $e_1, e_2, \ldots, e_l$ be a complementary sequence of $G$. Let $e_i = xy$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$ such that its ends $x, y$ belong to the same colour class of $G$. (Notice, that the colour classes of $G$ are uniquely determined because of connectivity of $G$.) Since the edge $e_l$ belongs to an even cycle $C_{2k}$ then there is an edge $e_j$, $j < i$ of this cycle (and the sequence given above) with both ends in one colour class which is impossible. ■
Theorem 16. Let $k \geq 2$ and $n \geq 2k + 1$. Let $\mathcal{P}$ be the hereditary property such that $F(\mathcal{P}) = \{C_{2k}\}$. Then

$$\text{wsat}(n, \mathcal{P}) = n.$$  

Proof. Let $G \in \text{WSat}(n, \mathcal{P})$. By Theorem 3 and Lemma 15 it follows that $G$ is connected and contains an odd cycle. Thus $\text{wsat}(n, \mathcal{P}) \geq n$.

To prove that the inequality $\text{wsat}(n, \mathcal{P}) \leq n$ holds it is sufficient to show that there is a weakly $\mathcal{P}$-saturated graph of order $n$ with $n$ edges. First we prove that $C_{2k+1}$ is a weakly $\mathcal{P}$-saturated graph. Let $V(C_{2k+1}) = \{v_1, v_2, \ldots, v_{2k+1}\}$. It is easy to see that if we add the edge $v_1v_3$ or the edge $v_2v_{2k+1}$, a cycle (containing this edge) of order $2k$ appears. To prove that if we add any edge $v_1v_t$ ($3 \leq t \leq 2k$) then we obtain a new cycle of order $2k$ we use induction on $t$. This is true for $t = 3$. After adding edges $v_1v_i$ for $3 \leq i < t$ the vertices $v_1, v_{t-2}, v_{t-3}, \ldots, v_2, v_{2k+1}, v_{2k}, \ldots, v_t, v_1$ induce a cycle of order $2k$ which contains the edge $v_1v_t$. Then the vertex $v_1$ can be joined with all vertices of $C_{2k+1}$. In the similar manner we can show that we can join any vertex $v_t \in V(C_{2k+1})$ with all vertices of $C_{2k+1}$. Hence $C_{2k+1}$ is weakly $\mathcal{P}$-saturated.

Let $G$ be the graph with the following properties: $G$ contains an induced cycle of order $2k + 1$, remaining vertices of $G$ form an independent set and each vertex of this set is adjacent with exactly one vertex of the cycle. Since the cycle of order $2k + 1$ is weakly $\mathcal{P}$-saturated (can be extended to $K_{2k+1}$), it follows that the graph $G$ also has this property, i.e., $G$ is weakly $\mathcal{P}$-saturated. Hence $\text{wsat}(n, \mathcal{P}) \leq n$.

References


Received 20 August 2000
Revised 3 December 2001