

VERTEX-DISJOINT STARS IN GRAPHS

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Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least $k + t - 1$ and the order is at least $(t + 1)k + O(t^2)$, then the graph contains k vertex-disjoint copies of a star $K_{1,t}$. The condition on the minimum degree is sharp, and there is an example showing that the term $O(t^2)$ for the number of uncovered vertices is necessary in a sense.

Keywords: stars, vertex-disjoint copies, minimum degree.

2000 Mathematics Subject Classification: 05C35, 05C70.

1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G , we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G , respectively.

For a graph F and a positive integer k , kF denotes the vertex-disjoint union of k copies of F . A spanning subgraph isomorphic to kF for some integer k is called an F -factor. There are several results concerning minimum degree conditions for a graph to have an F -factor for several specific graphs F . The result of Corrádi and Hajnal [3] implies that $\delta(G) \geq \frac{2}{3}|V(G)|$ suffices for the existence of a K_3 -factor. (When we consider an F -factor of a graph G , we always assume that $|V(G)|$ is a multiple of $|V(F)|$). Dirac [4] generalized this result by showing that if $\delta(G) \geq \frac{1}{2}(|V(G)| + k)$,

then G contains k vertex-disjoint triangles for any integer k with $3k \leq |V(G)|$. Enomoto, Kaneko and Tuza [7] proved for $F = P_3$ (the path of order three) that $\delta(G) \geq \frac{1}{3}|V(G)|$ is sufficient for the existence of an F -factor if we assume that G is connected. Hajnal and Szemerédi [8] proved that for $F = K_t$ $\delta(G) \geq \frac{t-1}{t}|V(G)|$ suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that $\delta(G) \geq \left(\frac{\chi(F)-1}{\chi(F)} + o(1)\right)|V(G)|$ assures the existence of an F -factor, where $\chi(F)$ denotes the chromatic number of F .

On the other hand, if we want to find k vertex-disjoint copies of F in a graph G of order slightly larger than $k|F|$, and if F admits a $\chi(F)$ -coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of G is

$$\frac{1}{\chi(F)-1} \left(\chi(F) - 2 + \frac{\alpha}{|V(F)|} \right) |V(G)|,$$

where α is the smallest possible color class size in any $\chi(F)$ -coloring of F .

In the case where F is a star $K_{1,t}$, Alon and Yuster's result implies that $\delta(G) \geq \left(\frac{1}{2} + o(1)\right)|V(G)|$ is sufficient for the existence of a $K_{1,t}$ -factor, and Komlós, Alon and Fischer's result implies that $\delta(G) \geq \frac{1}{t+1}|V(G)|$ is sufficient for the existence of k copies of $K_{1,t}$ if $|V(G)|$ is large. In this paper, we prove the following theorem, in which the required minimum degree of G does not depend on $|V(G)|$. The proof is given in the next section.

Theorem 1. *Let t be an integer with $t \geq 3$. If G is a graph of order $n \geq (t+1)k + 2t^2 - 3t - 1$ with minimum degree at least $k + t - 1$, then G contains k vertex-disjoint copies of $K_{1,t}$.*

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let H be a $t-1$ regular graph of order sufficiently large, and let G be obtained from H by adding $k-1$ new vertices which are joined to all other vertices. Then $\delta(G) = k + t - 2$. Since any $K_{1,t}$ subgraph of G must contain one of the new vertices, G does not contain k vertex-disjoint copies of $K_{1,t}$.

On the other hand, the following example shows that the term $O(t^2)$ for the number of uncovered vertices is necessary. Let $k_1 + k_2 + \cdots + k_t = k - 1$ so that $|k_i - k_j| \leq 1$ for any i and j . We define the graph G to be the vertex-disjoint union of the complete graphs $K_{(t+1)k_1+t}, K_{(t+1)k_2+t}, \dots, K_{(t+1)k_t+t}$.

Then, $|V(G)| = (t + 1)(k - 1) + t^2 = (t + 1)k + t^2 - t - 1$ and $\delta(G) = (t + 1)\lfloor \frac{k-1}{t} \rfloor + t - 1 \geq k + t - 1$ (if $k \gg t$). However, it is obvious that G contains at most $k - 1$ copies of $K_{1,t}$.

This example suggests that the same conclusion as in Theorem 1 follows if $|V(G)| \geq (t + 1)k + t^2 - t$. In fact, it is known to be true for $t \leq 3$. The case $t = 1$ is an easy exercise. The case $t = 3$ is proved in [6].

Theorem 2 [6]. *If G is a graph with $|V(G)| \geq 4k + 6$ and $\delta(G) \geq k + 2$, then G contains k vertex-disjoint copies of $K_{1,3}$.*

The case $t = 2$ can be proved in the following way. We use the following theorem due to Enomoto [5].

Theorem 3 [5]. *Let G be a connected graph of order n and $n = n_1 + \dots + n_k$ with $n_i \geq 2$ ($1 \leq i \leq k$). If $\delta(G) \geq k$, then $V(G)$ can be partitioned into V_1, \dots, V_k so that for each i , $|V_i| = n_i$ and V_i induces a subgraph without isolated vertices.*

Corollary 4. *Let G be a connected graph of order n , and k be an integer with $3k \leq n$. If*

$$\delta(G) \geq \begin{cases} k, & \text{if } n - 3k \text{ is even,} \\ k + 1, & \text{if } n - 3k \text{ is odd,} \end{cases}$$

then G contains k vertex-disjoint copies of P_3 .

Proof. If $n - 3k$ is even, then put $n_1 = \dots = n_{k-1} = 3$ and $n_k = n - 3k + 3$, and apply Theorem 3. If $n - 3k$ is odd, then by deleting one vertex from G so that the resulting graph is connected, we can apply the previous case. ■

Now we can prove the following theorem.

Theorem 5. *Let G be a graph of order n with $n \geq 3k + 2$. If $\delta(G) \geq k + 1$, then G contains k vertex-disjoint copies of P_3 .*

Proof. If G is connected, or if G has a component of order at least $3k$, then the result follows immediately from Corollary 4. Suppose that G is disconnected and each component is order less than $3k$. Note that by Corollary 4, each component C of G contains $\lfloor |V(C)|/3 \rfloor$ vertex-disjoint copies of P_3 . Also, since $\delta(G) \geq k + 1$, each component has at least $k + 2$ vertices.

If G has at least three components, then G contains at least $3\lfloor \frac{k+2}{3} \rfloor \geq k$ copies of P_3 , and we are done. If G consists of two components of orders n_1 and n_2 , then the number of vertex-disjoint copies of P_3 in G is at least $\lfloor \frac{n_1}{3} \rfloor + \lfloor \frac{n_2}{3} \rfloor \geq \lceil \frac{n_1+n_2-4}{3} \rceil \geq \lceil \frac{3k+2-4}{3} \rceil = k$. ■

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

2. Proof of Theorem 1

Let t be an integer with $t \geq 3$, and let G be a graph of order at least $(t+1)k + 2t^2 - 3t - 1$ and minimum degree at least $k + t - 1$.

We use the following notation and terminology. For $S \subset V(G)$, we write $\langle S \rangle$ for the subgraph of G induced by S . For disjoint vertex sets S and T , we denote the set of edges joining S and T by $E(S, T)$.

We consider a partition $V(G) = X \cup Y \cup Z$ satisfying the following conditions:

- (a) $|X| = (t+1)p$ and X contains p vertex-disjoint copies of $K_{1,t}$, say C_1, C_2, \dots, C_p .
- (b) The vertices of Y can be labelled y_1, y_2, \dots, y_q so that for each r ($1 \leq r \leq q$), $|N_G(y_r) \cap Z| \geq rt + (2t - 1)$.

Note that $X = Y = \emptyset$ and $Z = V(G)$ satisfy the above conditions with $p = q = 0$. We choose such a partition so that $p + q$ is maximum, and subject to this condition, q is maximum possible.

Claim 1. For any subset $A \subset Z$ with $|A| \leq 2t - 1$, $G - X - A$ contains q vertex-disjoint copies of $K_{1,t}$. In particular, G contains $p + q$ vertex-disjoint copies of $K_{1,t}$.

Proof. By the condition (b), it follows that $|N_G(y_r) \cap (Z - A)| \geq rt + 2t - 1 - |A| \geq rt$ for each $1 \leq r \leq q$. Therefore we can complete to take q stars in $\langle Y \cup (Z - A) \rangle$ whose centers are y_1, y_2, \dots, y_q , respectively. ■

Claim 2. For any subset $A \subset Z$ with $|A| \leq 2t - 1$, $\langle X \cup A \rangle$ does not contain $p + 1$ vertex-disjoint copies of $K_{1,t}$.

Proof. Suppose that $\langle X \cup A \rangle$ contains $(p + 1)K_{1,t}$. Then by Claim 1, G contains $(p + q + 1)K_{1,t}$. Let $V(G) = X' \cup Y' \cup Z'$ be a partition such that X'

is the set of vertices contained in $(p + q + 1)K_{1,t}$ and $Y' = \emptyset$. This partition satisfies the condition (a) and (b), and contradicts the maximality of $p + q$. ■

In particular, we have the following.

Claim 3. The maximum degree of $\langle Z \rangle$ is less than t . ■

Let a be the center of any star C_i in X . If $|N_G(a) \cap Z| \geq tq + 2t - 1$, then we put $X' = X - V(C_i)$, $Y' = Y \cup \{a\}$ with $y_{q+1} = a$, and $Z' = Z \cup (V(C_i) - \{a\})$. Then $\langle X' \rangle$ contains $p - 1$ vertex-disjoint $K_{1,t}$'s, and $|N_G(y_{q+1}) \cap Z'| = |N_G(a) \cap Z| + t \geq t(q + 1) + 2t - 1$. This contradicts the maximality of q . Hence we have

$$(1) \quad |N_G(a) \cap Z| \leq tq + 2t - 2.$$

By a similar argument, for each leaf b of any star C_i in X , we have

$$(2) \quad |N_G(b) \cap Z| \leq tq + 3t - 3.$$

Claim 4. For each $1 \leq i \leq p$, $|E(C_i, Z)| \leq \max\{tq + t^2 + t - 2, 2t^2 - 2t\}$.

Proof. Let a be the center and b_1, b_2, \dots, b_t be the leaves of C_i .

Case 1. $|E(a, Z)| \geq t + 1$.

In this case, each b_j is adjacent to at most $t - 1$ vertices in Z . For otherwise, we can take $A \subset N(b_j) \cap Z$ with $|A| = t$ and $z \in N(a) \cap Z - A$ so that $\langle \{b_j\} \cup A \rangle$ and $\langle (V(C_i) - \{b_j\}) \cup \{z\} \rangle$ contain a $K_{1,t}$. This contradicts Claim 2. Hence $|E(b_j, Z)| \leq t - 1$ for all j ($1 \leq j \leq t$). Then, since $|E(a, Z)| \leq tq + 2t - 2$ by (1),

$$\begin{aligned} |E(V(C_i), Z)| &= |E(a, Z)| + \sum_{j=1}^t |E(b_j, Z)| \\ &\leq tq + 2t - 2 + t(t - 1) = tq + t^2 + t - 2. \end{aligned}$$

Case 2. $1 \leq |E(a, Z)| \leq t$.

If $|E(b_j, Z)| \geq t + 1$ for some j ($1 \leq j \leq t$), then we can take $z \in N(a) \cap Z$ and $A \subset N(b_j) \cap Z - \{z\}$ with $|A| = t$ so that $\langle V(C_i) \cup A \cup \{z\} \rangle$ contains $2K_{1,t}$, a contradiction. Hence $|E(b_j, Z)| \leq t$ for all j . Thus, $|E(V(C_i), Z)| \leq (t + 1)t \leq 2t^2 - 2t$, since $t \geq 3$.

Case 3. $|E(a, Z)| = 0$.

If each leaf of C_i is adjacent to at most $2t - 2$ vertices in Z , then we have $|E(V(C_i), Z)| \leq 2t^2 - 2t$. Hence we may assume that there exists a vertex b_h ($1 \leq h \leq t$) with $|E(b_h, Z)| \geq 2t - 1$. If $|E(b_j, Z)| \geq t - 1$ for some j with $j \neq h$, then we can take $A \subset N(b_j) \cap Z$ with $|A| = t - 1$ and $A' \subset N(b_h) \cap Z - A$ with $|A'| = t$ so that $\langle \{b_h, b_j, a\} \cup A \cup A' \rangle$ contains $2K_{1,t}$'s, a contradiction. Hence $|E(b_j, Z)| \leq t - 2$ for all $j \neq h$. Since $|E(b_h, Z)| \leq tq + 3t - 3$ by (2), we have $|E(V(C_i), Z)| \leq tq + 3t - 3 + (t - 1)(t - 2) < tq + t^2 + t - 2$.

This completes the proof of Claim 4. \blacksquare

Now, we shall estimate the number of edges joining X and Z in two ways, by assuming that G does not contain k vertex-disjoint $K_{1,t}$'s. By Claim 3, each vertex in Z is adjacent to at least $(k + t - 1) - (t - 1) - q = k - q$ vertices of X . Hence,

$$|E(X, Z)| \geq (k - q)|Z| = (k - q)(n - (t + 1)p - q).$$

On the other hand, it follows from Claim 4 that

$$\begin{aligned} |E(X, Z)| &= \sum_{i=1}^p |E(V(C_i), Z)| \\ &\leq p \cdot \max\{tq + t^2 + t - 2, 2t^2 - 2t\}. \end{aligned}$$

If $tq + t^2 + t - 2 \geq 2t^2 - 2t$, or equivalently if $q \geq t - 2$, then

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(tq + t^2 + t - 2),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)k + t^2 + t - 2 - q).$$

By Claim 1, we may assume that $p + q \leq k - 1$. Thus the above inequality implies that

$$(k - q)(n - q) \leq (k - 1 - q)((t + 1)k + t^2 + t - 2 - q),$$

and hence

$$\begin{aligned} n &\leq (t + 1)k + t^2 + t - 2 - \frac{(t + 1)k + t^2 + t - 2 - q}{k - q} \\ &= (t + 1)k + 2t^2 - 3t - 1 - \left(t^2 - 4t + 3 + \frac{(t - 1)k + t^2 + t - 2 + q}{k - q} \right) \\ &< (t + 1)k + 2t^2 - 3t - 1. \end{aligned}$$

This contradicts the assumption that $n \geq (t + 1)k + 2t^2 - 3t - 1$.

If $q \leq t - 3$, then since $tq + t^2 + t - 2 < 2t^2 - 2t$,

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(2t^2 - 2t),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)(k - q) + 2t^2 - 2t).$$

Since $p + q \leq k - 1$,

$$\begin{aligned} (k - q)(n - q) &\leq (k - 1 - q)((t + 1)(k - q) + 2t^2 - 2t), \\ n &\leq (t + 1)k + 2t^2 - 3t - 1 - tq - \frac{2t^2 - 2t}{k - q} \\ &< (t + 1)k + 2t^2 - 3t - 1. \end{aligned}$$

This is a contradiction.

This completes the proof of Theorem 1. ■

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Received 27 September 2000

Revised 19 March 2001