Discussiones Mathematicae
Graph Theory 21 (2001) 167–177

GALLAI'S INNEQUALITY FOR CRITICAL GRAPHS OF REDUCIBLE HEREDITARY PROPERTIES

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Abstract

In this paper Gallai's inequality on the number of edges in critical graphs is generalized for reducible additive induced-hereditary properties of graphs in the following way. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k \, (k \geq 2)$ be additive induced-hereditary properties, $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_k$ and $\delta = \sum_{i=1}^{k} \delta(\mathcal{P}_i)$. Suppose that G is an $\mathcal{R}$-critical graph with n vertices and m edges. Then $2m \geq \delta n + \frac{\delta-2}{\delta^2+2\delta-2}\, n + \frac{2\delta}{\delta^2+2\delta-2}$ unless $\mathcal{R} = \mathcal{O}^2$ or $G = K_{\delta+1}$. The generalization of Gallai's inequality for $\mathcal{P}$-choice critical graphs is also presented.

Keywords: additive induced-hereditary property of graphs, reducible property of graphs, critical graph, Gallai's Theorem.

2000 Mathematics Subject Classification: 05C15, 05C75.

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[1]Research supported in part by Slovak VEGA Grant 2/1131/21.

[2]Research supported in part by the Ministry of Science and Technology of Slovenia, Research Project P1–0210–101–00.

1. Introduction and Notation

A convenient language that may be used in formulating problems of graph colouring in a general setting is the language of reducible properties of graphs. Let us denote by $\mathcal{I}$ the class of all finite simple graphs. A property of graphs $\mathcal{P}$ is any nonempty proper isomorphism closed subclass of $\mathcal{I}$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A graph G is vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*colorable* ($G$ has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$) if the vertex set $V(G)$ of G can be partitioned into n sets $V_1, V_2, \ldots, V_n$ such that the subgraph $G[V_i]$ of G induced by V_i belongs to $\mathcal{P}_i$, $i = 1, 2, \ldots, n$. The corresponding vertex coloring f is defined by $f(v) = i$ whenever $v \in V_i, i = 1, 2, \ldots, n$. In the case $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$ we write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n = \mathcal{P}^n$ and we say that $G \in \mathcal{P}^n$ is $(\mathcal{P}, n)$-colorable. Let us denote by $\mathcal{O}$ the class of all edgeless graphs. The classical graph coloring problems deals with *proper* coloring where $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_n = \mathcal{O}$ so that a graph G is n-*colorable* if and only if $G \in \mathcal{O}^n$. The basic property of the proper coloring is that every induced subgraph of a n-colorable graph is n-colorable and if every connected component of a graph G is n-colorable, then G is n-colorable, too. In this paper we consider as the generalizations of the proper coloring only such vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colorings where the properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ preserve the above mentioned requirements i.e., they are closed to induced subgraphs and disjoint union of graphs. Such properties of graphs are called *induced-hereditary and additive.* The set of all (additive) induced-hereditary properties will be denoted by ($\mathbb{M}^a$) $\mathbb{M}$.

An additive induced-hereditary property $\mathcal{R}$ is said to be *reducible* if there exist additive induced-hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$, otherwise the property $\mathcal{R}$ is *irreducible.*

If $\mathcal{P}$ is an induced-hereditary property, then the set of minimal forbidden subgraphs of $\mathcal{P}$, called $\mathcal{P}$-*critical graphs*, is defined as follows:

$$\mathcal{C}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but for each proper induced subgraph } H \text{ of } G, \\ H \in \mathcal{P}\}.$$

Every additive induced-hereditary property $\mathcal{P}$ is uniquely determined by the set of connected minimal forbidden subgraphs. For the class $\mathcal{O}^k$ of all k-colorable graphs the set $\boldsymbol{C}(\mathcal{O}^k)$ consists of vertex-$(k+1)$-critical graphs.

To investigate the structure of $\mathcal{R}$-critical graphs the following invariants of properties are useful. For an arbitrary graph theoretical invariant ρ and an induced-hereditary property $\mathcal{P}$ let us define:

$$\rho(\mathcal{P}) = \min\{\rho(F) : F \in \boldsymbol{C}(\mathcal{P})\}.$$

E.g. the invariant $\chi(\mathcal{P})$ is used in extremal graph theory. It is quite easy to prove that for every $G \in \mathcal{C}(\mathcal{R})$, $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, the minimum degree $\delta(G)$ of G is at least $\delta = \delta(\mathcal{P}_1) + \delta(\mathcal{P}_2) + \cdots + \delta(\mathcal{P}_n)$ i.e., $\delta(\mathcal{R}) \geq \delta$. Let us call the vertices of degree $\delta = \delta(\mathcal{P}_1) + \delta(\mathcal{P}_2) + \cdots + \delta(\mathcal{P}_n)$ in the graph $G \in \mathcal{C}(\mathcal{R})$ *minor*.

Analogously as for $\mathcal{O}^n$-critical graphs, using the classical recoloring method of Gallai [6], generalizations of the well-known Gallai's theorem can be obtained.

Theorem 1 [4]**.** *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be additive induced-hereditary properties, $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ and $G \in \boldsymbol{C}(\mathcal{R})$. Then every block B of the subgraph induced by the set of minor vertices of G is one of the following types:*

(a) *B is a complete graph of order $\leq \delta + 1$,*

(b) *B is a $\delta(\mathcal{P}_i)$-regular graph and $B \in \mathcal{C}(\mathcal{P}_i)$ for some i,*

(c) *$\Delta(B) \leq \delta(P_i)$ and $B \in \mathcal{P}_i$,*

(d) *B is an odd cycle.*

An analogous result for $\mathcal{P}$-choice critical graphs have been obtained in [3]. The presented results can be considered as generalizations of Gallai's and Brooks' Theorems (see [2, 5, 8, 15, 16]).

Let G be a graph and let $L(v)$ be a list of colours prescribed for the vertex v, and $\mathcal{P} \in \mathbb{M}$. A $(\mathcal{P}, L)$-*colouring* is a graph $\mathcal{P}$-colouring f with the additional requirement that for all $v \in V(G), f(v) \in L(v)$. If G admits a $(\mathcal{P}, L)$-colouring, then G is said to be $(\mathcal{P}, L)$-*colourable.* The graph G is $(\mathcal{P}, k)$-*choosable* if it is $(\mathcal{P}, L)$-colourable for every list L of G satisfying $|L(v)| \geq k$ for every $v \in V(G)$. The $\mathcal{P}$-*choice number* $\mathrm{ch}_{\mathcal{P}}(G)$ of the graph G is the smallest natural number k such that G is $(\mathcal{P}, k)$-choosable.

For a property $\mathcal{P} \in \mathbb{M}$ a graph G is said to be $(\mathcal{P}, L)$-*critical* if G has no $(\mathcal{P}, L)$-colouring but $G - v$ is $(\mathcal{P}, L)$-colourable for all $v \in V(G)$. The following statement is easy to prove: If $\mathcal{P} \in \mathbb{M}$ and G is $(\mathcal{P}, L)$-critical, then $d_G(v) \geq \delta(\mathcal{P})|L(v)|$ for any vertex v of G. Let us denote by $S(G) = \{v : v \in V(G),\ \ d_G(v) = \delta(\mathcal{P})|L(v)|\}$. For a nontrivial property $\mathcal{P} \in \mathbb{M}$, a graph G is said to be (*vertex*) $(\mathcal{P}, k)$-*choice critical* if $\mathrm{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\mathrm{ch}_{\mathcal{P}}(G - v) < k$ for all vertices v of G. According to the previous definitions, it follows immediately that if G is $(\mathcal{P}, k+1)$-choice critical, then G is $(\mathcal{P}, L)$-critical with some list $|L(v)| = k$ for all $v \in V(G)$.

Theorem 2 [3]. *Let $\mathcal{P}$ be an additive induced-hereditary property and G be a $(\mathcal{P}^k, L)$-critical graph (i.e., a $(\mathcal{P}, k+1)$-choice critical graph). Then every block B of the subgraph of G induced by the set $S(G) = \{v : v \in V(G), deg_G(v) = \delta(\mathcal{P})|L(v)|\}$ of minor vertices is one of the following types:*

(a) *B is a complete graph,*

(b) *B is a $\delta(\mathcal{P})$-regular graph and $B \in \mathcal{C}(\mathcal{P})$,*

(c) *$\Delta(B) \leq \delta(\mathcal{P})$ and $B \in \mathcal{P}$,*

(d) *B is an odd cycle.*

As for chromatically critical graphs, the structure of the subgraph induced by minor vertices of a critical graphs G implies a lower bound on the number of edges of G, which will be considered in Section 3.

2. δ-Graphs

Denote by K_n^+ the graph comprising of two blocks where the first one is isomorphic to K_n and the second is isomorphic to K_2. The graph K_n^+ has only one vertex of degree 1, we call it a *pendant-vertex* of K_n^+, the subgraph K_n in K_n^+ is the *head* of K_n^+.

A connected graph G is a *δ-graph* ($\delta \geq 1$) if all cut-vertices of G are of degree δ and all other vertices are of degree $\delta - 1$. Thus K_δ is a δ-graph. Let G be a δ-graph and let B be an endblock of G isomorphic to K_δ. Note that if $B \neq G$, then B is a head of a subgraph, say H, isomorphic to K_δ^+. We will use to say that H is a *pendant* K_δ^+ of G. The graph H is *redundant*, if by deleting the head of H in G, the remaining graph is also a δ-graph. If G is a δ-graph with no redundant pendant K_δ^+ subgraphs, then G is a *compact* δ-graph.

Theorem 3. *Let G^* be a δ-graph with n vertices and c cut-vertices. Then,*

$$\frac{c}{2} \leq \frac{n}{\delta} - 1. \tag{1}$$

Proof. For the sake of simplicity in this proof, for an arbitrary δ-graph H on n_h vertices and with c_h cut-vertices, we define $\varphi(H) = n_h/\delta - c_h/2$. So, we should prove that $\varphi(G^*) \geq 1$.

Suppose that the claim is false and G^* is a counterexample with n minimum. Thus, $\varphi(G^*) < 1$. It is easy to see that G^* is not 2-connected, since in this case $c = 0$ and $n \geq \delta$. Note that the claim is valid for $\delta = 1$. So, we assume that $\delta > 1$. Let $\mathcal{B}$ be the set of blocks of G^*.

Claim 1. *G^* is a compact δ-graph.*

Suppose that it is false. Then there is a redundant pendant K_δ^+ subgraph H in G^*. Let $\widehat{G}$ be the graph constructed from G^* by deleting the head of H. Then $\widehat{G}$ is a δ-graph. Let $\widehat{n}$ and $\widehat{c}$ be the number of vertices and the number of cut-vertices of $\widehat{G}$. Obviously, $\widehat{n} = n - \delta$. Obsereve that the pendant-vertex v of H is incident with precisely two blocks of G^*. In the first block (that is the bridge of H) v has degree 1 and in the second block it has degree $\delta - 1$. So after deleting the head of H, in the remaining graph $\widehat{G}$ the vertex v is not a cut-vertex any more. Therefore, $\widehat{c} = c - 2$. By the minimality of G^*, $\varphi(\widehat{G}) \geq 1$. So,

$$\varphi(G^*) = \frac{n}{\delta} - \frac{c}{2} = \frac{\widehat{n}}{\delta} - \frac{\widehat{c}}{2} = \varphi(\widehat{G}) \geq 1.$$

But it is a contradiction.

Claim 2. *Every bridge of G^* is an edge of a pendant K_δ^+.*

Suppose that it is false. Let $e = u_1u_2$ be an bridge that is not an edge of a pendant K_δ^+ subgraph of G^*. Denote by G_1 and G_2 the components of the graph $G^* - e$ and let us assume that u_i be a vertex in G_i. Let G_i^* be the δ-graph constructed from G_i by gluing at u_i the pendant vertex of a K_δ^+. Since, e is not a part of a pendant K_δ^+ subgraph of G^*, it follows that G_1^* and G_2^* have smaller number of vertices than G^*. Now, by the minimality, we infer that $\varphi(G_1^*) \geq 1$ and $\varphi(G_2^*) \geq 1$. Denote by n_i and c_i the number of vertices and the number of cut-vertices in G_i^*. Then,

$$n = n_1 + n_2 - 2\delta \qquad \text{and} \qquad c = c_1 + c_2 - 2.$$

Now, we obtain a contradiction in the following way

$$\varphi(G^*) = \frac{n}{\delta} - \frac{c}{2} = \frac{n_1 + n_2}{\delta} - \frac{c_1 + c_2}{2} - 1 = \varphi(G_1^*) + \varphi(G_2^*) - 1 \geq 1.$$

Thus Claim 2 is proved.

Let B^* be a block of G^*. Since G^* is a compact δ-graph, we may assume that B^* is not a head of a pendant K_δ^+ subgraph of G^*. We consider the block structure of G^* as a kind of rooted tree, whose root is B^*. In other words, we define a function depth : $\mathcal{B} \to \boldsymbol{N}$, as it follows. First, set depth$(B) = \infty$ for every $B \in \mathcal{B}$. Now, apply the following steps until every block gets finite depth:

Step 0. depth$(B^*) = 0$.

Step i. $(i \geq 1)$ If B is a block incident with a block whose depth is $i - 1$, then depth$(B) :=$ min (depth$(B), i)$.

If blocks B_1 and B_2 have common cut-vertex and depth$(B_1) =$ depth$(B_2)-1$, then we will say that B_1 is a *parent* of B_2 and B_2 is a *son* of B_1. Note that every block different from B^* has precisely one parent and it may have many sons. In the sequel, we will denote by n_B and c_B the number of vertices and the number of cut-vertices of a block B.

We assign a charge $\varphi(B)$ to every block $B \in \mathcal{B}$ in the next way:

$$\varphi(B) = \begin{cases} \dfrac{n_B - 1}{\delta} - \dfrac{c_B - 1}{2}, & B \neq B^*; \\ \dfrac{n_B}{\delta} - \dfrac{c_B}{2}, & B = B^*. \end{cases} \tag{2}$$

In fact, we assign charge $\frac{1}{\delta} - \frac{1}{2}$ to every cut-vertex of G^* and $\frac{1}{\delta}$ to every other vertex of G^*. Then, $\varphi(B)$ is the sum of the charges of all of its vertices except the cut-vertex incident with its parent. Note that the total sum of the charges of all blocks (or all vertices) is equal to $\varphi(G^*)$.

Now, we apply to every block the following rule. First, we apply it on the blocks with highest depth, then on the blocks with depth smaller for one, and so on.

Rule R. *Suppose that B_1 is a son of B_2 attached at a vertex v. Then, B_1 sends* (*through v*) *its charge and the charge received from its sons to B_2.*

Note that the redistribution will stop at block B^* since it has no parent. The total charge $\varphi(G^*)$ will be accumulated in B^*. Denote by $\widehat{c}(B, v)$ the charge that a block B receives from its sons attached at the cut-vertex v by Rule R.

Claim 3. *Suppose that v is a cut-vertex of G^* incident with a block B and incident also with some sons of B. Then, $\widehat{c}(B,v) \geq \frac{\delta-1}{\delta}$.*

Suppose that the claim is false and the pair (B,v) is a counterexample. We may assume that depth(B) is as large as possible. Suppose also that $\widehat{B}$ is an arbitrary son of B attached at v. Let us consider the minimal possible value of charge that $\widehat{B}$ could send to B by Rule R.

Note that every end-block of G^* has $\geq \delta$ vertices. So, if $\widehat{B}$ is an end-block, then it sends $\varphi(\widehat{B}) \geq \frac{\delta-1}{\delta}$ charge to B.

Suppose now that $\widehat{B}$ is a bridge. By Claim 2, $\widehat{B}$ is an edge of K_δ^+ subgraph whose pendant-vertex is v. So, in this case $\widehat{B}$ sends

$$\frac{\delta-1}{\delta} + \left(\frac{1}{\delta} - \frac{1}{2}\right) = \frac{1}{2}$$

charge to B.

Finally, we may assume that $\widehat{B}$ is neither an end-block nor a bridge of G^*. Note that $c_{\widehat{B}} \geq 2$. By the maximality of the depth of B, we infer that $\widehat{B}$ sends at least

$$\frac{n_{\widehat{B}}-1}{\delta} - \frac{c_{\widehat{B}}-1}{2} + (c_{\widehat{B}}-1)\frac{\delta-1}{\delta} \tag{3}$$

charge to B. If $c_{\widehat{B}} \geq 3$, then by (3) and by $n_{\widehat{B}} \geq c_{\widehat{B}}$, we infer $\widehat{c}(B,v) \geq (c_{\widehat{B}}-1)\frac{1}{2} \geq 1$. So, let $c_{\widehat{B}} = 2$. Since $\widehat{B}$ is not a bridge, there is a vertex $\widehat{v}$ of $\widehat{B}$ which is not a cut-vertex of G^*. Since $\widehat{v}$ is of degree $\delta-1$ in $\widehat{B}$, it follows that $\widehat{B}$ has at least δ vertices, i.e., $n_{\widehat{B}} \geq \delta$. Thus, by (3) and by $\delta \geq 2$, we obtain that $\widehat{B}$ sends charge to B at least

$$2\left(\frac{\delta-1}{\delta}\right) - \frac{1}{2} \geq \frac{\delta-1}{\delta}.$$

By above, if B has a son which is not a bridge attached at v then $\widehat{c}(B,v) \geq \frac{\delta-1}{\delta}$. So assume that all sons of B attached at vertex v are bridges. Then, by Claims 1 and 2 and by the choice of B^*, it follows that $k \geq 2$, and hence $\widehat{c}(B,v) \geq 1$. Thus Claim 3 is proved.

Using Claim 3, we will prove that $\varphi(G^*) \geq 1$ in a similar way as we argue above. Note that

$$\varphi(G^*) \geq \varphi(B^*) + c_{B^*}\frac{\delta-1}{\delta} \geq \frac{n_{B^*}}{\delta} - \frac{c_{B^*}}{2} + c_{B^*}\frac{\delta-1}{\delta}. \tag{4}$$

If $c_{B^*} \geq 2$, then by (4), we obtain $\varphi(G^*) \geq \frac{c_{B^*}}{2} \geq 1$. Thus let us assume that $c_{B^*} = 1$. In this case, B^* is an end-block and so $n_{B^*} \geq \delta$. Thus we infer that $\varphi(G^*) \geq \frac{\delta}{\delta} - \frac{1}{2} + \frac{\delta-1}{\delta} \geq 1$. This completes the proof of the theorem. ■

The following result is a generalization of Gallai's technical lemma [6, Lemma 4.5].

Corollary 4. *Let G be a graph with n vertices, m edges and let $\delta \geq 1$. Suppose that $\Delta(G) \leq \delta$ and each block B of G has maximum degree $\Delta(B) < \delta$. Then,*

$$m \leq \left(\frac{\delta - 1}{2} + \frac{1}{\delta}\right)n - 1. \tag{5}$$

Proof. Let us remark, that if G is 2-connected, then $\Delta(G) \leq \delta - 1$. Let G^* be the graph constructed from G in the following way: at every cut-vertex $v \in V(G)$ glue $\delta - d(v)$ copies of pendant K_δ^+ and at every other vertex of degree $< \delta - 1$ glue also $\delta - d(v)$ copies of pendant K_δ^+. Note that G^* is a δ-graph.

Suppose that we have added k copies of K_δ^+ in G in order to construct G^*. Denote by n^*, c^*, and m^* the number of vertices, the number of cut-vertices, and the number of edges of G^*, respectively. By Theorem 3, $\frac{c^*}{2} \leq \frac{n^*}{\delta} - 1$. Then,

$$\begin{aligned} m^* &= \frac{(\delta-1)}{2}(n^* - c^*) + \frac{\delta}{2}c^* = \frac{(\delta-1)n^*}{2} + \frac{c^*}{2} \\ &\leq \frac{(\delta-1)n^*}{2} + \frac{n^*}{\delta} - 1 = \left(\frac{\delta-1}{2} + \frac{1}{\delta}\right)n^* - 1. \end{aligned}$$

Thus we have proved the claim for G^*. Since,

$$n^* = n + \delta k \qquad \text{and} \qquad m^* = m + k\binom{\delta}{2} + k,$$

we have

$$\begin{aligned} m &= m^* - k\binom{\delta}{2} - k \leq \left(\frac{\delta-1}{2} + \frac{1}{\delta}\right)n^* - 1 - k\binom{\delta}{2} - k \\ &= \left(\frac{\delta-1}{2} + \frac{1}{\delta}\right)n - 1. \end{aligned}$$

■

3. Gallai's Inequality

Gallai [6] proved that a k-ciritical graph ($k \geq 4$) on n vertices and m edges, different from K_k satisfies the following inequality

$$2m \geq (k-1)\,n + \frac{k-3}{k^2-3}\,n.$$

This clasical result was later improved by Krivelevich [11, 12] and Kostochka and Stiebitz [9, 10]. See also the book of Jansen and Toft [7] for critical graphs with few edges.

Theorem 5. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ $(k \geq 2)$ be additive induced-hereditary properties, $\mathcal{R} = \mathcal{P}_1 \circ\, \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_k$ and $\delta = \sum_{i=1}^{k} \delta(\mathcal{P}_i)$. Suppose that G is an $\mathcal{R}$-critical graph with n vertices and m edges. Then*

$$2m \geq \delta n + \frac{\delta-2}{\delta^2+2\delta-2}\,n + \frac{2\delta}{\delta^2+2\delta-2} \tag{6}$$

unless $\mathcal{R} = \mathcal{O}^2$ or $G = K_{\delta+1}$.

Proof. Obviously, if $G = K_{\delta+1}$ then (6) is not satisfied. It is easy to see that if $\mathcal{R} = \mathcal{O}^2$, then G is an odd cycle. In this case ineqaulity (6) is also not satisfied. So, assume that $\mathcal{R} \neq \mathcal{O}^2$ and $G \neq K_{\delta+1}$. It is easy to see, that innequality (6) is satisfied for $\delta = 2$, since a cycle can be critical only for $\mathcal{R} = \mathcal{O}^2$. Hence we infer that $\delta \geq 3$. Denote by H the subgraph of G induced by the minor vertices i.e., vertices of degree δ. Let n' and m' be the number of vertices and the number of edges of H. It is not hard to see that

$$m \geq \delta n' - m'. \tag{7}$$

Since $\mathcal{R} \neq \mathcal{O}^2$, by Theorem 1 it follows that $\Delta(H) \leq \delta$ and each block B of H has $\Delta(B) < \delta$.

$$m \geq \delta n' - \left(\frac{\delta-1}{2} + \frac{1}{\delta}\right) n' + 1 \geq \left(\frac{\delta+1}{2} - \frac{1}{\delta}\right) n' + 1. \tag{8}$$

Similarly, the following is satisfied

$$2m \geq \delta n' + (\delta+1)(n-n') = (\delta+1)n - n'. \tag{9}$$

After multiplying (9) by $(\frac{\delta+1}{2} - \frac{1}{\delta})$ and additing it to (8), we obtain:

$$\left(\delta + 2 - \frac{2}{\delta}\right) m \geq (\delta + 1)\left(\frac{\delta+1}{2} - \frac{1}{\delta}\right) n + 1. \tag{10}$$

Finally, from (10) by some calculations, we easily obtain (6). ■

Remark that a special case of the above theorem was proved in [14]. Also remark, that Corollary 4 is a generalization of the Gallai's technical lemma [6, Lemma 4.5].

Using Theorem 2, the same arguments give us the $\mathcal{P}$-choice version of Gallai's inequality (as it is mentioned for $\mathcal{P} = \mathcal{O}$ in [8]):

Theorem 6. *Let $\mathcal{P}$ be additive induced-hereditary property and let $k \geq 2$. Let G be a $(\mathcal{P}, k+1)$-choice critical graph, with n vertices and m edges and $\delta = \delta(\mathcal{P})\, k$. Then*

$$2m \geq \delta n + \frac{\delta - 2}{\delta^2 + 2\delta - 2}\, n + \frac{2\delta}{\delta^2 + 2\delta - 2} \tag{11}$$

unless $\mathcal{R} = \mathcal{O}^2$ or $G = K_{\delta+1}$.

Let us finish the paper with the following problem. Dirac [1] proved that for every k-critical graph $G \neq K_k$ $(k \geq 3)$ on n vertices for the number of edges m the following inequality holds:

$$2m \geq (k-1)n + (k-3).$$

So an intersting problem is to generalized the above inequality for reducible additive induced-hereditary properties of graphs.

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Received 2 August 2000
Revised 9 May 2001