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TOTAL DOMINATION EDGE CRITICAL GRAPHS WITH MAXIMUM DIAMETER

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Abstract

Denote the total domination number of a graph G by $\gamma_t(G)$. A graph G is said to be total domination edge critical, or simply γ_t-critical, if $\gamma_t(G+e) < \gamma_t(G)$ for each edge $e \in E(\overline{G})$. For 3_t-critical graphs G, that is, γ_t-critical graphs with $\gamma_t(G) = 3$, the diameter of G is either 2 or 3. We characterise the 3_t-critical graphs G with *diam* $G = 3$.

1. Introduction

Let $G = (V, E)$ be a graph with order $|V| = n$. The *open neighbourhood* of a vertex v is the set of vertices adjacent to v, that is, $N(v) = \{w \mid vw \in E(G)\}$, and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$ we define the *open* and *closed neighbourhoods* $N(S)$ and $N[S]$ of S by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$, respectively. The *private neighbourhood* of $x \in S$, $S \subseteq V(G)$, consists of all vertices in the closed neighbourhood of x but not in the closed neighbourhood of $S - \{x\}$, and is denoted by $pn(x, S)$, that is, $pn(x, S) = N[x] - N[S - \{x\}]$. If $v \in pn(x, S)$, then v is called a *private neighbour of x relative to S*, or simply a *private neighbour of x*, if confusion is unlikely. If G is a graph with *diam* $G = k$ and $d(u, v) = k$, then we say that u and v are *diametrical vertices.* A shortest u-v path in G is a *diametrical path.* Two subsets X and Y of V are called *diametrical sets* if $d(x,y) = diam\ G$ for each $x \in X$ and $y \in Y$. If X and Y are diametrical sets, then (X,Y) is a *maximal diametrical pair* if for each $z \in V - (X \cup Y)$, $d(x,z) < diam\ G$ for some $x \in X$ and $d(y,z) < diam\ G$ for some $y \in Y$.

For sets $S, X \subseteq V$, if $N[S] = X$ ($N(S) = X$, respectively), we say that S *dominates* X, written $S \succ X$ (S *totally dominates* X, respectively, written $S \succ_t X$). If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ X$, $S \succ_t x$, etc. If $S \succ V$ ($S \succ_t V$, respectively), we say that S is a *dominating set* (*total dominating set*) of G, and we also write $S \succ G$ ($S \succ_t G$, respectively). The cardinality of a minimum dominating (minimum total dominating) set of G is called the *domination number* (*total domination number*) of G and is denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively); if S is a minimum dominating (minimum total dominating) set, we also call S a γ-*set* (γ_t-*set*) of G. We note that the parameter $\gamma_t(G)$ is only defined for graphs G without isolated vertices. Domination-related concepts not defined here can be found in [2].

The addition of an edge to a graph can change the domination number by at most one. Sumner and Blitch [5, 6] studied *domination edge critical graphs* G, that is, graphs G for which $\gamma(G) = \gamma(G) - 1$ for each $e \in E(\overline{G})$. We consider the same concept for total domination. A graph G is *total domination edge critical* or just γ_t-*critical* if $\gamma_t(G+e) < \gamma_t(G)$ for any edge $e \in E(\overline{G}) \neq \emptyset$. It is shown in [3] that the addition of an edge to a graph can change the total domination number by at most two.

Proposition 1 [3]**.** *For any edge* $e \in E(\overline{G})$*,*

$$\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G).$$

Graphs G with the property $\gamma_t(G+e) = \gamma_t(G) - 2$ for any $e \in E(\overline{G})$ are called *supercritical* and are characterised in [4].

In this paper, we restrict our attention to 3_t-critical graphs G, that is, γ_t-critical graphs G with $\gamma_t(G) = 3$. Note that since $\gamma_t(G) \geq 2$ for any graph G, the addition of an edge to a 3_t-critical graph reduces the total domination number by exactly one. Also, observe that any graph G with $\gamma_t(G) = 3$ is connected. Sharp bounds on the diameter of a 3_t-critical graph are determined in [3].

Proposition 2 [3]**.** *If* G *is a* 3_t*-critical graph, then*

$$2 \leq diam\ G \leq 3.$$

The graphs in Figures 1 and 2 illustrate sharpness of these bounds. Our goal is to investigate the 3_t-critical graphs with diameter three.

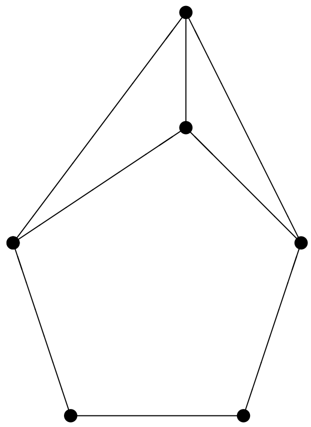


Figure 1. A 3_t-critical graph G with $diam\ G = 2$

2. 3_t-Critical Graphs with Diameter Three

In [3] the authors showed that any 3_t-critical graph G with a cutvertex has exactly one cutvertex and it is adjacent to an endvertex. Moreover, they proved that such graphs G have $diam\ G = 3$ and are the only 3_t-critical graphs with an endvertex. Figure 2 illustrates a 3_t-critical graph with an endvertex.

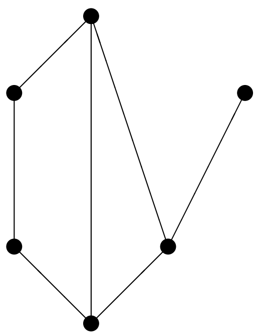


Figure 2. A 3_t-critical graph with an endvertex

Theorem 3 [3]**.** *A graph G with a cutvertex v is 3_t-critical if and only if v is adjacent to an endvertex x, and for $W = N(v) - \{x\}$ and $Y = V - N[v]$,*

(1) *$\langle W\rangle$ is complete and $|W| \geq 2$,*

(2) *$\langle Y\rangle$ is complete and $|Y| \geq 2$,*

and

(3) *every vertex in W is adjacent to $|Y| - 1$ vertices in Y and every vertex in Y is adjacent to at least one vertex in W.*

We begin with a straightforward but useful observation.

Observation 4. *For any 3_t-critical graph G and non-adjacent vertices u and v, either*

(1) *$\{u, v\}$ dominates G*
or

(2) *(without loss of generality) $\{u,w\}$ dominates $G-v$, but not v, for some $w \in N(u)$. In this case, we write $uw \mapsto v$.*

Next we develop some structural properties of 3_t-critical graphs G with *diam* $G = 3$. Although it is possible in a 3_t-critical graph G of diameter two for every pair of nonadjacent vertices to dominate G (see Figure 1, for example), we now show this is not possible if *diam* $G = 3$.

Proposition 5. *If G is a 3_t-critical graph with diam $G = 3$, then G has a pair of nonadjacent vertices that does not dominate G.*

Proof. Let G be a 3_t-critical graph with *diam* $G = 3$ and suppose that every pair of nonadjacent vertices of G dominates G. Let x and y be diametrical vertices of G where x, a, b, y is a shortest x-y path. Since $\{x, b\} \succ G$, every neighbour of y is also dominated by b. Similarly, every neighbour of x is dominated by a. Hence $\{a, b\}$ is a total dominating set of G, contradicting the fact that $\gamma_t(G) = 3$. ■

Also, it is possible for a 3_t-critical graph G with *diam* $G = 2$ to have the property that for every pair of nonadjacent vertices u and v, there is a vertex x such that $ux \mapsto v$, and there is a vertex y such that $vy \mapsto u$. See Figure 3 for an example. We now show that a 3_t-critical graph with diameter three cannot have this property.

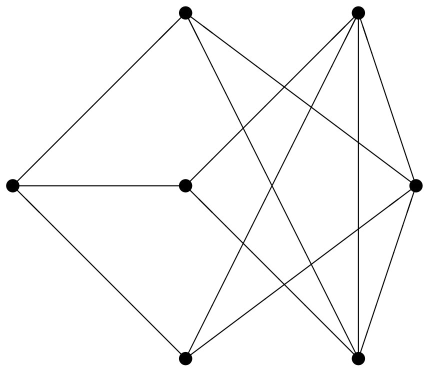


Figure 3. A 3_t-critical graph with *diam* $G = 2$

Proposition 6. *If G is a 3_t-critical graph with diam $G = 3$, then G has a pair of nonadjacent vertices u and v such that $ux \mapsto v$, for some $x \in V$, but there is no vertex y such that $vy \mapsto u$.*

Proof. Let G be a 3_t-critical graph with diameter three. Let x and y be diametrical vertices of G where x, a, b, y is a shortest x-y path. By the proof of Proposition 5, at least one of $\{x, b\}$ and $\{a, y\}$ does not dominate G. Assume then, without loss of generality, that $\{x, b\}$ does not dominate G. If $xw \mapsto b$, then $w \in N(x)$ by Observation 4 and $w \in N(y)$ to dominate y, thus $d(x, y) \leq 2$, a contradiction. Hence the only possibility is that $bw \mapsto x$. ■

It is useful to know more about the diametrical sets of vertices of a 3_t-critical graph with diameter three.

Theorem 7. *If G is a 3_t-critical graph with diam $G = 3$, then G has a unique maximal diametrical pair (X, Y). Moreover, X (say) has cardinality one and $\langle Y \rangle$ is complete.*

Proof. Let G be a 3_t-critical graph with *diam* $G = 3$. The proof of the theorem is a direct consequence of the following three lemmas.

Lemma 8. *For any maximal diametrical pair (Y_1, Y_2) of G, $\langle Y_i \rangle$ is complete for each i and $|Y_i| = 1$ for at least one i.*

Proof. Let (Y_1, Y_2) be a maximal diametrical pair of G. First we show that if $|Y_i| \geq 2$, then $\langle Y_i \rangle$ is complete. Let $x \in Y_1$ and $\{y, z\} \subseteq Y_2$ and suppose that $yz \notin E(G)$. Since $\{y,z\} \not\succ G$, we may assume without loss of generality that $yw \mapsto z$ for some vertex w, contradicting the fact that $d(x,y) = 3$. Hence $\langle Y_2 \rangle$ is complete. A similar argument shows that $\langle Y_1 \rangle$ is complete.

Next we show that $|Y_i| = 1$ for at least one i. Suppose to the contrary that both Y_1 and Y_2 have cardinality at least two. Let $x \in Y_1$ and $y \in Y_2$ and consider $\{x,y\}$. Since $|Y_i| \geq 2$ for $i \in \{1,2\}$, there is no vertex w such that $xw \mapsto y$ or $yw \mapsto x$. It follows that $\{x,y\} \succ G$. This is the case for every $x \in Y_1$ and every $y \in Y_2$. Let A (B, respectively) be the set of vertices that are distance one from every vertex of Y_1 (Y_2, respectively). If both $\langle A \rangle$ and $\langle B \rangle$ are complete, then $\gamma_t(G) = 2$, a contradiction. Thus let $a, b \in A$ where $ab \notin E(G)$. Consider $\{a,y\}$. Since neither a nor y is adjacent to b, $\{a,y\} \not\succ G$. Hence, $yc \mapsto a$ or $ac \mapsto y$. Since no vertex in $N[y]$ dominates Y_1, $ac \mapsto y$. Therefore, c dominates $Y_2 - \{y\}$. Furthermore, since $\{x,y\} \succ G$, c is adjacent to x, implying that y is the only vertex at distance three from x, contradicting the fact that $|Y_i| > 1$ for $i \in \{1,2\}$. ■

Consider the maximal diametrical pair $(\{x\}, Y)$ of G. Note that by Lemma 8 and the definition of maximal diametrical pair, $Y = \{y \in V |\ d(x,y) = 3\}$.

Lemma 9. *For every vertex $u \in V - \{x\}$, $d(u,y) \leq 2$ for every $y \in Y$.*

Proof. If $|Y| = 1$, then x is the only vertex at distance three from Y. Assume then that $|Y| \geq 2$. Let $y, z \in Y$ and suppose there is a vertex u such that $d(u,y) = 3$ and $d(u,z) = 2$; note that $u \neq x$. Let $uaby$ be a u-y path and let ucz be a u-z path (c may equal a). Note that $cy \notin E(G)$. Since neither x nor y is adjacent to c, $xw \mapsto y$ or $yw \mapsto x$. If $xw \mapsto y$, then $d(x,z) = 2$, contradicting that $z \in Y$ and that $\{x\}$ and Y are diametrical sets. If $yw \mapsto x$, then $d(u,y) = 2$, again a contradiction. ■

Lemma 10. *$(\{x\}, Y)$ is the unique maximal diametrical pair of G.*

Proof. Consider any maximal diametrical pair $(\{u\}, W)$ of G. If $u = x$, then $W = \{w \in V |\ d(u,w) = 3\} = \{w \in V |\ d(x,w) = 3\} = Y$ and we are done. If $u \in Y$, then $d(x,u) = 3$, i.e., $x \in W$ and by Lemma 9, $d(u,z) \leq 2$ for each $z \in V - \{x\}$. Hence $W = \{x\}$ and since $(\{u\}, \{x\})$ is a maximal diametrical pair, it follows that $Y = \{u\}$ and the result follows. Hence we may assume that $u \notin Y \cup \{x\}$. It follows from Lemma 9 that $W \cap (Y \cup \{x\}) = \emptyset$.

Consider any $w \in W$ and suppose firstly that $\{u, w\} \succ G$. Note that no vertex is adjacent to x as well as to a vertex in Y. Hence either $ux \in E(G)$ and $wy \in E(G)$ for each $y \in Y$, or $wx \in E(G)$ and $uy \in E(G)$ for each $y \in Y$. Suppose the former case holds and consider an arbitrary vertex $y \in Y$. By Lemma 9, $d(u, y) = 2$ and $d(w, x) = 2$. Let uay and wbx be a u-y path and a w-x path, respectively and note that $\{ub, yb\} \cap E(G) = \emptyset$. Thus $\{u, y\} \not\succ G$ and so $uc \mapsto y$ or $yc \mapsto u$ for some vertex c. If $uc \mapsto y$, then $cw \in E(G)$ and so $d(u, w) = 2$, a contradiction since u and w are diametrical vertices. If $yc \mapsto u$, then $d(x, y) = 2$, also a contradiction. Similarly, the case $wx \in E(G)$ and $uy \in E(G)$ for each $y \in Y$ is impossible. We conclude that $\{u, w\} \not\succ G$.

Thus there is some vertex d such that $\{u, w, d\}$ is independent. Since neither d nor u is adjacent to w, $uc \mapsto d$ or $dc \mapsto u$. If $uc \mapsto d$, then $d(u, w) = 2$, a contradiction. Thus we may assume that $dc \mapsto u$. Then without loss of generality, $d \in N(Y)$ and $c \in N(x)$. Now we consider $\{x, d\}$. Since d is not adjacent to u or w, and x cannot be adjacent to both u and w, xd is not a dominating edge for $G + xd$. Then $xs \mapsto d$ or $ds \mapsto x$. If $xs \mapsto d$, then $d(x, y) = 2$, a contradiction. If $ds \mapsto x$, then s is adjacent to both u and w, contradicting the fact that $d(u, w) = 3$. Hence $(\{x\}, Y)$ is the unique diametrical pair of G. ■

3. Characterisation

In the rest of this paper we characterise the 3_t-critical graphs with diameter three. We introduce more notation to simplify the characterisation. Let G be a graph with *diam* $G = 3$ and let $(\{x\}, Y)$ be a maximal diametrical pair of G. Let $A = N(x)$, $B = \{b \mid b \notin Y \text{ and } b \succ Y\}$, and $C = V - (A \cup B \cup Y \cup \{x\})$. Note that at least one of B and C is not empty. Let $\mathcal{F}$ be the family of all graphs G with *diam* $G = 3$ and the maximal diametrical pair $(\{x\}, Y)$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$, where

$G \in \mathcal{F}_1$ if $C = \emptyset$ and $|Y| \geq 2$,

$G \in \mathcal{F}_2$ if $C = \emptyset$ and $|Y| = 1$,

$G \in \mathcal{F}_3$ if $B = \emptyset$,

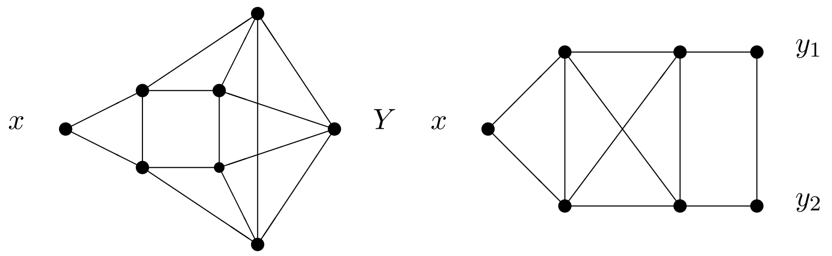
$G \in \mathcal{F}_4$ if $B \neq \emptyset$ and $C \neq \emptyset$.

To characterise the 3_t-critical graphs with diameter 3, we characterise the 3_t-critical graphs in each family $\mathcal{F}_i$, $1 \leq i \leq 4$. We begin with a lemma.

Lemma 11. *Let $G \in \mathcal{F}$ be 3_t-critical with $|Y| \geq 2$. If either $B = \emptyset$ or $C = \emptyset$, then $\langle A \rangle$ is complete.*

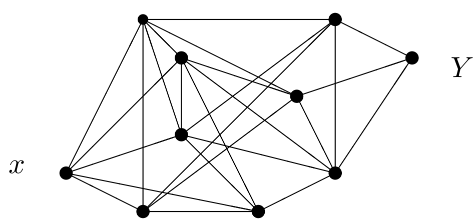
Proof. Let $G \in \mathcal{F}$ with $|Y| \geq 2$ and suppose that $\langle A \rangle$ is not complete. First assume that $C = \emptyset$. Let $u, v \in A$ with $uv \notin E(G)$. Consider $\{u, y\}$ for some vertex $y \in Y$. Since neither u nor y is adjacent to v, $uw \mapsto y$ or $yw \mapsto u$ for some vertex w. If $uw \mapsto y$, then $w \in A \cup \{x\}$ since $w \notin N(y)$. But then $Y - \{y\}$ is not dominated by $\{u, w\}$, a contradiction. If $yw \mapsto u$, then $d(x, y) \leq 2$, again a contradiction. Next assume that $B = \emptyset$. Since $\{u, v\} \not\succ G$, we may assume, without loss of generality, that $uw \mapsto v$. But this implies that $w \succ Y$, contradicting the fact that $B = \emptyset$. ■

Lemma 11 requires that the graph G has a diametrical set Y with cardinality greater than one. (See Figure 4(b)). The graph in Figure 4(a) is an example of a graph with a diametrical set Y such that $|Y| = 1$ and $\langle A \rangle$ complete. However, the condition of the lemma is necessary as can be seen by the 3_t-critical graph in Figure 5 that has $|Y| = 1$ and $\langle A \rangle$ is not complete.



(a) $|X| = |Y| = 1$ (b) $|X| = 1$ and $|Y| > 1$

Figure 4. Two 3_t-critical graphs with diameter three



$|X| = |Y| = 1$

Figure 5. A 3_t-critical graph with $\langle A \rangle$ not complete

We first characterise the 3_t-critical graphs $G \in \mathcal{F}_1$.

Theorem 12. *A graph $G \in \mathcal{F}_1$ is 3_t-critical if and only if the following conditions hold:*

(1) *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $\langle Y \rangle$ is complete.*

(2) *$\langle A \rangle$ is complete.*

(3) *For every nonadjacent pair $u, v \in B$, there is a vertex $a \in A$ such that $ua \mapsto v$. Also, no pair of adjacent vertices dominates G.*

(4) *For every vertex $b \in B$, there is a vertex $d \in B \cup Y$ such that $bd \mapsto x$.*

(5) *For every pair $\{a, b\}$ of nonadjacent vertices where $a \in A$ and $b \in B$, $\{a, b\} \succ G$ or $aw \mapsto b$ for some $w \in B$.*

Proof. Let $G \in \mathcal{F}_1$ be 3_t-critical. By Theorem 10, $(\{x\}, Y)$ is the unique maximal diametrical pair of G and $\langle Y \rangle$ is complete.

Since $C = \emptyset$, it follows that $\{x, y\} \succ G$ for every $y \in Y$. From Lemma 11 we have that $\langle A \rangle$ is complete. Furthermore, since $(\{x\}, Y)$ is a maximal diametrical pair, each $b \in B$ is adjacent to at least one vertex $a \in A$. If there is a vertex $b \in B$ that dominates B, then $\{a, b\} \succ_t G$ for an $a \in A$, contradicting the fact that $\gamma_t(G) = 3$. Let $u, v \in B$ with $uv \notin E(G)$. Obviously, $\{u, v\} \not\succ x$, so without loss of generality, there is a vertex $a \in A$ such that $au \mapsto v$. Since $\gamma_t(G) = 3$, no pair of adjacent vertices dominates G. To show that (4) holds, let b be any vertex in B. Since there is at least one vertex in B not adjacent to b, $\{x, b\} \not\succ G$. No vertex in $N[x]$ dominates Y, so $bd \mapsto x$ for some $d \in B \cup Y$. Condition (5) follows directly from Observation 4 and the fact that if $bw \mapsto a$, then $w \in A$ to dominate x; hence $w \succ a$ since $\langle A \rangle$ is complete, a contradiction.

Conversely, let $G \in \mathcal{F}_1$ such that the stated properties hold. Since no pair of adjacent vertices dominates G, $\gamma_t(G) \geq 3$. Further, $\{a, b, y\}$ is a γ_t-set for every $a \in A$, $b \in B$, $y \in Y$ where $ab \in E(G)$, implying that $\gamma_t(G) \leq 3$. Hence $\gamma_t(G) = 3$. To show that G is 3_t-critical we consider first $\{x, y\}$ for $y \in Y$. Since $C = \emptyset$, $\{x, y\} \succ G$. Similarly, $\{a, y\} \succ G$ for every $a \in A$. We next consider $\{x, b\}$. Since condition (4) holds, there is a vertex $d \in B \cup Y$ such that $bd \mapsto x$. We also consider $\{a, b\}$, $a \in A$ and $b \in B$. Property (5) implies that either $\{a, b\} \succ G$ or there is a vertex $w \in B$ such that $aw \mapsto b$. Finally we consider $\{b, c\}$, where $b, c \in B$. Since condition (3) holds, there is a vertex $a \in A$ such that $ab \mapsto c$. Thus G is 3_t-critical. ■

Note that $\{x, y\} \succ G$ for every $y \in Y$. We state this result as a corollary.

Corollary 13. *If $G \in \mathcal{F}_1$ is 3_t-critical, then $\gamma(G) = 2$.*

We now give a more descriptive characterisation of the 3_t-critical graphs $G \in \mathcal{F}_1$ with $\delta(G) = 2$. We first show that if $\delta(G) = 2$, then $deg(x) = 2$. Recall that $\langle A \rangle$ is complete.

Lemma 14. *If $G \in \mathcal{F}_1$ and G is 3_t-critical with $\delta(G) = 2$, then $deg(x) = 2$ and $deg(v) \geq 3$ for all $v \in V(G) - \{x\}$.*

Proof. Let $G \in \mathcal{F}_1$ be 3_t-critical. Since G has no cutvertices (Theorem 3), $|A|, |B| \geq 2$. Every vertex $b \in B$ is adjacent to some vertex $a \in A$ and to every vertex $y \in Y$. Thus $deg(b) \geq 3$ for every $b \in B$, since $|Y| \geq 2$. By Theorem 10, $\langle Y \rangle$ is complete. Therefore $deg(y) \geq 3$ for each $y \in Y$. Finally, every vertex $a \in A$ has at least one neighbour in A, implying that $deg(a) \geq 3$. ■

We use the following notation for the characterisation. Let $A = N(x) = \{x_1, x_2\}$ and $B_1 = (N(x_1) \cap N(x_2)) - \{x\}$, $B_2 = N(x_1) - (B_1 \cup \{x, x_2\})$, and $B_3 = N(x_2) - (B_1 \cup \{x, x_1\})$. Recall that $C = \emptyset$ and hence $B = B_1 \cup B_2 \cup B_3$.

We need the following lemmas for the characterisation. To simplify notation we refer to the 3_t-critical graphs $G \in \mathcal{F}_1$ with $\delta(G) = 2$ as family $\mathcal{G}_2$.

Lemma 15. *If $G \in \mathcal{G}_2$ and $B_i \neq \emptyset$, then $\langle B_i \rangle$ is complete for $i \in \{1, 2, 3\}$.*

Proof. Let $G \in \mathcal{G}_2$ and assume that $B_i \neq \emptyset$. Suppose that $u, v \in B_i$ and $uv \notin E(G)$. Since neither u nor v dominates x, without loss of generality, $uw \mapsto v$. Then $w \in N(u) \cap N(x)$. But since u and v are in B_i, $v \in N(w)$, contradicting that $uw \mapsto v$. ■

Lemma 16. *If $G \in \mathcal{G}_2$ and $B_1 \neq \emptyset$, then each vertex in B_1 dominates exactly $|B_i| - 1$ vertices in B_i for $i \in \{2, 3\}$.*

Proof. It is easy to see that no vertex $b \in B_1$ dominates B_2 or B_3. Suppose, without loss of generality, a vertex $b \in B_1$ is not adjacent to two vertices in B_2, say u and v, and consider $\{b, u\}$. Since neither b nor u dominates x, $\{b, u\} \not\succ G$. Furthermore, $ux_1 \not\mapsto b$ since $x_1 \in N(b)$. Hence $bx_2 \mapsto u$, implying that $v \in B_3$, a contradiction. ■

Lemma 17. *If $G \in \mathcal{G}_2$, then $|B_i| \geq 2$ for $i \in \{2, 3\}$.*

Proof. Let $G \in \mathcal{G}_2$. Since $(\{x\}, Y)$ is a maximal diametrical pair, each $a \in A$ is adjacent to some $b \in B$. Hence $B_1 \cup B_i \neq \emptyset$ for $i \in \{2,3\}$. If $B_2 = \emptyset$ (or $B_3 = \emptyset$, respectively), then $\{x_2, b_3\} \succ_t G$ for $b_3 \in B_1 \cup B_3$ ($\{x_1, b_2\} \succ_t G$ for $b_2 \in B_1 \cup B_2$, respectively). Hence neither B_2 nor B_3 is empty. Suppose without loss of generality that $|B_2| = 1$, say $B_2 = \{b_2\}$. By Lemma 16, b_2 is not adjacent to any vertex in B_1. Also, b_2 is not adjacent to any vertex in B_3, for otherwise $\{x_2, b_3\} \succ_t G$ for some $b_3 \in B_3 \cup N(b_2)$. Now consider $\{b_2, x\}$. Since $\{b_2, x\} \not\succ B_3 \neq \emptyset$ and $\{x, x_i\} \not\succ Y$, there exists a vertex w such that $b_2 w \mapsto x$. But no vertex adjacent to b_2 dominates x_2 as well as B_3, a contradiction. Hence $|B_i| \geq 2$ for $i \in \{2,3\}$. ∎

Lemma 18. *If $G \in \mathcal{G}_2$, then $\overline{\langle B_2 \cup B_3 \rangle}$ is the disjoint union of non-trivial stars.*

Proof. Note that $\overline{\langle B_2 \cup B_3 \rangle}$ has no isolates, for if $u \in B_2$ (say) dominates B_3, then $\{u, x_1\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 3$. Assume without loss of generality that a vertex $u \in B_2$ is not adjacent to vertices $b_1, \ldots, b_k \in B_3$, where $k \geq 2$ and where b_1 (say) is not adjacent to $v \in B_2$, $v \neq u$. Since $\{u, b_1\} \not\succ x$, we may assume without loss of generality that $uw \mapsto b_1$ for some vertex w. Then $w = x_1$ to dominate x, but $\{u, x_1\} \not\succ b_2$, a contradiction. The result follows since $\langle B_i \rangle$ is complete for $i = 2, 3$. ∎

Theorem 19. *A graph $G \in \mathcal{G}_2$ if and only if the following conditions hold:*

(1) $(\{x\}, Y)$ *is the unique maximal diametrical pair and* $\langle Y \rangle$ *is complete.*

(2) $deg(x) = 2$ *and* $\langle A \rangle$ *is complete.*

(3) $B_1 = \emptyset$ *or* $\langle B_1 \rangle$ *is complete.*

(4) $|B_i| \geq 2$ *and* $\langle B_i \rangle$ *for* $i \in \{2,3\}$ *is complete.*

(5) $\overline{\langle B_2 \cup B_3 \rangle}$ *is the disjoint union of non-trivial stars.*

(6) *If $B_1 \neq \emptyset$, then every vertex in B_1 dominates exactly $|B_i| - 1$ vertices in B_i for $i \in \{2,3\}$. Also, if $u \in B_2$ ($u \in B_3$, respectively) does not dominate B_1, then there is a vertex $v \in B_1 \cup B_3$ ($v \in B_1 \cup B_2$, respectively) such that $\{u, v\} \succ_t B$.*

Proof. Let $G \in \mathcal{G}_2$. By Theorem 12, $(\{x\}, Y)$ is the unique maximal diametrical pair of G, $\langle Y \rangle$ is complete, and $\langle A \rangle$ is complete. By Lemma 14, $deg(x) = 2$. By Lemmas 15, 17, and 18, conditions (3), (4), and (5) hold. Assume without loss of generality that $u \in B_2$ does not dominate B_1. Since

$\{x,u\} \not\succ G$ and $\{x,x_i\} \not\succ Y$, it follows that $uv \mapsto x$ for some v. To dominate x_2 but not x, $v \in B_1 \cup B_3$, and clearly $\{u,v\} \succ_t B$. Thus by Lemma 16, condition (6) holds.

Conversely, let G be graph such that all the conditions of the theorem hold. There is no edge $uv \in E(G)$ such that $\{u,v\} \succ G$. Hence $\gamma_t(G) \geq 3$. The path x_1, x_2, b_i, for $b_i \in B$, is a total dominating set. Therefore $\gamma_t(G) = 3$.

To show that G is γ_t-critical we first consider $\{x,y\}$ for any $y \in Y$. Since $C = \emptyset$, $\{x,y\} \succ G$ for every $y \in Y$. Next consider $\{x,b\}$ for any $b \in B_1$. Since $b \succ A \cup Y$, $by \mapsto x$ for any $y \in Y$. Now consider $\{x,u\}$ for any $u \in B_2$. If u is not adjacent to any vertex in B_3, then by (5), every $c \in B_2 - \{u\} \neq \emptyset$ is adjacent to all vertices in B_3, i.e., $\{x_1,c\} \succ_t G$, a contradiction. So, if $B_1 = \emptyset$ or $u \succ B_1$, let $v \in B_3$ be adjacent to u. Clearly, $uv \mapsto x$. If $u \not\succ B_1$, then by (6) there is a vertex $v \in B_1 \cup B_3$ such that $\{u,v\} \succ_t B$ and it is easy to see that $uv \mapsto x$. The set $\{x,u\}$ for any $u \in B_3$ is dealt with in exactly the same way. Further, it is easy to see that $\{x_1,v\}$ and $\{x_2,u\}$ dominate G for every $v \in B_3$ and every $u \in B_2$. Also, $\{x_i,y\} \succ G$ for $i = 1,2$ and every $y \in Y$. By Condition (6) a vertex $b \in B_1$ dominates exactly $|B_i| - 1$ vertices in B_i, $i = 2,3$. Let $u \in B_2$ be non-adjacent to $b \in B_1$. Then $bx_2 \mapsto u$. Similarly, $bx_1 \mapsto v$, for $v \in B_3$ and $bv \notin E(G)$. Finally, we consider $\{u,v\}$ with $u \in B_2$ and $v \in B_3$, where $uv \notin E$. Since $\overline{\langle B_2 \cup B_3 \rangle}$ is the disjoint union of non-trivial stars, we may assume without loss of generality that u has degree 1 in $\overline{\langle B_2 \cup B_3 \rangle}$. Then $ux_1 \mapsto v$. It now follows that $G \in \mathcal{G}_2$. ■

For an example of a 3_t-critical graph $G \in \mathcal{G}_2$, see Figure 6.

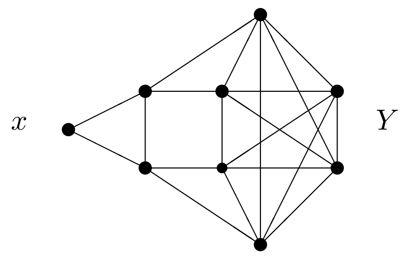


Figure 6. A 3_t-critical graph $G \in \mathcal{G}_2$

For 3_t-critical graphs $G \in \mathcal{F}_1$, the cardinality of Y is greater than one. A necessary condition for these graphs is that $\langle A \rangle$ is complete. However, when the cardinality of Y is equal to one, this condition is no longer required. Figure 4(a) is an example of $G \in \mathcal{F}_2$ and 3_t-critical with $\langle A \rangle$ complete and Figure 5 is an example of a graph $G \in \mathcal{F}_2$ and 3_t-critical with $|Y| = 1$ and $\langle A \rangle$ not complete.

Theorem 20. *A graph $G \in \mathcal{F}_2$ is 3_t-critical if and only if the following conditions hold:*

(1) *$(\{x\}, \{y\})$ is the unique diametrical pair of G.*

(2) *For each $a \in A$ and $b \in B$ with $ab \in E(G)$ there exists a vertex $w \notin N(a) \cup N(b)$.*

(3) *For each $a, a' \in A$, with $aa' \notin E(G)$, there exists $b' \in B$ such that $ab' \mapsto a'$. A similar statement holds for each $b, b' \in B$ with $bb' \notin E(G)$.*

(4) *For every $a \in A$, $\{a, y\} \succ G$ or there exists $a' \in A$ such that $aa' \mapsto y$. A similar statement holds for every $b \in B$ and $\{x\}$.*

(5) *For each $a \in A$ and $b \in B$ with $ab \notin E(G)$, $\{a, b\} \succ G$ or, without loss of generality, there exists $b' \in B$ such that $ab' \mapsto b$.*

Proof. Let $G \in \mathcal{F}_2$ be 3_t-critical. By Theorem 7 $(\{x\}, \{y\})$ is the unique diametrical pair of G. Condition (2) follows from the fact that $\gamma_t(G) = 3$. Since $\langle A \rangle$ and $\langle B \rangle$ cannot both be complete, let $a, a' \in A$ with $aa' \notin E(G)$. Neither a nor a' is adjacent to y. Therefore without loss of generality there exists $b' \in B$ such that $ab' \mapsto a'$. Let $a \in A$ be an arbitrary vertex. If $\{a, y\} \succ G$, then Condition (4) holds. Otherwise there exists w such that $yw \mapsto a$ or $aw \mapsto y$. If $yw \mapsto a$, then $x \in N(w)$ implying $d(x, y) = 2$, a contradiction. Hence $aw \mapsto y$ for some $w \in A$. A similar argument shows that for every $b \in B$, $\{b, x\} \succ G$ or there exists $b' \in B$ such that $bb' \mapsto x$. Let $a \in A$ and $b \in B$ with $ab \notin E(G)$. If $\{a, b\} \succ G$, then Condition (5) holds. Otherwise, without loss of generality, there exists $b' \in B$ such that $ab' \mapsto b$.

Conversely, let G be a graph such that the stated conditions hold. By Condition (2) there is no edge that dominates G. Thus, $\gamma_t(G) \geq 3$. Consider $\{a, y\}$ for any $a \in A$. If $\{a, y\} \succ G$, then with $b \in N(a) \cap N(y)$, $\{a, b, y\}$ is a total dominating set. If $\{a, y\} \not\succ G$, then by Condition (4) there exists $a' \in A$ such that $aa' \mapsto y$. Again with $b \in N(a) \cap N(y)$, $\{a, a', b\}$ is a total dominating set, so $\gamma_t(G) \leq 3$. Hence $\gamma_t(G) = 3$. That G is γ_t-critical follows from the fact that $\{x, y\} \succ G$ and from Conditions (2) through (5). ■

Two additional lemmas are needed for the remaining characterisations.

Lemma 21. *If $G \in \mathcal{F}$ is 3_t-critical, then every vertex in C is adjacent to exactly $|Y|-1$ vertices in Y.*

Proof. By definition, there is no vertex in C that dominates Y. Suppose there is a vertex $c \in C$ that is not adjacent to at least two vertices in Y, say u and v. Clearly, $\{c,u\} \not\succ G$. Therefore $cw \mapsto u$ or $uw \mapsto c$ for some vertex w. If $cw \mapsto u$, then $w \in N(x)$ and $w \succ v$, contradicting the fact that $d(x,v)=3$. If $uw \mapsto c$, then $w \succ x$, again contradicting that $d(x,u)=3$. ■

It was shown in Theorem 7 that $\langle Y \rangle$ is complete. We now consider $\langle C \rangle$.

Lemma 22. *If $G \in \mathcal{F}$ is 3_t-critical and $C \neq \emptyset$, then $\langle C \rangle$ is complete.*

Proof. Let $u,v \in C$ and $uv \notin E(G)$. Since $\{u,v\} \not\succ G$, assume without loss of generality that $uw \mapsto v$. By definition there is a vertex $y \in Y$ not adjacent to u. Therefore, $w \succ y$ and $w \succ x$. But this contradicts the fact that $d(x,y)=3$. ■

We now characterise the 3_t-critical graphs in family $\mathcal{F}_3$.

Theorem 23. *A graph $G \in \mathcal{F}_3$ is 3_t-critical if and only if the following conditions hold*:

(1) *$(\{x\},Y)$ is the unique maximal diametrical pair of G and $\langle Y \rangle$ is complete.*

(2) *$\langle A \cup C \rangle$ is complete.*

(3) *$|C| \geq 2$, $|Y| \geq 2$ and every vertex in C is adjacent to exactly $|Y|-1$ vertices in Y.*

Proof. Let $G \in \mathcal{F}_3$ be 3_t-critical. From Theorem 7 we have that $(\{x\},Y)$ is the unique maximal diametrical pair and $\langle Y \rangle$ is complete.

By Lemmas 11 and 22, $\langle A \rangle$ and $\langle C \rangle$ are complete. We show that $\langle A \cup C \rangle$ is complete. Let $a \in A$ and $c \in C$ with $ac \notin E(G)$. Since there is at least one vertex in Y not adjacent to c, $\{a,c\} \not\succ G$. The only possibility is that $aw \mapsto c$. Thus $w \succ Y$, contradicting the fact that $B=\emptyset$.

By Lemma 21, if $Y=\{y\}$ (say), then no vertex in C is adjacent to y and since $B=\emptyset$, it follows that y is isolated in G, which is impossible. Hence $|Y| \geq 2$. Suppose that $|C|=1$. Since $|Y| \geq 2$, there is a vertex $y \in Y$ that is not adjacent to a vertex of C. But then $diam(G) > 3$, a contradiction. Hence $|C| \geq 2$.

For the necessity, let $G \in \mathcal{F}_3$ and assume that the conditions of the theorem hold. It is easy to see that there is no edge $ac \in E(G)$ such that $\{a, c\}$ dominates G. Thus $\gamma_t(G) \geq 3$. On the other hand, every shortest y-a path, $y \in Y$ and $a \in A$, is a total dominating set of cardinality three, implying that $\gamma_t(G) = 3$. We now show that G is 3_t-critical. First consider $\{x, c\}$, for any $c \in C$. Since $c \succ A \cup C$, $cy \mapsto x$ for any $y \in Y$ adjacent to c. Next, consider $\{x, y\}$, for any $y \in Y$. Here it is also easy to see that $yc \mapsto x$ for any $c \in N(y) \cap C$. For any $a \in A$ and $y \in Y$, $\{a, y\} \succ G$. Finally we consider $\{c, y\}$ with $cy \notin E$. Since y is the only vertex in Y not adjacent to c, $ca \mapsto y$ for any $a \in A$. ■

Corollary 24. *If $G \in \mathcal{F}_3$ is 3_t-critical, then $\gamma(G) = 2$.*

See Figures 2 and 4(b) for examples of 3_t-critical graphs in $\mathcal{F}_3$. Note that this family of 3_t-critical graphs includes those graphs with minimum degree one characterised in Theorem 3 where x is the endvertex of G.

Next we consider the family $\mathcal{F}_4$. See Figure 7 for an example.

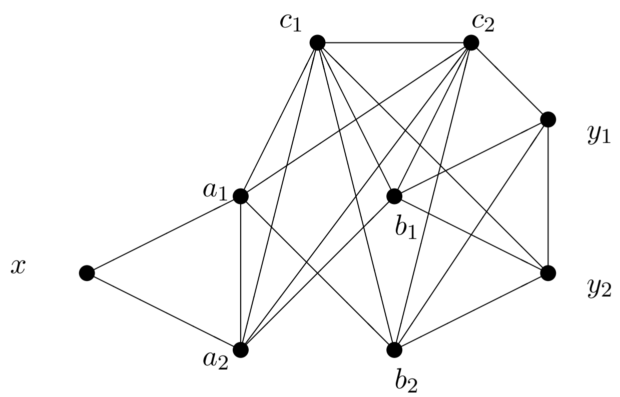


Figure 7. A 3_t-critical graph $G \in \mathcal{F}_4$

We now characterise the 3_t-critical graphs $G \in F_4$ using the same notation as before.

Theorem 25. *A graph $G \in F_4$ is 3_t-critical if and only if the following conditions hold:*

(1) *(x, Y) is the unique maximal diametrical pair of G and $\langle Y \rangle$ is complete.*

(2) *$\langle C \rangle$ is complete and each $c \in C$ dominates exactly $|Y| - 1$ vertices in Y.*

(3) *If* $|Y| \geq 2$, *then for every* $y \in Y$, $\{x, y\} \succ G$ *or there exists* $w \in B \cup C$ *such that* $yw \mapsto x$. *If* $|Y| = 1$ (*say* $Y = \{y\}$), *then* $\{x, y\} \not\succ G$ *and there exists* $y' \in B$ *such that* $y' \succ A \cup C$ *or* $x' \in A$ *such that* $x' \succ B \cup C$.

(4) *For every* $c \in C$, *there exists* $w \in B \cup C \cup Y$ *such that* $cw \mapsto x$.

(5) *For every* $b \in B$, $\{x, b\} \succ G$ *or there exists* $w \in B \cup C \cup Y$ *such that* $bw \mapsto x$.

(6) *For every* $a \in A$ *and* $y \in Y$, $\{a, y\} \succ G$ *or there exists* $w \in A \cup C$ *if* $Y = \{y\}$ ($w \in C$ *if* $|Y| \geq 2$) *such that* $aw \mapsto y$.

(7) *For each* $a \in A$ *and* $c \in C$ *with* $ac \notin E(G)$, *there exists* $b \in B$ *such that* $ab \mapsto c$.

(8) *For each* $a \in A$ *and* $b \in B$ *with* $ab \notin E(G)$, $\{a, b\} \succ G$ *or there exists* $a' \in A$ *such that* $a'b \mapsto a$ *or* $b' \in B$ *such that* $ab' \mapsto b$. *For each* $ab \in E(G)$ *with* $a \in A$ *and* $b \in B$, *there exists* $w \in A \cup B \cup C$ *such that* $w \notin (N(a) \cup N(b))$.

(9) *For each* $b \in B$ *and* $c \in C$ *with* $bc \notin E(G)$, *there exists* $a \in A$ *such that* $ab \mapsto c$.

(10) *For each* $c \in C$ *and* $y \in Y$ *with* $cy \notin E(G)$, *there exists* $a \in A$ *such that* $ac \mapsto y$.

Proof. Let $G \in \mathcal{F}_4$ be 3_t-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, $\langle C \rangle$ is complete. By Lemma 21, each vertex in C is adjacent to exactly $|Y| - 1$ vertices in Y.

Consider arbitrary $y \in Y$. If $\{x, y\} \succ G$, then $|Y| \geq 2$ since $C \neq \emptyset$ and y must dominate C. Hence Condition (3) holds in this case. Therefore we may assume that $\{x, y\}$ does not dominate G. Since G is 3_t-critical, $xw \mapsto y$ or $yw \mapsto x$. If $xw \mapsto y$, then $w \in A$ implying that $w \succ B \cup C$ and that $Y = \{y\}$. Thus if $|Y| \geq 2$, then $yw \mapsto x$ and we have shown that Condition (3) holds if $|Y| \geq 2$. Therefore we may assume that $|Y| = 1$. Now G has the unique maximal diametrical pair $(\{x\}, \{y\})$ and neither x nor y dominates any vertex in C. Hence $xx' \mapsto y$ with $x' \in A$ or $yy' \mapsto x$ with $y' \in B$, and Condition (3) follows.

Condition (4) follows from the fact that each $c \in C$ dominates at most $|Y| - 1$ vertices in Y and there is no $x' \in A$ such that $xx' \mapsto c$ for any $c \in C$.

Let b be an arbitrary vertex in B. If $\{x, b\} \succ G$, then Condition (5) holds. Otherwise $xx' \mapsto b$ for $x' \in A$ or $bb' \mapsto x$ for $b' \in B \cup C \cup Y$. If $xx' \mapsto b$, then $x' \succ Y$ implying $d(x, y) = 2$, a contradiction. Hence $bb' \mapsto x$.

If for $a \in A$ and $y \in Y$, $\{a, y\} \succ G$, then Condition (6) holds. Otherwise $yy' \mapsto a$ for $y' \in N(y)$ or $aa' \mapsto y$ for $a' \in A \cup C$. If $yy' \mapsto a$, then $x \in N(y')$ implying $d(x, y) < 3$, a contradiction. Hence $aa' \mapsto y$.

Consider $\{a, c\}$ where $a \in A$ and $c \in C$ are not adjacent. Since neither a nor c dominates Y, $\{a, c\} \not\succ G$. Therefore, $ca' \mapsto a$ with $a' \in A$ (to dominate x) or $ab' \mapsto c$ with $b' \in B$ (to dominate Y). If $ca' \mapsto a$, then $c \succ Y$, contradicting that each $c \in C$ dominates at most $|Y| - 1$ vertices in Y. Hence $ab' \mapsto c$ and Condition (7) holds.

Condition (8) follows directly from the definition of 3_t-critical graphs. If $b \in B$ and $c \in C$ with $bc \notin E(G)$, then $\{b, c\} \not\succ G$ since neither b nor c is adjacent to x. Since there is no $c' \in N(c)$ such that $cc' \mapsto b$, $ba' \mapsto c$ with $a' \in A$. Hence Condition (9) holds.

Finally we consider $\{c, y\}$ with $cy \notin E(G)$. Again since neither c nor y is adjacent to x, $\{c, y\} \not\succ G$. Also, since y has no neighbour y' such that $y' \succ x$, $ca' \mapsto y$ with $a' \in A$.

Let G be a graph such that the stated properties hold. By Condition (8) there is no $ab \in E(G)$ with $a \in A$ and $b \in B$ such that $\{a, b\} \succ G$, and since no other edge dominates G, $\gamma_t(G) \geq 3$. By Condition (10), there is $a \in A$ for every $c \in C$ such that $ac \mapsto y$ for some $y \in Y$. Further, each $a \in A$ is adjacent to some $b \in B$ since $(\{x\}, Y)$ is the unique maximal diametrical pair. Therefore, $\{a, b, c\}$ is a total dominating set of G, implying that $\gamma_t(G) \leq 3$. Hence $\gamma_t(G) = 3$. That G is γ_t-critical, follows from Conditions (3) through (10). ■

Finally we consider a subclass of the family $\mathcal{F}_4$.

Lemma 26. *If $G \in \mathcal{F}_4$ is 3_t-critical and $\langle A \rangle$ is not complete, then every $y \in Y$ dominates at most $|C| - 1$ vertices in C.*

Proof. Let $u, v \in A$ with $uv \notin E(G)$ and suppose there is a vertex $y \in Y$ such that $y \succ C$. Consider $\{u, y\}$. Since $\{u, y\} \not\succ G$ and there is no vertex $c \in C$ such that $uc \mapsto y$, there must be a vertex $w \in N(y)$ such that $yw \mapsto u$. But then $d(y, x) \leq 2$, contradicting $diam(G) = 3$. ■

Lemma 27. *If $G \in \mathcal{F}_4$ is 3_t-critical and $\langle A \rangle$ is not complete, then $|C| \geq |Y|$.*

Proof. Let $|C| = k$ and $|Y| = p$. Since every vertex in C is adjacent to exactly $|Y| - 1$ vertices in Y, there are exactly $k(p - 1)$ edges from C to Y. By Lemma 26, every $y \in Y$ dominates at most $|C| - 1$ vertices in C.

Therefore there are at most $p(k-1)-s$ edges from Y to C, $s \geq 0$. Thus

$$p(k-1)-s = k(p-1),$$

hence

$$k-s = p$$

and it follows that $k \geq p$. ■

Restricting our attention to the graphs $G \in \mathcal{F}_4$ with $\langle A \rangle$ not complete and $|Y| = |C|$, we are able to obtain a more concise and descriptive characterisation than the one given for the family $\mathcal{F}_4$.

Theorem 28. *Let G be a graph in $\mathcal{F}_4$ with $\langle A \rangle$ not complete and $|Y| = |C|$. Then G is 3_t-critical if and only if the following conditions hold:*

(1) *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $\langle Y \rangle$ is complete.*

(2) *$\langle C \rangle$ is complete and $\langle C \cup Y \rangle$ is complete minus a perfect matching between C and Y.*

(3) *Every vertex $c \in C$ dominates $A \cup B$.*

(4) *For every $ab \in E(\langle A \cup B \rangle)$, there is a vertex $a_i \in A$ or $b_j \in B$ not adjacent to a and b and if a_1, a_2 (b_1, b_2, respectively) are nonadjacent vertices in A (B, respectively), then there is a vertex $b \in B$ ($a \in A$, respectively) such that $a_1 b \mapsto a_2$ ($b_1 a \mapsto b_2$, respectively). Also for every $a \in A$ and $b \in B$ that are not adjacent, $\{a, b\} \succ G$ or there is a vertex w such that $aw \mapsto b$ or $bw \mapsto a$.*

Proof. Let $G \in \mathcal{F}_4$ with $\langle A \rangle$ not complete and $|Y| = |C|$ be 3_t-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, $\langle C \rangle$ is complete. By Lemmas 21 and 26, we have that each vertex in C is adjacent to $|Y|-1$ vertices in Y and if $\langle A \rangle$ is not complete, then each vertex in Y is adjacent to at most $|C|-1$ vertices is C. Thus there are $|C|(|Y|-1)$ edges from C to Y and at most $|Y|(|C|-1)$ edges from Y to C. Since $|C| = |Y|$, there are exactly $|Y|(|C|-1)$ edges from Y to C and so every vertex in Y is adjacent to exactly $|C|-1$ vertices in C. Therefore, all edges minus a perfect matching are present between C and Y.

To show that (3) holds, suppose that $ac \notin E(G)$, $a \in A$ and $c \in C$. Consider $\{a, y\}$ where $y \in Y$ and $cy \notin E(G)$. Obviously, $\{a, y\} \not\succ G$. Since no vertex in $N[y]$ dominates x, it follows that $az \mapsto y$ and $z \in C$. But this contradicts condition (2). Hence $c \succ A$ for each $c \in C$.

Now suppose that $cb \notin E(G)$, $c \in C$ and $b \in B$, and consider $\{b, c\}$. Since neither b nor c is adjacent to x, $\{c, b\} \not\succ G$. Therefore $cw \mapsto b$ or $bw \mapsto c$ for $w \in A$ (to dominate x). But if $cw \mapsto b$, then Y is not dominated, a contradiction. And if $bw \mapsto c$, then $w \notin N(c)$, contradicting the fact that every vertex $c \in C$ dominates A. Hence $c \succ B$ for each $c \in C$. Thus, condition (3) holds. Condition (4) follows from the fact that every $b \in B$ dominates $C \cup Y$ and every $a \in A$ dominates C.

Let $G \in \mathcal{F}_4$ with $\langle A \rangle$ not complete and $|Y| = |C|$ and assume that the conditions of the theorem hold. Since no pair of adjacent vertices dominate G, $\gamma_t(G) \geq 3$. Further, $\{a, b, c\}$, where $a \in A$, $b \in B$ and $c \in C$, is a total dominating set, so $\gamma_t(G) = 3$. To show that G is 3_t-critical, we first consider $\{x, y\}$ for $y \in Y$. Then $cy \mapsto x$ where $c \in N(y)$. A similar argument holds for $\{x, c\}$. Next consider $\{x, b\}$ for $b \in B$. Then $bc \mapsto x$ for any $c \in C$. For $\{a, y\}$, $ac \mapsto y$ where $c \in C - N(y)$. It now follows from condition (4) that G is 3_t-critical. ■

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