

DOMINATION PARAMETERS OF A GRAPH WITH DELETED SPECIAL SUBSET OF EDGES

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Abstract

This paper contains a number of estimations of the split domination number and the maximal domination number of a graph with a deleted subset of edges which induces a complete subgraph K_p . We discuss noncomplete graphs having or not having hanging vertices. In particular, for $p = 2$ the edge deleted graphs are considered. The motivation of these problems comes from [2] and [6], where the authors, among other things, gave the lower and upper bounds on irredundance, independence and domination numbers of an edge deleted graph.

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1. Introduction

We shall consider in this paper only finite, undirected, noncomplete graphs, without loops and multiple edges, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G . For an arbitrary vertex $x \in V(G)$ we denote by $N_G(x)$ the neighbourhood of x in G , that is, the subset of all vertices adjacent to x in G . By $\delta_G(x)$ we denote the degree of the vertex x in G and note that $\delta_G(x) = |N_G(x)|$. Further, denote by $\delta(G)$ the minimum degree of G . We recall that if $\delta_G(x) = |V(G)| - 1$ or $\delta_G(x) = 1$ or $\delta_G(x) = 0$, then the vertex x said to be a *dominating* vertex, a *hanging* vertex and an *isolated*

vertex of G , respectively. By $G - E_0$ we mean a spanning subgraph of G with the edge set $E(G) - E_0$. The notation $\langle X \rangle_G$ denotes the subgraph of G induced by a subset $X \subseteq V(G)$ or $X \subseteq E(G)$. For short, the fact that G_0 is an induced (or an induced proper) subgraph of G we write $G_0 \leq G$ or $G_0 < G$, respectively.

A *path* joining vertices x_1 and x_n in G is the sequence of vertices $x_1, x_2, \dots, x_n \in V(G)$ such that, $(x_i, x_{i+1}) \in E(G)$, for $i = 1, 2, \dots, n - 1$ and $n \geq 2$. We shall denote it by $P_G(x_1, x_n)$.

A subset $D \subseteq V(G)$ is called a *dominating* set of G (in abbreviation $D \text{ dom } G$) if every vertex $x \in V(G) - D$ is adjacent to at least one vertex from D . The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G . In this paper we study two domination parameters. A dominating set D of G is called a *split dominating set* of G if the induced subgraph $\langle V(G) - D \rangle_G$ is disconnected (for short: $D \text{ sdom } G$). We note that the existence of a split dominating set in a connected graph is possible whenever the graph is different from a complete graph. The *split domination number* $\gamma_s(G)$ of G is the minimum cardinality taken over all split dominating sets of G . It follows immediately from the definition of $\gamma_s(G)$ that $\gamma_s(G) \geq \gamma(G)$. Evidently, $\gamma_s(G) \leq |V(G)| - 2$, for an arbitrary noncomplete graph. A dominating set D of G is called a *maximal dominating set* of G if $V(G) - D$ is not a dominating set of G . The *maximal domination number* $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set of G and note that $\gamma_m(G) \geq \gamma(G)$. For convenience, a subset which realizes $\gamma(G)$, $\gamma_s(G)$, $\gamma_m(G)$ will be called a $\gamma(G)$ -set, a $\gamma_s(G)$ -set, and a $\gamma_m(G)$ -set, respectively. For more information about split dominating sets and maximal dominating sets, the reader is referred to [4] and [3], respectively.

Undefined notation and terminology can be found in [1].

In this paper we shall give the lower and upper bounds on the split domination number and the maximal domination number of a spanning subgraph $G - E_0$ of a graph G , where $\langle E_0 \rangle_G$ is isomorphic to the complete graph K_p , for $p \geq 2$. In the case when $p = 2$, the resulting spanning subgraph $G - E_0$ is meant as an edge deleted graph $G - e$.

2. Preliminaries

Let $K_p < G$, $p \geq 2$ and D be a split dominating set of a graph G . Putting $E_0 = E(K_p)$ we start with some observations which will be useful in further investigations.

Proposition 1. *If $V(K_p) \subseteq D$, then D $sdom$ $G - E_0$.*

Proof. Assuming that D $sdom$ G we deduce that D dom G . This means that for any $x \in V(G) - D$, $N_G(x) \cap D \neq \emptyset$. Moreover, $N_{G-E_0}(x) \cap D \neq \emptyset$, since $V(K_p) \subseteq D$. This implies that D dom $G - E_0$. To complete the proof we must show that $\langle V(G) - D \rangle_{G-E_0}$ is disconnected. It is obvious that $\langle V(G) - D \rangle_{G-E_0} = \langle V(G) - D \rangle_G$, because of $E_0 \subseteq E(\langle D \rangle_G)$. Hence $\langle V(G) - D \rangle_{G-E_0}$ is disconnected, since the subgraph $\langle V(G) - D \rangle_G$ is disconnected by the definition of the split dominating set. ■

Proposition 2. *If $V(K_p) \subset V(G) - D$, then D $sdom$ $G - E_0$.*

Proof. Let D $sdom$ G . Note that after removing the subset E_0 from G , we can observe that D dom $G - E_0$. Moreover, the subgraph $\langle V(G) - D \rangle_{G-E_0}$ is disconnected as a spanning subgraph of a disconnected graph $\langle V(G) - D \rangle_G$. All this together gives that D $sdom$ $G - E_0$, as required. ■

Note that it is not possible $\langle V(G) - D \rangle_G = K_p$, since the subgraph $\langle V(G) - D \rangle_G$ is disconnected.

3. The Split Domination Number of a Graph with a Deleted Subset of Edges

In this section we give several lower and upper bounds for $\gamma_s(G - E_0)$. From the results follow estimations for the split domination number of the graph with a removed edge included in [5]. Note that through all sections of the paper $K_p < G$ with $p \geq 2$, $E_0 = E(K_p)$ and $G - E_0$ is a spanning subgraph of G .

Theorem 3. *If $G - E_0$ is connected, then*

$$(1) \quad \gamma_s(G) - p + 1 \leq \gamma_s(G - E_0) \leq \gamma_s(G) + p - 1.$$

Proof. We now verify, that

$$(2) \quad \gamma_s(G - E_0) \leq \gamma_s(G) + p - 1.$$

Let D be a $\gamma_s(G)$ -set. Then the subgraph $\langle V(G) - D \rangle_G$ is disconnected. If $V(K_p) \subseteq D$ or $V(K_p) \subset V(G) - D$, then by Proposition 1 and Proposition 2,

respectively we obtain that D $\text{sdom } G - E_0$. As a consequence we have $\gamma_s(G - E_0) \leq \gamma_s(G)$. Since $p \geq 2$, then the inequality in (2) holds.

Assume that $V(K_p) \cap D \neq \emptyset$, $V(K_p) \cap D \neq V(K_p)$ and put $D^* = V(K_p) \cap (V(G) - D) \neq \emptyset$. Since $H = \langle V(G) - D \rangle_G$ is disconnected, then there exist two vertices, say $x_1, x_2 \in V(G) - D$, such that there is no path $P_H(x_1, x_2)$. It is not possible that $x_1, x_2 \in D^*$, otherwise x_1, x_2 would be adjacent in H . If $x_1, x_2 \notin D^*$, then also $D \cup D^* \text{ dom } G - E_0$. Moreover, since $x_1, x_2 \in V(G) - (D \cup D^*)$, there is no path joining them in $\langle V(G) - (D \cup D^*) \rangle_{G-E_0}$. In conclusion $D \cup D^* \text{ sdom } G - E_0$ and $\gamma_s(G - E_0) \leq |D \cup D^*| = |D| + |D^*| \leq \gamma_s(G) + p - 1$.

Now, suppose that $x_1 \in D^*$ and $x_2 \notin D^*$. Since $G - E_0$ is connected, every vertex from $V(G) - D$ has a neighbour in $G - E_0$, otherwise $G - E_0$ would have an isolated vertex. Assume that every vertex from D^* is adjacent to some vertex from D in $G - E_0$. This means that also $D \text{ dom } G - E_0$. Since disconnectedness of the graph $\langle V(G) - D \rangle_{G-E_0}$ is assured by disconnectedness of $\langle V(G) - D \rangle_G$ we obtain that also $D \text{ sdom } G - E_0$. As a consequence we have $\gamma_s(G - E_0) \leq \gamma_s(G)$. Now suppose that there exists a vertex $y \in D^*$ such that $N_{G-E_0}(y) \cap D = \emptyset$. This means there exists a vertex $z \in V(G) - (D \cup D^*)$ adjacent to y in $G - E_0$. Noting that $x_2 \in V(G) - (D \cup D^*)$ we state that there is no path joining z and x_2 in $\langle V(G) - (D \cup D^*) \rangle_{G-E_0}$ (otherwise x_1 and x_2 would be joined by a path in $H = \langle V(G) - D \rangle_G$ because of $(x_1, y), (y, z) \in E(H)$). Therefore $D \cup D^* \text{ sdom } G - E_0$ and $\gamma_s(G - E_0) \leq \gamma_s(G) + p - 1$.

Next we shall show that

$$(3) \quad \begin{aligned} \gamma_s(G) - p + 1 &\leq \gamma_s(G - E_0) \text{ or equivalently} \\ \gamma_s(G) &\leq \gamma_s(G - E_0) + p - 1. \end{aligned}$$

Let D_0 be a $\gamma_s(G - E_0)$ -set.

If $V(K_p) \subseteq D_0$, then D_0 also is a split dominating set of G and as a consequence

$$\gamma_s(G) \leq |D_0| = \gamma_s(G - E_0).$$

Since $p \geq 2$, the inequality in (3) holds.

Assume, that $V(K_p) \not\subseteq D_0$. This means that $V(K_p) \cap (V(G) - D_0) \neq \emptyset$. If $V(K_p) = V(G) - D_0$, then $|V(G)| = p + |D_0|$. As was mentioned in the Introduction, $\gamma_s(G) \leq |V(G)| - 2$. Thus $\gamma_s(G) \leq p + |D_0| - 2$ and satisfies the inequality in (3). Now let $V(K_p) \neq V(G) - D_0$. Then there exists a vertex $y \in V(G) - D_0$, such that $y \notin V(K_p)$. Since $H_0 = \langle V(G) - D_0 \rangle_{G-E_0}$

is disconnected, there exist two vertices, say x_1, x_2 in $V(H_0)$, not joined by a path in H_0 . Moreover, at least one of $P_{H_0}(x_1, y)$ or $P_{H_0}(x_2, y)$ does not exist (otherwise x_1 and x_2 would be joined by a path in H_0). Without loss of generality, we assume that there is no $P_{H_0}(x_1, y)$. Furthermore, there is no path $P_H(x_1, y)$, where $H = \langle V(G) - (D_0 \cup V(K_p) - \{x_1\}) \rangle_G$, because of $y \notin V(K_p)$. This means that H is disconnected. Since the subset $D_0 \cup V(K_p) - \{x_1\}$ dom G , then it is a split dominating set of G . Therefore we have $\gamma_s(G) \leq |D_0 \cup V(K_p) - \{x_1\}| \leq \gamma_s(G - E_0) + p - 1$, proving the theorem. ■

We can observe that for $p = 2$, $G - E_0 = G - e$ and the theorem immediately yields the following corollary:

Corollary 4. *If $G - e$ is connected, then*

$$\gamma_s(G) - 1 \leq \gamma_s(G - e) \leq \gamma_s(G) + 1.$$

Note that we did not use the assumption of connectedness of $G - E_0$, in the proof of the inequality in (3). Therefore as a consequence we obtain:

Remark 5. *If $K_p < G$ and $E_0 = E(K_p)$, then $\gamma_s(G) - p + 1 \leq \gamma_s(G - E_0)$.*

And as a consequence we state

Corollary 6. *For any arbitrary edge $e \in E(G)$, $\gamma_s(G) - 1 \leq \gamma_s(G - e)$.*

For some graphs we are able to give a better lower bound for a split domination number of a graph with deleted edges. To do it we use the following results:

Theorem 7 [4]. *For any graph G with hanging vertices*

1. $\gamma_s(G) = \gamma(G)$.
2. *Furthermore, there exists a $\gamma_s(G)$ -set containing all vertices adjacent to hanging vertices.*

A simple verification shows that every hanging vertex of G belongs to $V(G) - D$, where D is a $\gamma_s(G)$ -set from the second part of Theorem 7. Inspired by the above result we discuss a graph G having hanging vertices taking into account the number $\gamma_s(G - E_0)$.

The next theorem provides estimations of the parameter $\gamma_s(G - e)$ which will be used in the proof of Theorem 9.

Theorem 8 [5]. *Let G be a connected graph with $|V(G)| \geq 4$. If G has at least two hanging vertices, then*

$$\gamma_s(G) \leq \gamma_s(G - e) \leq \gamma_s(G) + 1,$$

for any $e \in E(G)$.

We recall that $K_p < G$ and $E_0 = E(K_p)$, $p \geq 2$. If G has at least two hanging vertices and less than 4 vertices, then $G \cong K_2$ or $G \cong P_3$. But, then there does not exist a split dominating set in $G - E_0$, for $p = 2$. Because of this we consider graphs with more than three vertices.

Theorem 9. *Let G be a connected graph with $|V(G)| \geq 4$. If G has at least two hanging vertices, then*

$$(4) \quad \gamma_s(G) \leq \gamma_s(G - E_0) \leq \gamma_s(G) + p - 1.$$

Proof. Suppose that D is a $\gamma_s(G)$ -set.

First we shall verify that

$$(5) \quad \gamma_s(G - E_0) \leq \gamma_s(G) + p - 1.$$

If $V(K_p) \subseteq D$ or $V(K_p) \subset V(G) - D$, then according to Proposition 1 and Proposition 2, respectively, we have that D sdom $G - E_0$ and as a consequence $\gamma_s(G - E_0) \leq \gamma_s(G)$. Since $p \geq 2$, we obtain $\gamma_s(G) < \gamma_s(G) + p - 1$. Hence the inequality in (5) holds.

It remains to consider the case when $V(K_p) \cap D \neq \emptyset$ and $V(K_p) \cap (V(G) - D) \neq \emptyset$.

Let D be a $\gamma_s(G)$ -set, such that no hanging vertex belongs to D . This means that all vertices adjacent to hanging vertices belong to D . The existence of such a $\gamma_s(G)$ -set is guaranteed by Theorem 7. Let x_1, x_2 be two hanging vertices of G . If K_p contains a hanging vertex, then $p = 2$, which implies that $E_0 = \{e\}$ and further by Theorem 8 we obtain the inequality in (5). Assume that $p \geq 3$ and put $D^* = V(K_p) \cap (V(G) - D)$. Further, we conclude that $x_1, x_2 \in V(G) - (D \cup D^*)$, since $x_1, x_2 \notin V(K_p)$, for $p \geq 3$. Hence we can state that x_1, x_2 are hanging vertices of $G - E_0$. Moreover x_1 and x_2 are isolated vertices of $\langle V(G) - (D \cup D^*) \rangle_{G - E_0}$, because they are isolated

in $\langle V(G) - D \rangle_G$. It follows that the subgraph $\langle V(G) - (D \cup D^*) \rangle_{G-E_0}$ is disconnected. In addition the subset $D \cup D^* \text{ dom } G - E_0$ and for this reason the subset $D \cup D^* \text{ sdom } G - E_0$. Since $V(K_p) \cap D \neq \emptyset$, we have $|D^*| \leq p - 1$ and consequently

$$\gamma_s(G - E_0) \leq |D \cup D^*| \leq \gamma_s(G) + p - 1,$$

as desired.

Now we shall prove that

$$(6) \quad \gamma_s(G) \leq \gamma_s(G - E_0).$$

Let D_0 be a $\gamma_s(G - E_0)$ -set containing no hanging vertex of $G - E_0$ (such a subset exists by Theorem 7). First we note that $G - E_0$ has at least one hanging vertex, say x , such that it is a hanging vertex of G . Note that K_p does not contain any hanging vertex of G , i.e., $p \geq 3$; otherwise $G - E_0$ would have an isolated vertex not belonging to D_0 , a contradiction to the fact that D_0 is dominating in $G - E_0$. Therefore $x \in V(G) - (D_0 \cup V(K_p))$ and it is an isolated vertex in a subgraph $\langle V(G) - D_0 \rangle_G$ which means that $\langle V(G) - D_0 \rangle_G$ is disconnected. It is obvious that $D_0 \text{ dom } G$. All this together gives that $D_0 \text{ sdom } G$. Consequently $\gamma_s(G) \leq |D_0| = \gamma_s(G - E_0)$ and the theorem is completely proved. ■

Theorem 10. *If $G - E_0$ is disconnected and has at least two connected components different from K_1 , then*

$$\gamma_s(G) - p + 1 \leq \gamma_s(G - E_0) \leq \gamma_s(G) + p - 1.$$

Proof. Let D be a $\gamma_s(G)$ -set. According to Proposition 1 and Proposition 2, it remains to consider the case that $V(K_p) \cap D \neq \emptyset$ and $V(K_p) \cap (V(G) - D) \neq \emptyset$ (since in other cases $\gamma_s(G - E_0) \leq \gamma_s(G)$ and the result follows). Denote by H_1 and H_2 two connected components of $G - E_0$, such that $|V(H_i)| \geq 2$, ($i = 1, 2$). Our first goal is to show that $V(H_i) \cap (V(G) - D) \neq \emptyset$, for $i = 1, 2$. Without loss of generality assume that $V(H_1) \cap (V(G) - D) = \emptyset$. Then $H_1 \leq \langle D \rangle_{G-E_0}$. Consider the following possibilities. If there exists a vertex x such that $x \in V(H_1)$ and $x \notin V(K_p)$, then $\emptyset \neq N_G(x) \subseteq D$. Further $D - \{x\} \text{ dom } G$. Moreover $\langle V(G) - (D - \{x\}) \rangle_G$ is disconnected since $\langle V(G) - D \rangle_G$ is disconnected and x is isolated in $\langle V(G) - (D - \{x\}) \rangle_G$. Hence $D - \{x\} \text{ sdom } G$, a contradiction to the fact that D is a minimum split dominating set of G . Further, suppose there is no such x , i.e.,

$V(H_1) \subseteq V(K_p)$. But then H_1 contains at least two isolated vertices as a subgraph of $\langle V(K_p) \rangle_{G-E_0}$ which has no edge, a contradiction, since H_1 is the connected component of $G - E_0$.

From the above investigations it follows that $V(H_i) \cap (V(G) - D) \neq \emptyset$, for $i = 1, 2$. Let $D^* = V(K_p) \cap (V(G) - D)$. Consider the subset $D \cup D^*$. Since $D \text{ dom } G$, so $D \cup D^* \text{ dom } G$. Moreover, $D \cup D^* \text{ dom } G - E_0$. If the induced subgraph $\langle V(G) - (D \cup D^*) \rangle_{G-E_0}$ is disconnected, then $D \cup D^*$ is a split dominating set of $G - E_0$ and $\gamma_s(G - E_0) \leq \gamma_s(G) + p - 1$, since $|D^*| \leq p - 1$.

Now assume that $\langle V(G) - (D \cup D^*) \rangle_{G-E_0}$ is connected. It must be that every vertex from $V(H_1)$ or $V(H_2)$ belongs to $D \cup D^*$, say $V(H_1) \subseteq D \cup D^*$, (or H_1 and H_2 would not be distinct components). Further, since as we noticed above $V(H_1) \cap (V(G) - D) \neq \emptyset$, let $x_1 \in V(H_1) \cap D^*$. Moreover, x_1 is adjacent to some vertex from $D \cup (D^* - \{x_1\})$ in H_1 , since H_1 is connected and $|V(H_1)| \geq 2$. If $V(H_2) \subseteq D \cup D^*$, then we choose a vertex, say x_2 from the subset $V(H_2) \cap D^*$ (this subset is not empty by the similar argue as for $V(H_1)$). For otherwise we choose x_2 from $V(H_2) \cap (V(G) - D)$. Notice that the vertex x_2 is adjacent to some vertex from $D \cup (D^* - \{x_1\})$ in $G - E_0$. Consequently, there are two vertices x_1 and x_2 , such that the subset $D_2 = D \cup (D^* - \{x_1, x_2\}) \text{ dom } G - E_0$. Furthermore, x_1 and x_2 are not joined by any path in $\langle V(G) - D_2 \rangle_{G-E_0}$, therefore $D_2 \text{ sdom } G - E_0$ and as a consequence we have $\gamma_s(G - E_0) \leq \gamma_s(G) + p - 3 \leq \gamma_s(G) + p - 1$, as required.

Since by Remark 5 we obtain $\gamma_s(G) - p + 1 \leq \gamma_s(G - E_0)$, the proof is complete. \blacksquare

4. The Maximal Domination Number of a Graph with DELETED EDGES

We start with two simple observations which follow straightforward from the definition of the maximal dominating set.

First we note that deleting or adding edges to the subgraph induced by a maximal dominating set of G does not destroy the property of being a maximal dominating set of the resulting graph. The same situation holds if the edges belong to the subgraph induced by the subset containing the vertices not belonging to a maximal dominating set. As a consequence we obtain:

Proposition 11. *If $V(K_p) \subseteq D$ or $V(K_p) \subseteq V(G) - D$, then $\gamma_m(G - E_0) \leq \gamma_m(G)$, where D is a $\gamma_m(G)$ -set.*

Proposition 12. *If $V(K_p) \subseteq D_0$ or $V(K_p) \subseteq V(G) - D_0$, then $\gamma_m(G) \leq \gamma_m(G - E_0)$, where D_0 is a $\gamma_m(G - E_0)$ -set.*

Next we note

Proposition 13. *Let D be a dominating set of G . The subset D is a maximal dominating set of G if and only if there exists a vertex $x \in D$, such that $N_G(x) \subseteq D$.*

Theorem 14. *If $K_p < G$, $p \geq 2$, then*

$$\gamma_m(G) - p + 1 \leq \gamma_m(G - E_0) \leq \gamma_m(G) + p - 1.$$

Proof. Let D be a $\gamma_m(G)$ -set. Because of Proposition 11, to show upper bound we need consider only the case when $V(K_p) \cap D \neq \emptyset$ and $V(K_p) \cap (V(G) - D) \neq \emptyset$. Denote $D^* = V(K_p) \cap (V(G) - D)$. Since D is maximal dominating in G , there exists a vertex $x \in D$ such that $N_G(x) \subseteq D$ (see Proposition 13) and hence $N_{G-E_0}(x) \subseteq D \cup D^*$. Further $D \cup D^* \text{ dom } G - E_0$ and hence it is a maximal dominating set of $G - E_0$. We see that $|D^*| \leq p - 1$ and consequently $\gamma_m(G - E_0) \leq |D \cup D^*| = \gamma_m(G) + p - 1$, as desired.

Next, assume that D_0 is a $\gamma_m(G - E_0)$ -set. According to Proposition 12 we consider only the case when $V(K_p) \cap D_0 \neq \emptyset$, $V(K_p) \cap (V(G) - D_0) \neq \emptyset$. Then it is easy to observe that $D_0 \cup D^*$ is a maximal dominating set of G , where $D^* = V(K_p) \cap (V(G) - D)$. Further, since $|D^*| \leq p - 1$, then $\gamma_m(G) \leq |D_0 \cup D^*| = \gamma_m(G - E_0) + p - 1$, which completes the proof. ■

Putting $G - E_0 = G - e$, there is the immediate corollary to Theorem 14.

Corollary 15. $\gamma_m(G) - 1 \leq \gamma_m(G - e) \leq \gamma_m(G) + 1$.

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