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# DESTROYING SYMMETRY BY ORIENTING EDGES: COMPLETE GRAPHS AND COMPLETE BIGRAPHS

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Dedicated to the memory of "Uncle" Paul Erdős who stimulated and the research careers of many mathematicians.

**Abstract**

Our purpose is to introduce the concept of determining the smallest number of edges of a graph which can be oriented so that the resulting mixed graph has the trivial automorphism group. We find that this number for complete graphs is related to the number of identity oriented trees. For complete bipartite graphs $K_{s,t}$, $s \leq t$, this number does not always exist. We determine for $s \leq 4$ the values of $t$ for which this number does exist.

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## 1. Introduction

Following the notation and terminology of the books [1, 2], a graph $G = (V, E)$ has node set $V$ and edge set $E$ with $|V| = n$, the *order* of $G$, and $|E| = m$, its *size*. An *automorphism* of $G$ is a permutation of $V$ which preserves adjacency. The set $\Gamma(G)$ of all automorphisms is obviously a permution group acting on the node set $V$. This group is called the *automorphism group* of $G$ or more briefly, *the group* of $G$. When $\Gamma(G)$ is the trivial group consisting only of the identity permutation, $G$ is called an *identity graph*.

An orientation of an edge $uv$ of $G$ changes this edge to one of the two arcs $(u, v)$ or $(v, u)$. A *mixed graph* is obtained from $G$ when some of the edges of $G$ (ranging from none to all) are oriented. In an *orientation* of $G$, every edge of $G$ is oriented, resulting in an *oriented graph*.

For a mixed graph $M$, an automorphism $\alpha$ is a permutation of $V$ which preserves both edges and arcs. We write $\Gamma(M)$ for its automorphism group. Then $M$ is called an *identity mixed graph* when $\Gamma(M)$ is trivial. Analogous to an identity graph and an identity mixed graph, one can consider an identity oriented tree or forest.

Now we can define the *identity orientation number* of a graph $G$, denoted $io\,(G)$, as the smallest number (if any) of edges of $G$ having orientations that result in an identity mixed graph $M$. We call a set of edges of $G$ whose orientations give $M$ the trivial automorphism group an *io-set*. Note that not all graphs have an *io*-set: the star $K_{1,3}$ has one while $K_{1,4}$ does not.

We illustrate this concept with a few examples. Obviously for $P_n$, a path of order $n \geq 2$, we have $io\,(P_n) = 1$, as any one edge of $P_n$ can be oriented arbitrarily to obtain an identity mixed graph. Similarly the cycle $C_n$ has $i\,(C_n) = 1$ for the same reason. Our object is to study the subtle problems of considering graphs which contain an *io*-set and of determining the values of the invariant $io\,(G)$ for complete bipartite graphs and complete graphs.

This concept is closely related to two elegant extremal results of Louis Quintas. In [10] he determined exactly the minimum size of an identity graph of order $n$. Then in [9], with D.J. Mc Carthy, the result of [10] was generalized to an arbitrary finite group. Such a generalization is also possible for our problem with the identity group.

## 2. Bipartite Graphs

Given a mixed graph $G$, for each node $x$, we have three types of degree: the in-degree $d^-(x)$, out-degree $d^+(x)$ and unoriented degree, $d(x)$. Note that in an oriented graph $G$, if $\phi$ is an automorphism which maps node $x$ to node $y$, then necessarily

$$d^+(x) = d^+(y)\,,\ d^-(x) = d^-(y)\,,\ \text{and}\ d(x) = d(y)\,.$$

We denote the complete bipartite graph with $s$ white nodes and $t$ black ones by $K_{s,t}$ so that $s+t=n$, and without loss of generality let $s \le t$.

Consider first the stars $K_{1,t}$ with $t \ge 2$. We see at once from Figure 2.1 that $io(K_{1,2}) = 1$, $io(K_{1,3}) = 2$, and $io(K_{1,t})$ with $t > 3$ does not exist.

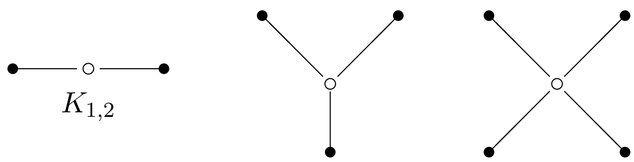


Figure 2.1. Three stars, just the first two having an *io*-set

Our first result gives an inequality on $s$ and $t$ which precludes the existence of an *io*-set of edges in $K_{s,t}$.

**Lemma 1.** *If $t \ge 3^s$, then $K_{s,t}$ does not have an identity orientation.*

***Proof.*** Let $t \ge 3^s$ and consider an orientation of a subset of $E(K_{s,t})$, resulting in a mixed graph $M$. Denote the two parts of $V(K_{s,t})$ by $X, Y$ with $X = \{x_1, \cdots, x_s\}$, $Y = \{y_1, \cdots, y_t\}$.

With each node of $Y$ we associate the $s$-tuple $(b_1, b_2, \ldots, b_s)$ where each $b_i \in \{+1, 0, -1\}$ as follows, illustrated for node $y_1$:

$$b_i = \begin{cases} \ \ 1 \text{ if edge } x_i y_1 \text{ is oriented } (y_1, x_i) \text{ in } M, \\ \ \ 0 \text{ if edge } x_i y_1 \text{ is not oriented, and} \\ -1 \text{ if arc } (x_i, y_1) \text{ is in } M. \end{cases}$$

Since there are just $3^s$ different $s$-tuples of $1, 0, -1$ it follows that when $t > 3^s$ there must be two nodes $y, y'$ of $Y$ with the same $s$-tuple. Hence there exists a non-identity automorphism $\alpha$ of $M$ such that $\alpha(y) = y'$.

In this case, if a pair of nodes of $Y$ have the same $s$-tuple, there would be a nontrivial automorphism. Thus, every possible $s$-tuple must be present, and hence it follows that the respective degrees of each vertex of $Y$ are the same, namely

$$d^+(y_i) = d^-(y_i) = d^0(y_i) = 3^s - 1$$

for each $i = 1, 2, \ldots, m$. But then any permutation of the elements of $Y$, and the appropriate corresponding permutation of the elements of $X$, gives a nontrivial automorphism of $M$ and the result follows. ∎

**Corollary 2.** *The complete bipartite graph $K_{2,9}$ does not have an io-set while $K_{2,8}$ does.*

***Proof.*** The first part follows at once from Lemma 1. The $K_{2,9}$ assertion is easily constructed. ∎

**Theorem 3.** *When $G$ is the complete bipartite graph $K_{s,t}$, the following results for $io\,(G)$ are known:*

(a) $io\,(K_{1,3}) = 2$ *but* $io\,(K_{1,4})$ *does not exist.*

(b) $io\,(K_{2,8}) = 10$ *but* $io\,(K_{2,9})$ *does not exist.*

(c) $io\,(K_{3,26})$ *exists but* $io\,(K_{3,27})$ *does not exist.*

(d) $io\,(K_{4,79})$ *exists but* $io\,(K_{4,80})$ *does not exist.*

***Proof.*** (a) This part follows from Lemma 1 with $s = 1$, $t = 3$.

(b) That $io\,(K_{2,9})$ does not exist follows from Lemma 1. We now show that $io\,(K_{2,8})$=10.

We proceed by listing an assignment of the orientation of the edges from the nodes $x_1, x_2$ of the smaller partite set to the larger partite set $y_1, y_2, \ldots, y_8$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1 | 1 | 0 | 0 | 0 | −1 | −1 |
| $x_2$ | 1 | 0 | −1 | 1 | 0 | −1 | 0 | −1 |

Note that the only ordering omitted from the nine possibilities is $x_1 = -1$, $x_2 = 1$ but this forces the nodes $x_1$ and $x_2$ to have different in-and-out degrees $d^-(x_1) = 3$ and $d^-(x_1) = 2$ while $d^+(x) = 2$ and $d^-(x) = 3$. Hence the automorphisms of this mixed graph must fix $x_1$ and $x_2$. Although the $y_i$ have the same degrees, they can not be permuted since that would require $x_1$ and $x_2$ to be permuted. Thus $io\,(K_{2,8}) \leq 10$.

To prove $io\,(K_{2,8}) \geq 10$, we show that there cannot be more than six unoriented edges. If there were more than six, then without loss of generality we can assume that $d^0(x_1) \geq 4$. But then two of the four nodes must have the same orientation $(+, -, 0)$ from $x_2$ and thus can be permuted under an automorphism. Consequently, there can be at most six unoriented edges and thus at least ten oriented edges.

(c) Again the non-existence of an *io*-set for $K_{3,27}$ is a consequence of Lemma 1. The proof that $io\,(K_{3,26})$ does exist is similar to that of $K_{2,8}$ in (b). Associate to the 26 nodes of the larger partite set all the possible triples except $(1, 0, -1)$. As in the previous result, this forces the nodes of the smaller partite set to be fixed under any automorphism, which consequently fixes all the nodes of the larger part.

(d) To see that $K_{4,79}$ has an *io*-set, associate with the 79 nodes of the larger partite set all of the $3^4$ possible arrangements with the exception of $(+\,+\,0\,0)$ and $(+\,0\,0\,-)$. Now using the argument above, it is easy to see that the smaller partite set must be fixed under any permutation and consequently the larger set must be fixed.

To verify that $K_{4,80}$ does not have an *io*-set, let the nodes in the smaller part be $x_1, x_2, x_3$ and $x_4$ and $y_1, y_2 \ldots y_{80}$ be the larger part. So we can associate with each of the nodes in the larger part a 4-tuple of 1s, $-1$s and 0s as above. For convenience, label the 4-tuple associated with $y_i, z_1^i z_2^i z_3^i z_4^i$. Clearly if two of the 80 4-tuples are the same, then a nontrivial automorphism results. Since there are 81 different 4-tuples exactly one is not used. This implies that some pair, say $x_1$ and $x_2$ have $d^+(x_1) = d^+(x_2)$, $d^-(x_1) = d^-(x_2)$ and $d^0(x_1) = d^0(x_2)$.

Consider the mapping $\phi$ defined as follows:

$$\begin{aligned}
\phi\,(x_1) &= x_2, \phi\,(x_2) = x_1, \phi\,(x_3) = x_3, \phi\,(x_4) = x_4 \text{ and} \\
\phi\,(y_i) &= \begin{cases} y_i & \text{if } z_1^i = z_2^i, \\ y_j & \text{where } y_j \text{ is associated with the 4-tuple } z_1^i, z_2^i, z_3^i, z_4^i. \end{cases}
\end{aligned}$$

Note this is well defined; the $y_i$ always exist since the only 4-tuple not used has the first and second terms the same. Furthermore, one can show that $\phi$ is an automorphism.

**Theorem 4.** *If* $t = \lceil r/3 \rceil$ *then* $io\,(K_{m,3^m - t}) \leq 2m3^{m-1} - 2m^2/9$.

***Proof.*** Of the $3^m$ $m$-tuples use all but the following $t$ $m$-tuples for orientations from the $3^m - t$ nodes of the larger partite set.

$$
\begin{array}{ccc}
\overbrace{\phantom{+++\ldots++}}^{\frac{m}{3}} & \overbrace{\phantom{00\ldots000}}^{\frac{m}{3}} & \overbrace{\phantom{---\ldots---}}^{\frac{m}{3}} \\
+++\ldots++ & 00\ldots000 & ---\ldots--- \\
+++\ldots+0 & 00\ldots00- & ---\ldots--+ \\
+++\ldots+00 & 00\ldots0-- & ---\ldots-++ \\
\vdots & 00\ldots--- & \vdots \\
 & \vdots & \\
+000\ldots0 & 0--\ldots-- & -++\ldots+
\end{array}
$$

This forces the resulting mixed graph to have distinct "degrees" at each of the nodes of the small partite set, and thus any automorphism would necessarily fix those nodes. Having those nodes fixed, it is easy to observe that the nodes in the larger partite set must also be fixed, since the degrees are all distinct. The upper bound follows since all the $3^m$ possible 2/3 $m$-tuples have 2/3 of the $m\ 3^m$ edges oriented. ∎

## 3. Complete Graphs

Although it is well known [2, 3, 4] that almost all graphs have trivial automorphism groups, the problem at hand becomes quite complicated when the graph has a "rich" automorphism group. The complete graphs $K_m$ have the "richest possible" group, namely, the symmetric group $S_n$ consisting of all the permutations on $V$.

We begin by illustrating the problem for the smallest nontrivial $K_n$ with $2 \leq n \leq 5$ in Figure 3.1 and Table 3.1. These minimum numbers can always be attained by an *io*-forest.

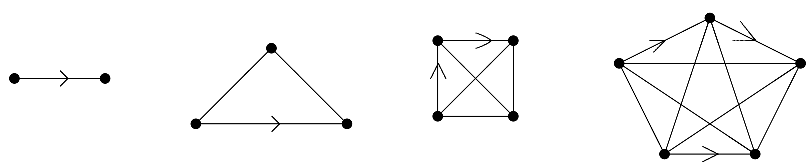

Figure 3.1. The identity orientations of $K_2$ to $K_5$

| Table 3.1. | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $io\,(K_n)$ | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 | 7 |

**Theorem 5.** *Every nontrivial complete graph $K_n$ has an io-set and the value of $io\,(K_n)$ is the smallest number of arcs in an identity oriented forest of order $n$.*

***Proof.*** It is well known that $\Gamma(D)$, the group of $D$, is identical with $\Gamma(\bar{D})$, the group of the complement of $D$. Thus to verify that $io\,(K_n)$ exists, we need only consider the directed path $D = \vec{P}_n$. Its complement $\bar{D}$ is an identity orientation of $K_n$. Hence $io\,(K_n) \leq n-1$, so it exists.

To evaluate $io\,(K_n)$ we need only point out that the smallest number of edges of $K_n$ in an *io*-set giving an identity $M$ is just the number of arcs in the complement $\bar{M}$. This is necessarily an identity oriented forest having the smallest possible number of arcs. ■

Appendix 1 shows all the identity oriented trees with $n = 1$ to 5 nodes. Appendix 2 uses these trees to depict all the identity oriented forests with $n = 1$ to 7 nodes. Finally, Appendix 3 lists all the partitions of $n = 2$ to 18 where each partition of $n$ has its parts giving the number of nodes in an identity oriented tree with $n$ nodes which is a component of an identity oriented forest with the minimum number of arcs. It is simple to check all the partitions.

There are four identity oriented trees with 4 nodes. Hence in Appendix 3 each part, 4, can be realized by any one of these four identity oriented trees. Further, each occurrence of two 4s in a partition can be realized in (42) = 6 ways, and the one appearance, for $n = 18$, of three 4s occurs in four ways. This gives the following Table 3.2 giving the number of smallest identity oriented forests for $n = 2$ to 18.

| Table 3.2. | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $\#io$-F | 1 | 1 | 1 | 5 | 1 | 4 | 14 | 54 | 4 | 16 | 83 | 378 | 6 |

In [5] several species of trees were counted. These include identity trees and oriented trees. The counting of identity oriented trees provide an algorithmic solution to the determination of the numbers $io(K_n)$. These have been counted by Harary and Robinson [8].

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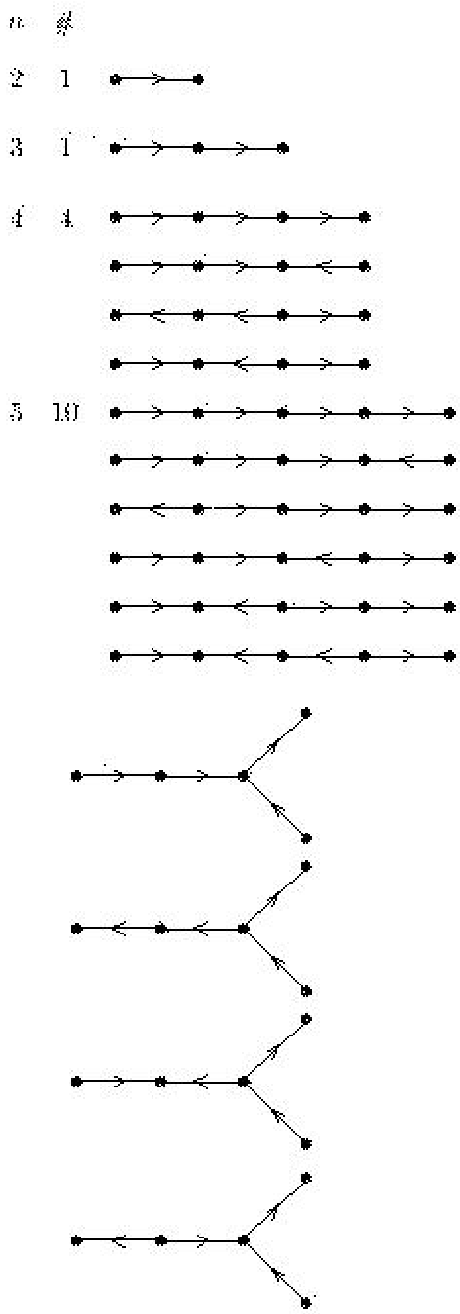
**Appendix 1.** Nontrivial identity oriented trees



**Appendix 2.** Minimum size nontrivial identity oriented forests