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ON THE STABILITY FOR PANCYCLICITY

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Abstract

A property P defined on all graphs of order n is said to be k-stable if for any graph of order n that does not satisfy P, the fact that uv is not an edge of G and that $G+uv$ satisfies P implies $d_G(u)+d_G(v) < k$. Every property is $(2n-3)$-stable and every k-stable property is $(k+1)$-stable. We denote by $s(P)$ the smallest integer k such that P is k-stable and call it the *stability* of P. This number usually depends on n and is at most $2n-3$. A graph of order n is said to be pancyclic if it contains cycles of all lengths from 3 to n. We show that the stability $s(P)$ for the graph property "G is pancyclic" satisfies $\max(\lceil \frac{6n}{5} \rceil - 5, n+t) \leq s(P) \leq \max(\lceil \frac{4n}{3} \rceil - 2, n+t)$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$.

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1. Introduction

We use [3] for terminology and notation not defined here and consider simple graphs only. For any integer k, denote by C_k a cycle of length k. A graph of order n is said to be *pancyclic* if it contains cycles of all lengths from 3 to n.

In [2], Bondy and Chvátal introduced the closure of a graph and the stability of a graph property. The *k-closure* $C_k(G)$ of a graph G is obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least k, until no such pair remains.

A property P defined on all graphs of order n is said to be *k-stable* if for any graph of order n that does not satisfy P, the fact that uv is not an edge of G and that $G+uv$ satisfies P implies $d_G(u)+d_G(v)<k$. Vice versa, if $uv \notin E(G), d_G(u)+d_G(v) \geq k$ and $G+uv$ has property P, then G itself has property P. Every property is $(2n-3)$-stable and every k-stable property is $(k+1)$-stable. We denote by $s(P)$ the smallest integer k such that P is k-stable and call it the *stability* of P. This number usually depends on n and is at most $2n-3$.

Theorem 1 [2]**.** *The property P: "G contains a cycle C_k" satisfies $s(P) = 2n-k$ for $4 \leq k \leq n$ and $s(P) = 2n-k-1$ for $4 \leq k < n$ if k is even.*

Question 1. *What is the stability for the property "G is pancyclic"?*

In 1971 Bondy [1] has posed the interesting "metaconjecture".

Conjecture 1 (*metaconjecture*)**.** *Almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic* (*except for maybe a simple family of exceptional graphs*).

By Theorem 1, $s(P)=n$ for the property "G is hamiltonian". The complete bipartite graphs $K_{\frac{n}{2},\frac{n}{2}}$ for n even, $n \geq 4$, and $K_{\frac{n+1}{2},\frac{n-1}{2}}$ for n odd, $n \geq 5$, show that the stability $s(P)$ for the property "G is pancyclic" satisfies $s(P) \geq n+t$ for all $n \geq 4$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$. In [5] the following Theorem was proved.

Theorem 2. *Let G be a hamiltonian graph of order $n \geq 32$ and u and v two nonadjacent vertices with $d(u)+d(v) \geq n+t$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$. Then G contains all cycles of length k where $3 \leq k \leq \frac{n+13}{5}$.*

Moreover, examples were presented showing one cannot expect G to contain cycles of length considerably longer than $\frac{n}{3}$ with the assumption of Theorem 2.

For the property P: "G is pancyclic" we will prove the following Theorem.

Theorem 3. *Let P be the property "G is pancyclic". Then the stability $s(P)$ satisfies $\max(\lceil \frac{6n}{5} \rceil - 5, n+t) \leq s(P) \leq \max(\lceil \frac{4n}{3} \rceil - 2, n+t)$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$.*

2. Exact Values and the Lower Bound

For a graph G of order n denote by $s(P,n)$ the stability of the property "G is pancyclic". Then it is not very difficult to check that $s(P,n) = n + t$ for $3 \leq n \leq 9$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$.

Next we will give a proof for the lower bound given in Theorem 3.

Proof. As mentioned in the introduction the complete bipartite graphs $K_{\frac{n}{2},\frac{n}{2}}$ for n even, $n \geq 4$, and $K_{\frac{n+1}{2},\frac{n-1}{2}}$ for n odd, $n \geq 5$, show that $s(P,n) \geq n + t$ for all $n \geq 4$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n+1)$.

1. For $k \geq 1$ let G_{5k} be the graph of order $n = 5k$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_n v_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\}$. Then $d(u) + d(v) = 6k - 6 = n + \frac{n-30}{5}$ and the graph $G + uv$ is pancyclic whereas G misses a cycle of length $2k + 3$.

2. For $k \geq 1$ let G_{5k+1} be the graph of order $n = 5k + 1$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_n v_1$. Define $u = v_1, v = v_{k+2}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k+1}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\}$. Then $d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5}$ and the graph $G + uv$ is pancyclic whereas G misses a cycle of length $2k + 3$.

3. For $k \geq 1$ let G_{5k+2} be the graph of order $n = 5k + 2$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_n v_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k+2}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\}$. Then $d(u) + d(v) = 6k - 2 = n + \frac{n-22}{5}$ and the graph $G + uv$ is pancyclic whereas G misses a cycle of length $2k + 3$.

4. For $k \geq 1$ let G_{5k+3} be the graph of order $n = 5k + 3$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_n v_1$. Define $u = v_1, v = v_{k+2}, a = v_{2k+2}, b = v_{2k+3}, c = v_{4k+4}, d = v_{4k+5}$. Let $Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+3}\}, S = \{v_{2k+4}, \ldots, v_{4k+3}\}$ and $P = \{v_{4k+4}, \ldots, v_{5k+3}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R -$

$\{c, d\}$. Then $d(u) + d(v) = 6k - 2 = n + \frac{n-28}{5}$ and the graph $G + uv$ is pancyclic whereas G misses a cycle of length $2k + 4$.

5. For $k \geq 0$ let G_{5k+4} be the graph of order $n = 5k + 4$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_n v_1$. Define $u = v_1, v = v_{k+2}, a = v_{2k+2}, b = v_{2k+3}, c = v_{4k+4}, d = v_{4k+5}$. Let $Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+3}\}, S = \{v_{2k+4}, \ldots, v_{4k+3}\}$ and $P = \{v_{4k+4}, \ldots, v_{5k+4}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\}$. Then $d(u) + d(v) = 6k = n + \frac{n-24}{5}$ and the graph $G + uv$ is pancyclic whereas G misses a cycle of length $2k + 4$.

Summarizing we obtain that $s(P) \geq \max(\lceil \frac{6n}{5} \rceil - 5, n + t)$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n + 1)$. ■

3. The Upper Bound

In this section we will give a proof for the upper bound given in Theorem 3. For this proof we will use the following results.

Corollary 1 [4]. *Let G be a hamiltonian graph of order n. If there exist two nonadjacent vertices u and v at distance $d \geq 3$ on a hamiltonian cycle of G such that $d(u) + d(v) \geq n + d - 2$, then G contains cycles of all lengths between 3 and $n - d + 1$.*

Lemma 1 [4]. *Let G contain a hamiltonian path $P = v_1 v_2 \ldots v_n$ such that $v_1 v_n \notin E(G)$ and $d(v_1) + d(v_n) \geq n + d$ for some integer d, $0 \leq d \leq n - 4$. Then for any l, $2 \leq l \leq d + 3$, there exists a (v_1, v_n)-path of length l.*

Theorem 4 [4]. *Let G be a graph of order n. If G has a hamiltonian (u, v)-path for a pair of nonadjacent vertices u and v such that $d(u) + d(v) \geq n$, then G is pancyclic.*

***Proof of Theorem* 3.** Suppose there is a graph G with nonadjacent vertices u, v such that $d(u) + d(v) \geq \max(\lceil \frac{4n}{3} \rceil - 2, n + t), G + uv$ is pancyclic, but G is not. Then $n \geq 10$. By Theorem 1, G is hamiltonian. Let $C : v_1 \ldots v_n v_1$ be a Hamilton cycle in G. Choose the labeling such that $u = v_1, v = v_{r+2}$ with $n = r + s + 2$ and $r \leq s$. Let $R = \{v_2, \ldots, v_{r+1}\}, S = \{v_{r+3}, \ldots, v_n\}$ and $d = d_C(u, v) = r + 1$. Set $d(u) + d(v) = r + p + s + q$, where $d_R(u) + d_R(v) = r + p$ and $d_S(u) + d_S(v) = s + q$. Recall that $d(u) + d(v) \geq \lceil \frac{4n}{3} \rceil - 2$. By Theorem 1, G contains cycles C_k for $\lfloor \frac{2}{3} n \rfloor + 2 \leq k \leq n$.

We distinguish several cases.

Case 1. $d \leq \lceil \frac{n}{3} \rceil$.
Since $n \geq 10$ we have $d(u) + d(v) \geq n + 2$. Thus $d_S(u) + d_S(v) \geq s + 2$ for $2 \leq d \leq 3$. By Theorem 4, G contains cycles $C_3, \ldots, C_{s+2}$. Hence G is pancyclic for $d = 2$, a contradiction.

So we may assume that $d \geq 3$. By Corollary 1, G contains cycles $C_3, \ldots, C_{n-d+1}$. Hence G is pancyclic since $n - d + 1 \geq \lfloor \frac{2n}{3} \rfloor + 1$, a contradiction.

Case 2. $d \geq \lceil \frac{n}{3} \rceil + 1$.

Subcase 2.1. $d_S(u) + d_S(v) \geq s + 2$.
By Theorem 4, G contains cycles $C_3, \ldots, C_{s+2}$. Note that $s + 2 \geq \frac{n}{2} + 1$.

Subcase 2.1.1. $p \geq \lfloor \frac{2n}{3} \rfloor - s$.
By Lemma 1 we can take (u, v)-paths of length l in $R \cup \{u, v\}$ for $2 \leq l \leq p + 1$ and a (v, u)-path of length $s + 1$ in $S \cup \{u, v\}$. This gives cycles $C_{s+3}, \ldots, C_{s+p+2}$. Hence G is pancyclic since $s + p + 2 \geq \lfloor \frac{2n}{3} \rfloor + 2$, a contradiction.

Subcase 2.1.2. $p \leq \lfloor \frac{2}{3} n \rfloor - s - 1$.
Then $q \geq \lceil \frac{n}{3} \rceil - 2 + 2 - p \geq \lceil \frac{n}{3} \rceil - \lfloor \frac{2n}{3} \rfloor + s + 1 \geq s + 1 - \lceil \frac{n}{3} \rceil \geq 2$ for $n \geq 11$. Take (v, u)-paths of length l for $2 \leq l \leq s - \lceil \frac{n}{3} \rceil + 2$ in $S \cup \{u, v\}$. This gives cycles $C_{n-s-1+2}, \ldots, C_{\lfloor \frac{2n}{3} \rfloor + 1}$. Hence G is pancyclic, a contradiction. It is easy to check that for $n = 10$ and $s = 4$ G is also pancyclic and we get a contradiction.

Subcase 2.2. $d_S(u) + d_S(v) \leq s + 1$.
Then $d_R(u) + d_R(v) \geq r + 1 + \lceil \frac{n}{3} \rceil - 2$. By Theorem 4, G contains cycles $C_3, \ldots, C_{r+2}$. Set $r + 2 = \lceil \frac{n}{3} \rceil + 1 + d'$. By Lemma 1 there are (u, v)-paths of lengths l for $2 \leq l \leq \lceil \frac{n}{3} \rceil$ in $R \cup \{u, v\}$. This gives cycles $C_{s+1+2}, \ldots, C_{s+1+\lceil \frac{n}{3} \rceil}$. So far cycles of lengths $\lceil \frac{n}{3} \rceil + d' + 2, \ldots, \lfloor \frac{2n}{3} \rfloor - d' + 1$ are missing.

Let $S = S_1 \cup S_2 \cup S_3$ with $S_1 = \{v_{\lceil \frac{n}{3} \rceil + d' + 2}, \ldots, v_{n - \lceil \frac{n}{3} \rceil}\}$, $S_2 = \{v_{n - \lceil \frac{n}{3} \rceil + 1}, \ldots, v_{2\lceil \frac{n}{3} \rceil + d' + 1}\}$ and $S_3 = \{v_{2\lceil \frac{n}{3} \rceil + d' + 2}, \ldots, v_n\}$. Then $|S_1| = n - 2\lceil \frac{n}{3} \rceil - d' - 1 = |S_3|$ and $|S_2| = d' + 1 + 3\lceil \frac{n}{3} \rceil - n$.

Suppose $uv_i \in E(G)$ for some i with $\lceil \frac{n}{3} \rceil + 2 + d' \leq i \leq n$. Then there is a path $uv_iv_{i-1} \ldots v$ of length $i - (\lceil \frac{n}{3} \rceil + d' + 1) + 1$. Together with the (u, v)-paths in $R \cup \{u, v\}$ we obtain cycles of lengths $i - \lceil \frac{n}{3} \rceil - d' + 2, \ldots, i - d'$. Hence, for $n - \lceil \frac{n}{3} \rceil + 1 \leq i \leq n - \lceil \frac{n}{3} \rceil + 2d'$, we obtain all missing cycles and G is pancyclic, a contradiction.

A symmetric argument applies for edges vv_i with $\lceil \frac{n}{3} \rceil + 2 + d' \leq i \leq n$. In this case, for $n - \lceil \frac{n}{3} \rceil - d' + 2 \leq i \leq 2\lceil \frac{n}{3} \rceil + d' + 1$, we obtain all missing cycles and G is pancyclic, a contradiction.

Hence we may assume that $N_{S_2}(u) = N_{S_2}(v) = \emptyset$. Suppose $N_S(u) \cap N_S(v) = \emptyset$. Then $(d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2(\lceil \frac{n}{3} \rceil + d' - 1) + 2(n - 2\lceil \frac{n}{3} \rceil - d' - 1) = 2n - 2\lceil \frac{n}{3} \rceil - 4 \leq n + \lceil \frac{n}{3} \rceil - 4 < \lceil \frac{4n}{3} \rceil - 2$, a contradiction. Hence $N_S(u) \cap N_S(v) \neq \emptyset$. Thus there is a cycle of length $\lceil \frac{n}{3} \rceil + d' + 2$.

Next consider two vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \lceil \frac{n}{3} \rceil$. If $|E(\{x, y\}, \{u, v\})| \geq 3$ then there is a (u, v)-path of length $\lceil \frac{n}{3} \rceil + 2$. Together with the (u, v)-paths through R we obtain cycles of lengths $\lceil \frac{n}{3} \rceil + 4, \ldots, 2\lceil \frac{n}{3} \rceil + 2$ and G is pancyclic (recall that $d' \geq 1$).

Hence we may further assume that $|E(\{x, y\}, \{u, v\})| \leq 2$ for all pairs of vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \lceil \frac{n}{3} \rceil$. But then $\lceil \frac{4n}{3} \rceil - 2 \leq (d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2(\lceil \frac{n}{3} \rceil + d' - 1) + 2(n - 2\lceil \frac{n}{3} \rceil - d' - 1) = 2n - 2\lceil \frac{n}{3} \rceil - 4 \leq n + \lceil \frac{n}{3} \rceil - 4 < \lceil \frac{4n}{3} \rceil - 2$, a final contradiction. ■

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