ON THE STABILITY FOR PANCYCLICITY

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Abstract

A property $P$ defined on all graphs of order $n$ is said to be $k$-stable if for any graph of order $n$ that does not satisfy $P$, the fact that $uv$ is not an edge of $G$ and that $G+uv$ satisfies $P$ implies $d_G(u)+d_G(v) < k$. Every property is $(2n-3)$-stable and every $k$-stable property is $(k+1)$-stable. We denote by $s(P)$ the smallest integer $k$ such that $P$ is $k$-stable and call it the stability of $P$. This number usually depends on $n$ and is at most $2n-3$. A graph of order $n$ is said to be pancyclic if it contains cycles of all lengths from 3 to $n$. We show that the stability $s(P)$ for the graph property ”$G$ is pancyclic” satisfies $\max(\lceil \frac{6n}{5} \rceil - 5, n + t) \leq s(P) \leq \max(\lceil \frac{4n}{3} \rceil - 2, n + t)$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n + 1)$.

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1. Introduction

We use [3] for terminology and notation not defined here and consider simple graphs only. For any integer $k$, denote by $C_k$ a cycle of length $k$. A graph of order $n$ is said to be pancyclic if it contains cycles of all lengths from 3 to $n$.

In [2], Bondy and Chvátal introduced the closure of a graph and the stability of a graph property. The $k$-closure $C_k(G)$ of a graph $G$ is obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least $k$, until no such pair remains.
A property $P$ defined on all graphs of order $n$ is said to be $k$-stable if for any graph of order $n$ that does not satisfy $P$, the fact that $uv$ is not an edge of $G$ and that $G + uv$ satisfies $P$ implies $d_G(u) + d_G(v) < k$. Vice versa, if $uv \notin E(G), d_G(u) + d_G(v) \geq k$ and $G + uv$ has property $P$, then $G$ itself has property $P$. Every property is $(2n - 3)$-stable and every $k$-stable property is $(k + 1)$-stable. We denote by $s(P)$ the smallest integer $k$ such that $P$ is $k$-stable and call it the stability of $P$. This number usually depends on $n$ and is at most $2n - 3$.

**Theorem 1** [2]. The property $P$: "$G$ contains a cycle $C_k$" satisfies $s(P) = 2n - k$ for $4 \leq k \leq n$ and $s(P) = 2n - k - 1$ for $4 \leq k < n$ if $k$ is even.

**Question 1.** What is the stability for the property "$G$ is pancyclic"?

In 1971 Bondy [1] has posed the interesting "metaconjecture".

**Conjecture 1** (metaconjecture). Almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic (except for maybe a simple family of exceptional graphs).

By Theorem 1, $s(P) = n$ for the property "$G$ is hamiltonian". The complete bipartite graphs $K_{\frac{n}{2}, \frac{n}{2}}$ for $n$ even, $n \geq 4$, and $K_{\frac{n-1}{2}, \frac{n-1}{2}}$ for $n$ odd, $n \geq 5$, show that the stability $s(P)$ for the property "$G$ is pancyclic" satisfies $s(P) \geq n + t$ for all $n \geq 4$, where $t = 2\lceil\frac{n+1}{2}\rceil - (n + 1)$. In [5] the following Theorem was proved.

**Theorem 2.** Let $G$ be a hamiltonian graph of order $n \geq 32$ and $u$ and $v$ two nonadjacent vertices with $d(u) + d(v) \geq n + t$, where $t = 2\lceil\frac{n+1}{2}\rceil - (n + 1)$. Then $G$ contains all cycles of length $k$ where $3 \leq k \leq \frac{n+13}{6}$.

Moreover, examples were presented showing one cannot expect $G$ to contain cycles of length considerably longer than $\frac{n}{3}$ with the assumption of Theorem 2.

For the property $P$: "$G$ is pancyclic" we will prove the following Theorem.

**Theorem 3.** Let $P$ be the property "$G$ is pancyclic". Then the stability $s(P)$ satisfies $\max(\lceil\frac{6n}{5}\rceil - 5, n + t) \leq s(P) \leq \max(\lceil\frac{4n}{3}\rceil - 2, n + t)$, where $t = 2\lceil\frac{n+1}{2}\rceil - (n + 1)$. 
2. Exact Values and the Lower Bound

For a graph $G$ of order $n$ denote by $s(P, n)$ the stability of the property "$G$ is pancyclic". Then it is not very difficult to check that $s(P, n) = n + t$ for $3 \leq n \leq 9$, where $t = 2\lceil\frac{n+1}{2}\rceil - (n + 1)$.

Next we will give a proof for the lower bound given in Theorem 3.

**Proof.** As mentioned in the introduction the complete bipartite graphs $K_{\frac{n}{2}, \frac{n}{2}}$ for $n$ even, $n \geq 4$, and $K_{\frac{n+1}{2}, \frac{n-1}{2}}$ for $n$ odd, $n \geq 5$, show that $s(P, n) \geq n + t$ for all $n \geq 4$, where $t = 2\lceil\frac{n+1}{2}\rceil - (n + 1)$.

1. For $k \geq 1$ let $G_{5k}$ be the graph of order $n = 5k$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_nv_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+1}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{a, b\}$. Then $d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5}$ and the graph $G + uv$ is pancyclic whereas $G$ misses a cycle of length $2k + 3$.

2. For $k \geq 1$ let $G_{5k+1}$ be the graph of order $n = 5k + 1$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_nv_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+1}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k+1}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{a, b\}$. Then $d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5}$ and the graph $G + uv$ is pancyclic whereas $G$ misses a cycle of length $2k + 3$.

3. For $k \geq 1$ let $G_{5k+2}$ be the graph of order $n = 5k + 2$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_nv_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+1}, d = v_{4k+3}$. Let $Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k+2}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{a, b\}$. Then $d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5}$ and the graph $G + uv$ is pancyclic whereas $G$ misses a cycle of length $2k + 3$.

4. For $k \geq 1$ let $G_{5k+3}$ be the graph of order $n = 5k + 3$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a Hamilton cycle $C : v_1 \ldots v_nv_1$. Define $u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+1}, d = v_{4k+5}$. Let $Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\}$ and $P = \{v_{4k+2}, \ldots, v_{5k+3}\}$. Define $N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{a, b\}$. Then $d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5}$ and the graph $G + uv$ is pancyclic whereas $G$ misses a cycle of length $2k + 3$. 

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\{c, d\}. Then \(d(u) + d(v) = 6k - 2 = n + \frac{n-28}{3}\) and the graph \(G + uv\) is pancyclic whereas \(G\) misses a cycle of length \(2k + 4\).

5. For \(k \geq 0\) let \(G_{5k+4}\) be the graph of order \(n = 5k + 4\) with vertex set \(V(G) = \{v_1, \ldots, v_n\}\) and a Hamilton cycle \(C : v_1 \ldots v_nv_1\). Define \(u = v_1, v = v_{k+2}, a = v_{2k+2}, b = v_{2k+3}, c = v_{4k+4}, d = v_{4k+5}\). Let \(Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+3}\}, S = \{v_{2k+4}, \ldots, v_{4k+3}\}\) and \(P = \{v_{4k+4}, \ldots, v_{5k+4}\}\). Define \(N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\}\). Then \(d(u) + d(v) = 6k = n + \frac{n-28}{3}\) and the graph \(G + uv\) is pancyclic whereas \(G\) misses a cycle of length \(2k + 4\).

Summarizing we obtain that \(s(P) \geq \max(\lceil \frac{4n}{3} \rceil - 5, n+t)\), where \(t = 2\lceil \frac{n+1}{2} \rceil - (n+1)\).

3. The Upper Bound

In this section we will give a proof for the upper bound given in Theorem 3. For this proof we will use the following results.

**Corollary 1** [4]. Let \(G\) be a hamiltonian graph of order \(n\). If there exist two nonadjacent vertices \(u\) and \(v\) at distance \(d \geq 3\) on a hamiltonian cycle of \(G\) such that \(d(u) + d(v) \geq n + d - 2\), then \(G\) contains cycles of all lengths between \(3\) and \(n - d + 1\).

**Lemma 1** [4]. Let \(G\) contain a hamiltonian path \(P = v_1v_2\ldots v_n\) such that \(v_1v_n \notin E(G)\) and \(d(v_1) + d(v_n) \geq n + d\) for some integer \(d\), \(0 \leq d \leq n - 4\). Then for any \(l\), \(2 \leq l \leq d + 3\), there exists a \((v_1, v_n)\)-path of length \(l\).

**Theorem 4** [4]. Let \(G\) be a graph of order \(n\). If \(G\) has a hamiltonian \((u, v)\)-path for a pair of nonadjacent vertices \(u\) and \(v\) such that \(d(u) + d(v) \geq n\), then \(G\) is pancyclic.

**Proof of Theorem 3.** Suppose there is a graph \(G\) with nonadjacent vertices \(u, v\) such that \(d(u) + d(v) \geq \max(\lceil \frac{4n}{3} \rceil - 2, n+t)\), \(G + uv\) is pancyclic, but \(G\) is not. Then \(n \geq 10\). By Theorem 1, \(G\) is hamiltonian. Let \(C : v_1 \ldots v_nv_1\) be a Hamilton cycle in \(G\). Choose the labeling such that \(u = v_1, v = v_{r+2}\) with \(n = r + s + 2\) and \(r \leq s\). Let \(R = \{v_2, \ldots, v_{r+1}\}, S = \{v_{r+3}, \ldots, v_n\}\) and \(d = d_C(u, v) = r+1\). Set \(d(u) + d(v) = r + p + s + q\), where \(d_R(u) + d_R(v) = r + p\) and \(d_S(u) + d_S(v) = s + q\). Recall that \(d(u) + d(v) \geq \lceil \frac{4n}{3} \rceil - 2\). By Theorem 1, \(G\) contains cycles \(C_k\) for \(\lceil \frac{2}{3}n \rceil + 2 \leq k \leq n\).
We distinguish several cases.

Case 1. \( d \leq \left\lceil \frac{n}{3} \right\rceil \).

Since \( n \geq 10 \) we have \( d(u) + d(v) \geq n + 2 \). Thus \( d_S(u) + d_S(v) \geq s + 2 \) for \( 2 \leq d \leq 3 \). By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{s+2} \). Hence \( G \) is pancyclic for \( d = 2 \), a contradiction.

So we may assume that \( d \geq 3 \). By Corollary 1, \( G \) contains cycles \( C_3, \ldots, C_{n-d+1} \). Hence \( G \) is pancyclic since \( n-d+1 \geq \left\lceil \frac{2n}{3} \right\rceil + 1 \), a contradiction.

Case 2. \( d \geq \left\lceil \frac{n}{3} \right\rceil + 1 \).

Subcase 2.1. \( d_S(u) + d_S(v) \geq s + 2 \).

By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{s+2} \). Note that \( s + 2 \geq \frac{n}{2} + 1 \).

Subcase 2.1.1. \( p \geq \left\lceil \frac{2n}{3} \right\rceil - s \).

By Lemma 1 we can take \( (u, v) \)-paths of length \( l \) in \( R \cup \{ u, v \} \) for \( 2 \leq l \leq p + 1 \) and a \( (v, u) \)-path of length \( s + 1 \) in \( S \cup \{ u, v \} \). This gives cycles \( C_{s+3}, \ldots, C_{s+p+2} \). Hence \( G \) is pancyclic since \( s + p + 2 \geq \left\lceil \frac{2n}{3} \right\rceil + 2 \), a contradiction.

Subcase 2.1.2. \( p \leq \left\lceil \frac{2}{3} n \right\rceil - s - 1 \).

Then \( q \geq \left\lceil \frac{2}{3} n \right\rceil - 2 + 2 - p \geq \left\lceil \frac{2}{3} n \right\rceil - \frac{2n}{3} + s + 1 \geq s + 1 - \frac{n}{3} \geq 2 \) for \( n \geq 11 \). Take \( (v, u) \)-paths of length \( l \) for \( 2 \leq l \leq \left\lceil \frac{n}{3} \right\rceil + 2 \) in \( S \cup \{ u, v \} \). This gives cycles \( C_{n-s+1+2}, \ldots, C_{n-s+1+\left\lceil \frac{n}{3} \right\rceil} \). Hence \( G \) is pancyclic, a contradiction. It is easy to check that for \( n = 10 \) and \( s = 4 \) \( G \) is also pancyclic and we get a contradiction.

Subcase 2.2. \( d_S(u) + d_S(v) \leq s + 1 \).

Then \( d_R(u) + d_R(v) \geq r + 1 + \left\lceil \frac{n}{3} \right\rceil - 2 \). By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{r+2} \). Set \( r + 2 = \left\lceil \frac{n}{3} \right\rceil + 1 + d' \). By Lemma 1 there are \( (u, v) \)-paths of lengths \( l \) for \( 2 \leq l \leq \left\lceil \frac{n}{3} \right\rceil \) in \( R \cup \{ u, v \} \). This gives cycles \( C_{s+1+2}, \ldots, C_{s+1+\left\lceil \frac{n}{3} \right\rceil} \). So far cycles of lengths \( \left\lceil \frac{n}{3} \right\rceil + d' + 2, \ldots, \left\lceil \frac{2n}{3} \right\rceil - d' - 1 \) are missing.

Let \( S = S_1 \cup S_2 \cup S_3 \) with \( S_1 = \{ v_{\left\lceil \frac{n}{3} \right\rceil + d' + 2}, \ldots, v_{n - \left\lceil \frac{n}{3} \right\rceil} \} \), \( S_2 = \{ v_{n - \left\lceil \frac{n}{3} \right\rceil + 1}, \ldots, v_{2 \left\lceil \frac{n}{3} \right\rceil + d' + 1} \} \) and \( S_3 = \{ v_{2 \left\lceil \frac{n}{3} \right\rceil + d' + 2}, \ldots, v_n \} \). Then \( |S_1| = n - 2 \left\lceil \frac{n}{3} \right\rceil - d' - 1 = |S_3| \) and \( |S_2| = d' + 1 + 3 \left\lceil \frac{n}{3} \right\rceil - n \).

Suppose \( uv_i \in E(G) \) for some \( i \) with \( \left\lceil \frac{n}{3} \right\rceil + 2 + d' \leq i \leq n \). Then there is a path \( uv_{i-1} \ldots v \) of length \( i - (\left\lceil \frac{n}{3} \right\rceil + d' + 1) + 1 \). Together with the \( (u, v) \)-paths in \( R \cup \{ u, v \} \) we obtain cycles of lengths \( i - \left\lceil \frac{n}{3} \right\rceil - d' + 2, \ldots, i - d' \). Hence, for \( n - \left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq n - \left\lceil \frac{n}{3} \right\rceil + 2d' \), we obtain all missing cycles and \( G \) is pancyclic, a contradiction.
A symmetric argument applies for edges $vv_i$ with $\left\lceil \frac{n}{3} \right\rceil + 2 + d' \leq i \leq n$. In this case, for $n - \left\lceil \frac{n}{3} \right\rceil - d' + 2 \leq i \leq 2\left\lceil \frac{n}{3} \right\rceil + d' + 1$, we obtain all missing cycles and $G$ is pancyclic, a contradiction.

Hence we may assume that $N_{S_2}(u) = N_{S_2}(v) = \emptyset$. Suppose $N_S(u) \cap N_S(v) = \emptyset$. Then $(d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2\left(\frac{n}{3} + d' - 1\right) + 2(n - 2\left\lceil \frac{n}{3} \right\rceil - d' - 1) = 2n - 2\left\lceil \frac{n}{3} \right\rceil - 4 \leq n + \left\lceil \frac{n}{3} \right\rceil - 4 < \left\lceil \frac{4n}{3} \right\rceil - 2$, a contradiction.

Hence we may assume that $N_S(u) \cap N_S(v) \neq \emptyset$. Thus there is a cycle of length $\left\lceil \frac{n}{3} \right\rceil + d' + 2$.

Next consider two vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \left\lceil \frac{n}{3} \right\rceil$. If $|E(\{x, y\}, \{u, v\})| \geq 3$ then there is a $(u, v)$-path of length $\left\lceil \frac{n}{3} \right\rceil + 2$. Together with the $(u, v)$-paths through $R$ we obtain cycles of lengths $\left\lceil \frac{n}{3} \right\rceil + 4, \ldots, 2\left\lceil \frac{n}{3} \right\rceil + 2$ and $G$ is pancyclic (recall that $d' \geq 1$).

Hence we may further assume that $|E(\{x, y\}, \{u, v\})| \leq 2$ for all pairs of vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \left\lceil \frac{n}{3} \right\rceil$. But then $\left\lceil \frac{4n}{3} \right\rceil - 2 \leq (d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2\left(\left\lceil \frac{n}{3} \right\rceil + d' - 1\right) + 2(n - 2\left\lceil \frac{n}{3} \right\rceil - d' - 1) = 2n - 2\left\lceil \frac{n}{3} \right\rceil - 4 \leq n + \left\lceil \frac{n}{3} \right\rceil - 4 < \left\lceil \frac{4n}{3} \right\rceil - 2$, a final contradiction.

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References


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