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FULL DOMINATION IN GRAPHS

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Abstract

For each vertex v in a graph G, let there be associated a subgraph H_v of G. The vertex v is said to dominate H_v as well as dominate each vertex and edge of H_v. A set S of vertices of G is called a full dominating set if every vertex of G is dominated by some vertex of S, as is every edge of G. The minimum cardinality of a full dominating set of G is its full domination number $\gamma_{FH}(G)$. A full dominating set of G of cardinality $\gamma_{FH}(G)$ is called a γ_{FH}-set of G. We study three types of full domination in graphs: full star domination, where H_v is the maximum star centered at v, full closed domination, where H_v is the subgraph induced by the closed neighborhood of v, and full open domination, where H_v is the subgraph induced by the open neighborhood of v.

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1. Introduction

A vertex v in a graph G is said to *dominate* itself and each of its neighbors. A set $S \subseteq V(G)$ is called a *dominating set* for G if every vertex of G is dominated by some vertex of S. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$ of G. A dominating set of cardinality $\gamma(G)$ is a γ-set for G. There has been increased interest in recent years in the study of domination in graphs. Indeed, the books [2, 3] by Haynes, Hedetniemi, and Slater are devoted exclusively to this subject. In domination, a vertex dominates a set of vertices (according to some rule); while in covering, a vertex covers the edges incident with it. We combine these concepts to describe another variation of domination.

For a graph G, let H be a function that maps each vertex v of G into a subgraph H_v of G. In this context, the vertex v is said to *dominate* H_v as well as dominate each vertex and edge of H_v. A set S of vertices of G is called a *full dominating set* if every vertex and every edge of G is dominated by some vertex of S. For each full dominating set S of G and $v \in V(G) - S$, the set $S \cup \{v\}$ is also a full dominating set. If G has no isolated vertices, then we need only be concerned with each edge of G being dominated by some vertex of S. The minimum cardinality of a full dominating set of G is its *full domination number* (with respect to the function H) and is denoted by $\gamma_{FH}(G)$. A full dominating set of G of cardinality $\gamma_{FH}(G)$ is called a γ_{FH}*-set* of G. Certainly, $\gamma_{FH}(G)$ is defined for a graph G if and only if $V(G)$ is a full dominating set for G.

In this paper we study three examples of full domination, namely *full star domination*, where H_v is the maximum star S_v centered at v, *full closed domination*, where $H_v = \langle N[v] \rangle$, the subgraph induced by the closed neighborhood of v, and *full open domination*, where $H_v = \langle N(v) \rangle$, the subgraph induced by the open neighborhood of v.

2. Full Star Domination in Graphs

We denote the *full star domination number* of a graph G by $\gamma_{FS}(G)$. Certainly, $\gamma_{FS}(G)$ is defined for every graph G. Indeed, if G is a graph without isolated vertices, then $\gamma_{FS}(G) = \alpha_o(G)$, the vertex covering number of G (the minimum number of vertices that cover all edges of G). If G has $I(G)$ isolated vertices, then $\gamma_{FS}(G) = \alpha_o(G) + I(G)$. Therefore, the full star domination number is not a new parameter; it only provides a new setting for an old one. A well-known theorem of Gallai [1] states that if G is a graph of order n without isolated vertices, then $\alpha_o(G) + \beta_o(G) = n$, where $\beta_o(G)$ is the vertex independence number of G. This gives us the following.

Observation 2.1. *For every graph G of order n without isolated vertices,*

$$\gamma_{FS}(G) = n - \beta_o(G).$$

Since every full star dominating set of a graph is also a dominating set, it follows that $\gamma(G) \leq \gamma_{FS}(G)$. By Observation 2.1,

$$1 \leq \gamma(G) \leq \gamma_{FS}(G) \leq n - 1$$

for every graph G of order n with at most $n-2$ isolated vertices. We now consider the realizablility of three integers a, b, n as the domination number, full star domination number, and order, respectively, of some graph without isolated vertices. Thus any such triple a, b, n described above must satisfy $1 \leq a \leq b \leq n-1$. By Observation 2.1, however, $\gamma_{FS}(G) = n-1$ if and only if $G = K_n$, which implies that $\gamma(G) = 1$. Hence we may assume that $1 \leq a \leq b \leq n-2$. On the other hand, the independent domination number $i(G)$ satisfies

$$\gamma(G) \leq i(G) \leq \beta_o(G) = n - \gamma_{FS}(G).$$

This implies that $\gamma(G) + \gamma_{FS}(G) \leq n$, thereby obtaining Ore's [6] well-known inequality $\gamma(G) \leq n/2$ for graphs G of order n without isolated vertices. We now present the desired realization result.

Proposition 2.2. *For every triple a, b, n of integers with $n \geq 3$, $1 \leq a \leq b \leq n-2$, and $a + b \leq n$, there exists a graph G of order n without isolated vertices such that $\gamma(G) = a$ and $\gamma_{FS}(G) = b$.*

Proof. We consider two cases.

Case 1. $a + b \leq n - 1$. Let K_{b+1} be the complete graph with vertex set $\{u_1, u_2, \ldots, u_{b+1}\}$ and let G be the graph obtained from K_{b+1} by adding $n-b-1$ new vertices $v_1, v_2, \ldots, v_{n-b-1}$, the $a-1$ edges u_iv_i $(1 \leq i \leq a-1)$, and the $n-b-a$ edges u_av_i $(a \leq i \leq n-b-1)$. The graph G is shown in Figure 1. Since $\{u_1, u_2, \ldots, u_a\}$ is a γ-set and $\{u_1, u_2, \ldots, u_b\}$ is a γ_{FS}-set for G, it follows that $\gamma(G) = a$ and $\gamma_{FS}(G) = b$.

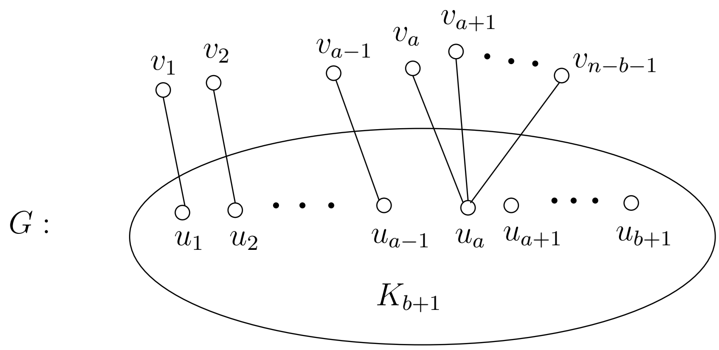

Figure 1. The graph G in Case 1

Case 2. $a + b = n$. Let $K_{\ell_1}, K_{\ell_2}, \ldots, K_{\ell_a}$ be complete graphs, where $\ell_i \geq 2$ for all i and $\sum_{i=1}^{a} \ell_i = n$. Also, let v_{i1} and v_{i2} be distinct vertices in K_{ℓ_i} $(1 \leq i \leq a)$. Then the graph G is obtained from the graph $\bigcup_{i=1}^{a} K_{\ell_i}$ by adding the $a-1$ edges $v_{i1}\ v_{i+1,2}$ for $1 \leq i \leq a-1$. For $a = 4$, the graph G is shown in Figure 2. Since $\{v_{11}, v_{21}, \ldots, v_{a1}\}$ is a γ-set, $\gamma(G) = a$. On the other hand,

$$\bigcup_{i=1}^{a} V(K_{\ell_i}) - \{v_{12}, v_{22}, \ldots, v_{a2}\}$$

is a γ_{FS}-set and so $\gamma_{FS}(G) = n - a = b$. ■

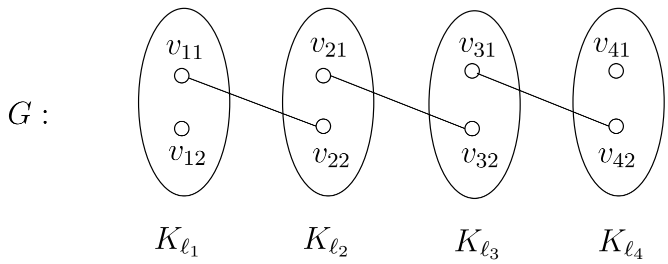

Figure 2. The graph G in Case 2 when $a = 4$

3. Full Closed Domination in Graphs

Recall that a set S of vertices in a graph G is a *full closed dominating set* if every vertex and edge of G belongs to $\langle N[v] \rangle$ for some $v \in S$. The minimum cardinality of a full closed dominating set is the *full closed domination number* $\gamma_{FC}(G)$. A full closed dominating set of cardinality $\gamma_{FC}(G)$ is referred to as a *γ_{FC}-set.* This parameter was first introduced by Sampathkumar and Neeralagi in [5], where it was called the neighborhood number of a graph, and further studied by Jayaram, Kwong, and Straight in [4]. The following two propositions appeared in [5].

Proposition 3.1. *For every graph G, $\gamma(G) \le \gamma_{FC}(G) \le \gamma_{FS}(G)$.*

Proposition 3.2. *If G is a triangle-free graph, then $\gamma_{FC}(G) = \gamma_{FS}(G)$.*

The converse of Proposition 3.2 is not true in general unless $\gamma_{FC}(G) = \gamma_{FS}(G) = 1$, in which case G is a star. To see this, we recall that the *corona* of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is that graph of order $2n$ obtained from G by adding n new vertices $u_1, u_2, \ldots, u_n$ and the n new edges u_iv_i $(1 \le i \le n)$. For $n \ge 3$, let G_n denote the corona of K_n. Define G_2 as the graph obtained from G_3 by deleting an end-vertex. Then $\gamma_{FS}(G_n) = \gamma_{FC}(G_n) = n$ for $n \ge 2$ but certainly G_n is not triangle-free.

If $\gamma(G) = 1$, then $\gamma_{FC}(G) = 1$ while $1 \le \gamma_{FS}(G) \le n-1$. For each integer k with $1 \le k \le n-1$, the graph H obtained by deleting the edges of a complete subgraph of order $n-k$ from K_n has $\gamma(H) = \gamma_{FC}(H) = 1$ and $\gamma_{FS}(H) = k$. For $\gamma(G) \ge 2$, the following realization result appeared in [4].

Theorem 3.3. *For every triple a, b, c of integers with $2 \le a \le b \le c$, there exists a graph G with $\gamma(G) = a$, $\gamma_{FC}(G) = b$, and $\gamma_{FS}(G) = c$.*

It is often of interest to know how the value of a graphical parameter is affected when a small change is made in a graph. In this connection, we now consider this question in the case of $\gamma_{FC}(G)$ when an edge is deleted from G. We show, in fact, that such an operation produces a graph whose full closed domination number differs from that of the original graph by at most 1.

Proposition 3.4. *For each edge e of a graph G,*

$$|\gamma_{FC}(G) - \gamma_{FC}(G-e)| \le 1.$$

Proof. Let $e = uv$ be an edge of G and let S be a γ_{FC}-set of $G-e$. Then $S \cup \{u\}$ is a full closed dominating set of G. So $\gamma_{FC}(G) \le |S \cup \{u\}| \le \gamma_{FC}(G-e)+1$. Next we show that $\gamma_{FC}(G-e) \le \gamma_{FC}(G)+1$. We consider two cases.

Case 1. *There exists a γ_{FC}-set S' of G such that $u, v \notin S'$.* Then S' is a full closed dominating set of $G - e$ as well. Therefore, $\gamma_{FC}(G - e) \leq |S'| = \gamma_{FC}(G) < \gamma_{FC}(G) + 1$.

Case 2. *For every γ_{FC}-set S of G, at least one of u and v belongs to S.* Since $S \cup \{u, v\}$ is a full closed dominating set of $G - e$, it follows that

$$\gamma_{FC}(G - e) \leq |S \cup \{u, v\}| \leq |S| + 1 = \gamma_{FC}(G) + 1. \qquad \blacksquare$$

The bounds presented in Proposition 3.4 are sharp. To see this, we consider the graph G of Figure 3, where $\gamma_{FC}(G) = 3$ and the vertices of a γ_{FC}-set in G are indicated by solid circles. Observe that

$$\begin{aligned}
\gamma_{FC}(G - e_0) &= \gamma_{FC}(G) = 3, \\
\gamma_{FC}(G - e_1) &= \gamma_{FC}(G) - 1 = 2, \\
\gamma_{FC}(G - e_2) &= \gamma_{FC}(G) + 1 = 4.
\end{aligned}$$

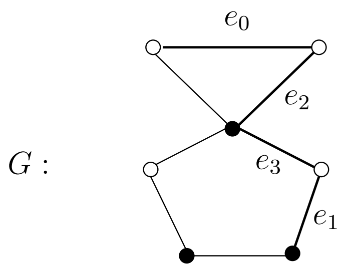

Figure 3. How the full closed domination number is affected by the removal of an edge

In view of Proposition 3.4, the edge set of a graph G can be partitioned into the following subsets:

$$\begin{aligned}
E_0(G) &= \{e \in E(G) \ : \ \gamma_{FC}(G - e) = \gamma_{FC}(G)\}, \\
E_-(G) &= \{e \in E(G) \ : \ \gamma_{FC}(G - e) = \gamma_{FC}(G) - 1\}, \\
E_+(G) &= \{e \in E(G) \ : \ \gamma_{FC}(G - e) = \gamma_{FC}(G) + 1\}.
\end{aligned}$$

The graph G of Figure 3 shows that it is possible for all three of these subsets to be nonempty for a single graph G. We now present some facts concerning elements in $E_-(G)$ and $E_+(G)$.

Proposition 3.5. *Let G be a graph containing an edge $e = uv$. Then $e \in E_-(G)$ if and only if for every γ_{FC}-set S' of $G - e$,*

(a) *neither u, v, nor any common neighbor of u and v belongs to S', and*

(b) *for each $w \in N_G[u] \cap N_G[v]$, the set $S' \cup \{w\}$ is a γ_{FC}-set of G.*

Proof. Suppose that $e = uv \in E_-(G)$. Let S' be a γ_{FC}-set of $G - e$. We first verify (a). Assume, to the contrary, that either u, v, or some common neighbor of u and v belongs to S'. Thus S' is also a full closed dominating set of G. So $\gamma_{FC}(G) \leq |S'|$, a contradiction. To verify (b), let $w \in N_G[u] \cap N_G[v]$. Then $w \notin S'$ by (a). Let $S = S' \cup \{w\}$. Thus $|S| = |S'| + 1 = \gamma_{FC}(G)$. Since S is a full closed dominating set of G, it follows that S is a γ_{FC}-set for G and so (b) holds.

For the converse, let S' be a γ_{FC}-set of $G - e$, satisfying (a) and (b). It then follows from (a) that $S' \cap (N_G[u] \cap N_G[v]) = \emptyset$. Let $w \in N_G[u] \cap N_G[v]$. By (b) the set $S = S' \cup \{w\}$ is a γ_{FC}-set of G and so $|S| = |S'| + 1 = \gamma_{FC}(G)$. Thus $|S| = \gamma_{FC}(G) - 1$, implying that $e \in E_-(G)$. ■

Proposition 3.6. *Let G be a graph containing an edge $e = uv$. Then $e \in E_+(G)$ if and only if for every γ_{FC}-set S of G,*

(a) *$|S \cap \{u, v\}| = 1$, and*

(b) *$S \cup \{u, v\}$ is a γ_{FC}-set of $G - e$.*

Proof. Let $e = uv \in E_+(G)$ and let S be a γ_{FC}-set of G. First we verify (a). Assume, to the contrary, that $|S \cap \{u, v\}| \neq 1$. If $S \cap \{u, v\} = \emptyset$, then, since e is dominated by some vertex in S, there is a vertex $w \in S$ adjacent to both u and v. However, then, S is a full closed dominating set for $G - e$, contradicting the fact that $e \in E_+(G)$. On the other hand, if $\{u, v\} \subseteq S$, then, once again, S is full closed dominating set for $G - e$, a contradiction. Next we verify (b). Certainly, $S \cup \{u, v\}$ is a full closed dominating set of $G - e$. By (a), however,

$$|S \cup \{u, v\}| = |S| + 1 = \gamma_{FC}(G) + 1 = \gamma_{FC}(G - e)$$

and so $S \cup \{u, v\}$ is a γ_{FC}-set for $G - e$.

For the converse, let S be a γ_{FC}-set of G that satisfies (a) and (b). By (a), S contains exactly one of u and v. Let $S' = S \cup \{u, v\}$ and so $|S'| = |S| + 1$. By (b), S' is a γ_{FC}-set of $G - e$. Thus $\gamma_{FC}(G - e) = |S'| = |S| + 1 = \gamma_{FC}(G) + 1$, implying that $e \in E_+(G)$. ■

If we were to delete two edges from G, one belonging to $E_-(G)$ and the other belonging to $E_+(G)$, then the full closed domination number of the resulting graph is the same as $\gamma_{FC}(G)$.

Proposition 3.7. *Let G be a graph. If $e_1 \in E_-(G)$ and $e_2 \in E_+(G)$, then*

$$\gamma_{FC}(G - e_1 - e_2) = \gamma_{FC}(G)$$

Proof. Removing e_1 first and then e_2 shows that $\gamma_{FC}(G - e_1 - e_2) \leq \gamma_{FC}(G)$; while removing e_2 first and then e_1 produces the inequality $\gamma_{FC}(G - e_2 - e_1) \geq \gamma_{FC}(G)$. ■

Accordingly, if the edges e_1 and e_2 of graph G of Figure 3 are deleted, then $\gamma_{FC}(G - e_1 - e_2) = 3$ since $\gamma_{FC}(G) = 3$. Observe that the edges e_1 and e_2 of this graph are not adjacent. This is no coincidence as we next show.

Proposition 3.8. *Let G be a graph. If $e_1 \in E_-(G)$ and $e_2 \in E_+(G)$, then e_1 and e_2 are not adjacent in G.*

Proof. Assume, to the contrary, that there exists a graph G containing adjacent edges e_1 and e_2 with $e_1 \in E_-(G)$ and $e_2 \in E_+(G)$. Let $e_1 = uv$ and $e_2 = vw$. Let S' be a γ_{FC}-set for $G - uv$. By Proposition 3.5, $v \notin S'$. Let $S'' = S' \cup \{v\}$ and consider the graph $G - vw$. The edge uv is dominated by $v \in S''$. Since vw is dominated by some vertex of S', it follows that either vw is dominated by $w \in S'$ or dominated by some $x \in S'$, where x is adjacent to both v and w. In either case, w is dominated in $G - vw$ by some vertex of S''. Hence, S'' is a full closed dominating set for $G - vw$. However,

$$|S''| = \gamma_{FC}(G - uv) + 1 = \gamma_{FC}(G) < \gamma_{FC}(G - vw),$$

which is impossible. ■

By Proposition 3.8 then, for the graph G of Figure 3, it follows that $e_3 \in E_0(G)$. Indeed, if G is a connected graph in which $E_+(G) \neq \emptyset$ and $E_-(G) \neq \emptyset$, then $E_0(G) \neq \emptyset$. There are numerous graphs G in which every edge of G belongs to $E_0(G)$, such as even cycles and K_n ($n \geq 3$). There is, however, no graph G in which every edge belongs to $E_+(G)$.

Proposition 3.9. *No graph G exists every edge of which belongs to $E_+(G)$.*

Proof. Assume, to the contrary, that there exists a graph G such that $E(G) = E_+(G)$. Let S be a γ_{FC}-set of G. Then, by Proposition 3.6, for every edge uv in G, one of u and v belongs to S and the other to $V(G) - S$. This implies that G is a bipartite graph with partite sets S and $V(G) - S$. Thus G is triangle-free. By Observation 2.1, $\gamma_{FC}(G) = \gamma_{FS}(G) = \alpha_o(G)$. However,

$$\gamma_{FC}(G) + 1 = \gamma_{FC}(G - e) = \alpha_o(G - e) \leq \alpha_o(G) = \gamma_{FC}(G)$$

for every edge e in G, which contradicts the fact that $e \in E_+(G)$. ■

There are graphs G, though, every edge of which belongs to $E_-(G)$. For example, odd cycles of order at least 5 have this property.

For a set S of vertices of a graph G and a vertex v of G, the *distance between v and S* is defined as

$$d(v, S) = \min\{d(v, u) : \ u \in S\}.$$

The *diameter* of S is defined as

$$\mathrm{diam} S = \max\{d(u, v) : u, v \in S\}.$$

Thus $\mathrm{diam} V(G) = \mathrm{diam} G$.

For a nonempty set S of vertices in a connected graph G, a *Steiner S-tree* is a tree of minimum size in G containing S. Certainly, every end-vertex of a Steiner S-tree belongs to S. An edge $e = uv$ in a Steiner S-tree T is called *S-free* if both $u \notin S$ and $v \notin S$.

Lemma 3.10. *For every γ_{FC}-set S of a connected graph G, there exists a Steiner S-tree containing no S-free edges.*

Proof. Assume, to the contrary, that there is a connected graph G and a γ_{FC}-set S of G such that every Steiner S-tree in G contains S-free edges. Among all Steiner S-trees, let T be a Steiner S-tree containing a minimum number of S-free edges. Then T contains an S-free edge $e = uv$ and a vertex $x \in S$ such that x, u, v is a path in T. Since S is a γ_{FC}-set of G, it follows that e is dominated by some vertex in S. If e is dominated by x, then necessarily x is adjacent to both u and v. Hence $(T - uv) + xv$ is a Steiner S-tree in G containing fewer S-free edges than T, which is impossible. Thus e is dominated by some vertex $w \in S$, where $w \neq x$. Let T_u and T_v be the two components of $T - uv$, where T_u contains u and T_v contains v. Necessarily, w belongs to exactly one of T_u and T_v, say T_u. Then $(T - uv) + wv$ is a Steiner S-tree in G containing fewer S-free edges than T, again an impossibility. ■

Lemma 3.11. *If S is a γ_{FC}-set of a connected graph G, then the order of any Steiner S-tree is at most $2\gamma_{FC}(G) - 1$.*

Proof. Let T be a Steiner S-tree containing no S-free edges. Then $V(T) = S \cup W$, where $S \cap W = \emptyset$. Let $|W| = a$. Thus the order of T is $a + \gamma_{FC}(G)$ and the size of T is $a + \gamma_{FC}(G) - 1$. Assume that $a \geq \gamma_{FC}(G)$. Since T is a Steiner S-tree, every end-vertex in T belongs to S. Thus every vertex in W has degree at least 2 in T. Also, since T has no S-free edge, every vertex in W is adjacent only to vertices of S in T.

Therefore, the size of T is at least

$$\sum_{w \in W} \deg_T w \geq 2a \geq a + \gamma_{FC},$$

producing a contradiction. ■

Corollary 3.12. *If S is a γ_{FC}-set in a connected graph G, then*

$$\text{diam} S \leq 2\gamma_{FC}(G) - 2.$$

Proof. Let T be a Steiner S-tree of G and suppose that that order of T is k. By Lemma 3.11, $k \leq 2\gamma_{FC}(G) - 1$. Among all trees of order k, the path P_k has the greatest diameter, namely $k - 1$, and $k - 1 \leq 2\gamma_{FC}(G) - 2$. ■

Theorem 3.13. *If G is a graph of diameter d, then*

$$\gamma_{FC}(G) \geq d/2.$$

Proof. Let x and y be vertices of G such that $d(x, y) = d$ and let S be a γ_{FC}-set in G. Then x is dominated by some vertex $u \in S$ and y is dominated by some $v \in S$. Either $u = x$ or $ux \in E(G)$. Similarly, either $v = y$ or $vy \in E(G)$. Thus, using Corollary 3.12, we have

$$d = \text{diam} G = d(x, y) \leq d(u, v) + 2 \leq \text{diam} S + 2 \leq 2\gamma_{FC}(G),$$

producing the desired result. ■

To show that the bound presented in Theorem 3.13 is sharp, let $G = P_{2k+1}$ be the path of order $2k + 1$. Then $\text{diam} G = 2k$ and $\gamma_{FC}(G) = k$, as desired.

A set S of vertices in a graph G is an *open dominating set* (or *total dominating set*) if every vertex of G is adjacent to at least one vertex of S. An open dominating set of minimum cardinality is a *minimum open dominating* *set* and its cardinality is the *open domination number* $\gamma_t(G)$, also called the *total domination number*. The open domination number is also referred to as the *total domination number*. No graph containing isolated vertices has an open dominating set.

In order to obtain a relationship between the open domination number and the full closed domination number, we present the following lemma.

Lemma 3.14. *For every γ_{FC}-set S in a connected graph and each vertex $v \in S$,*

$$d(v, S - \{v\}) \leq 2.$$

Proof. Assume, to the contrary, that there is a γ_{FC}-set S in a connected graph G and a vertex $v \in S$ such that $d(v, S - \{v\}) = k \geq 3$. Let $w \in S$ such that $d(v, w) = d(v, S - \{v\})$ and let $P : v = u_0, u_1, \ldots, u_k = w$ be a $v - w$ geodesic in G. Thus, neither u_1 nor u_2 is in S; for otherwise, $d(v, S - \{v\}) \leq 2 < k$. Hence the edge $e = u_1u_2$ is dominated by some vertex $y \in S$ (that is necessarily adjacent to both u_1 and u_2). Hence $d(v, y) \leq 2 < k$, a contradiction. ■

We have already seen (in Corollary 3.12) that if S is a γ_{FC}-set in a connected graph G, then $\text{diam} S \leq 2\gamma_{FC}(G) - 2$. We now show that $\gamma_t(G)$ has a similar upper bound.

Theorem 3.15. *For every connected graph G,*

$$\gamma_t(G) \leq 2\gamma_{FC}(G) - 1.$$

Proof. Let S be a γ_{FC}-set. Since S is also a dominating set for G, every vertex in $V(G) - S$ is dominated by and therefore adjacent to some vertex in S. Consequently, S openly dominates all vertices in $V(G) - S$. By Lemma 3.14, for every vertex $u \in S$, there is a vertex v ($\neq u$) in S such that $d(u, v) \leq 2$. If every vertex in S is adjacent to some vertex in S, then S is also an open dominating set and so $\gamma_t(G) \leq \gamma_{FC}(G)$. On the other hand, suppose that there is a vertex $x \in S$ that is adjacent to no vertex in S. Then there is a vertex $y \in S$ such that $d(x, y) = 2$. Let w be a vertex of G adjacent to x and y. Hence $w \notin S$. For each vertex $u \in S - \{x, y\}$, let u' be a vertex of G that is adjacent to u. So u' may or may not be in S. Let $S' = \{u' \mid u \in S - \{x, y\}\}$. Then $S \cup S' \cup \{w\}$ is an open dominating set, $|S \cup S' \cup \{w\}| \leq 2\gamma_{FC}(G) - 1$, and so $\gamma_t(G) \leq 2\gamma_{FC}(G) - 1$. ■

To see that the upper bound in Theorem 3.15 is sharp, we show that for each integer $k \geq 5$, there exists a connected graph G_k such that $\gamma_{FC}(G_k) = k$

and $\gamma_t(G) = 2k - 1$. Let G_k be the graph obtained from a cycle C_k : $v_0, v_1, v_2, \ldots, v_{k-1}, v_0$ by (i) adding a vertex a_0 and the edges a_0v_0 and v_0v_i for $2 \leq i \leq k-2$ and (ii) adding the vertices a_i and b_i $(1 \leq i \leq k-1)$ and the edges a_ib_i and b_iv_i for $1 \leq i \leq k-1$. The graph G_5 is shown in Figure 4.

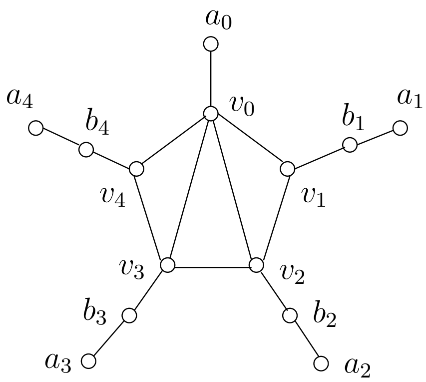

Figure 4. The graph G_5

4. Full Open Domination in Graphs

A vertex v in a graph G *openly dominates* the subgraph $\langle N(v) \rangle$ induced by the (open) neighborhood $N(v)$ of v, but v does not openly dominate itself or any edge incident with it. A set S of vertices in G is a *full open dominating set* if every vertex and every edge of G belongs to $\langle N(v) \rangle$ for some $v \in S$. The minimum cardinality of a full open dominating set is the *full open domination number* $\gamma_{FO}(G)$. A full open dominating set of cardinality $\gamma_{FO}(G)$ is referred to as a γ_{FO}*-set.* Note that a graph G has a full open dominating set if and only if G contains no isolated vertices and every edge of G lies on a triangle in G. Consequently, we have the following.

Observation 4.1. *Let S be a full open dominating set in a graph G. Every vertex of S (and consequently every edge joining two vertices of S) belongs to a triangle every vertex of which belongs to S.*

To illustrate these concepts, consider the graphs $G_1 = P_5 + K_1$ and $G_2 = K_{2,2,2}$ shown in Figure 5. In G_1, since each edge v_iv_{i+1} $(1 \leq i \leq 4)$ is openly dominated only by u, the edge uv_1 is openly dominated only by v_2, and the edge uv_5 is openly dominated only by v_4, it follows that u, v_2, v_4

belong to every γ_{FO}-set of G_1. However, the set $\{u, v_2, v_4\}$ is not a full open dominating set of G_1 as the edges uv_2 and uv_4 are not openly dominated by any vertex in $\{u, v_2, v_4\}$. Since $S_1 = \{u, v_2, v_3, v_4\}$ is a full open dominating set, S_1 is a γ_{FO}-set of G_1 and so $\gamma_{FO}(G_1) = 4$. In G_2, the set $S_2 = \{x_1, x_2, x_3\}$ is a full open dominating set. Moreover, there is no 2-element full open dominating set. Thus, S_2 is a γ_{FO}-set of G_2 and $\gamma_{FO}(G_2) = 3$.

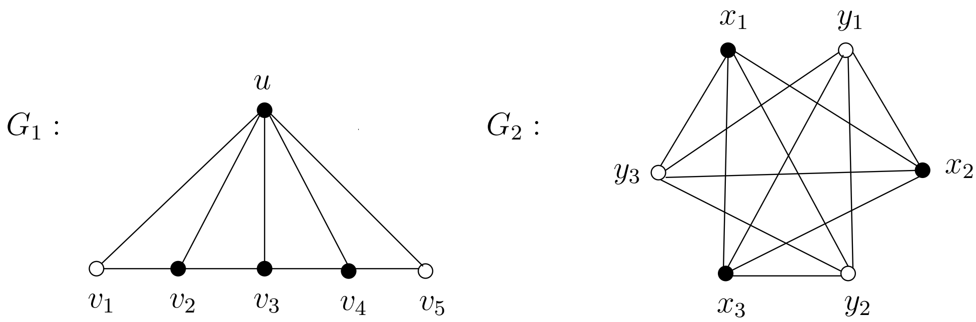

Figure 5. Graphs $G_1 = P_5 + K_1$ and $G_2 = K_{2,2,2}$

By Observation 4.1, every full open dominating set of a graph G must contain at least three vertices and so $\gamma_{FO}(G) \geq 3$. Certainly, every full open dominating set of a graph G is also a full closed dominating set and so $\gamma_{FO}(G) \geq \gamma_{FC}(G)$. This observation yields the following lower bound for $\gamma_{FO}(G)$.

Corollary 4.2. *For a graph G without isolated vertices and in which every edge belongs to a triangle,*

$$\gamma_{FO}(G) \geq \max\{3, \gamma_{FC}(G)\}.$$

Certainly, if G is a nontrivial connected graph of order n, then $1 \leq \gamma(G) \leq \gamma_{FC}(G) \leq \gamma_{FS}(G) \leq n-1$. Hence no nontrivial connected graph G of order n has $\gamma_{FC}(G) = n$ or $\gamma_{FS}(G) = n$. However, this is not true for $\gamma_{FO}(G)$, as we show next. For a graph G consisting of k $(k \geq 1)$ disjoint copies of a graph H, we write $G = kH$. In particular, $G = H$ for $k = 1$.

Theorem 4.3. *For $n \geq 3$, there exists a connected graph G of order n such that $\gamma_{FO}(G) = n$ if and only if $n \notin \{4, 6\}$.*

Proof. We first show that for $n = 4$ or $n = 6$, there is no connected graph G of order n with $\gamma_{FO}(G) = n$. If $n = 4$, then $K_4 - e$ and K_4 are the only graphs of order 4 in which every edge belongs to a triangle. However, $\gamma_{FO}(K_4-e) = \gamma_{FO}(K_4) = 3$. Next we show that there is no connected graph of order 6 with full open domination number 6. Assume, to the contrary, that there is a connected graph G of order 6 such that $\gamma_{FO}(G) = 6$. Let $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since every edge of G belongs to a triangle in G, there exist at least two triangles in G that have a common edge. Thus G contains $K_4 - e$ as a subgraph. Assume, without loss of generality, that $v_2v_4 \in E(G)$, $v_iv_{i+1} \in E(G)$ for $i = 1, 2, 3$, and $v_1v_4 \in E(G)$. Since $\{v_1, v_2, v_3, v_4\}$ openly dominates the induced subgraph $\langle\{v_1, v_2, v_3, v_4\}\rangle$, which is $K_4 - e$ or K_4, the vertex v_5 must openly dominate v_6 or some edge incident with v_6. If v_5 openly dominates v_6, then $v_5v_6 \in E(G)$. However, then, the edge v_5v_6 must be openly dominated by some vertex v_i for $i \in \{1, 2, 3, 4\}$. If v_5 openly dominates some edge e that is incident with v_6, then $e = v_iv_6$ for some $i \in \{1, 2, 3, 4\}$ and v_5 is adjacent to both v_i and v_6. In either case, G contains a triangle v_i, v_5, v_6, v_i for some $i \in \{1, 2, 3, 4\}$. Thus, we may assume that G contains at least one of the two graphs G_1 or G_2 of Figure 6 as a subgraph. Let $S = V(G) - \{v_1\}$. Since $\langle S\rangle$ contains $2K_2 + K_1$ as a subgraph, every edge of $\langle S\rangle$ belongs to a triangle. Thus S is a full open dominating set of $\langle S\rangle$. Moreover, the vertex v_1 and all edges incident with v_1 are openly dominated by S. This implies that S is also a full open dominating set of G. Therefore, $\gamma_{FO}(G) \le |S| = 5$, which is a contradiction.

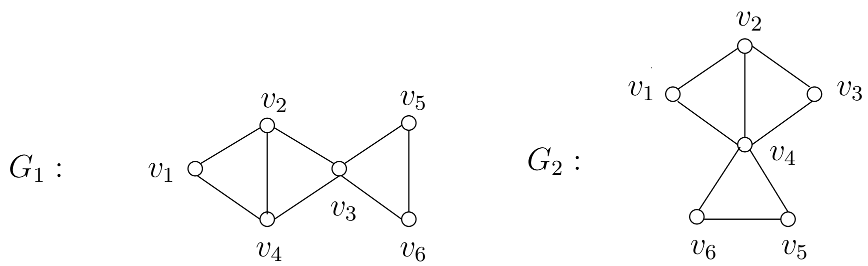

Figure 6. Subgraphs G_1 and G_2

For the converse, assume that $n \ge 3$ but $n \ne 4, 6$. We construct a graph G of order n with $\gamma_{FO}(G) = n$. If $n = 2k + 1$ for some integer $k \ge 1$, let $G = kK_2 + K_1$ for some positive integer k. Then the order of G is $2k + 1$. Since $V(G)$ is the only full open dominating set, $\gamma_{FO}(G) = 2k + 1$. Now let $n = 2k$ some integer $k \ge 4$. For $k = 4$, let F be the graph of Figure 7. Note that for every vertex w in G, there is an edge that is only openly dominated by w. Hence $V(G)$ is the only full open dominating set and so $\gamma_{FO}(G) = 8$.

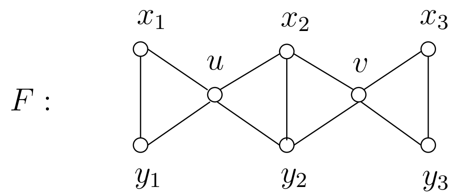

Figure 7. A graph F of order 8 with $\gamma_{FO}(F) = 8$

For $k \geq 5$, let G be obtained from the graph F of Figure 7 and the graph $(k-4)K_2$ by joining every vertex of $(k-4)K_2$ to the vertex v in F. Then the order of G is $2k$ and $\gamma_{FO}(F) = 2k$, as desired. ■

Since there is no graph G of order 4 and $\gamma_{FO}(G) = 4$ while the disconnected graph $G = 2K_3$ has order 6 and $\gamma_{FO}(G) = 6$, we have the following corollary.

Corollary 4.4. *For $n \geq 3$, there exists a graph G of order n such that $\gamma_{FO}(G) = n$ if and only if $n \neq 4$.*

We have seen that if G is a graph in which every edge belongs to a triangle and $\gamma_{FC}(G) = a$ and $\gamma_{FO}(G) = b$, then $1 \leq a \leq b$ and $b \geq 3$. Next we show that the converse of this fact is also true.

Theorem 4.5. *For each pair a, b of integers with $1 \leq a \leq b$ and $b \geq 3$, there exists a connected graph G in which every edge belongs to a triangle with $\gamma_{FC}(G) = a$ and $\gamma_{FO}(G) = b$.*

Proof. We consider three cases, according to whether $a = 1$, $a = 2$, or $a \geq 3$.

Case 1. $a = 1$. Suppose, first, that b is odd. Then $b = 2k+1$ for some integer $k \geq 1$. Let $G = kK_2 + K_1$, where $\deg_G u = 2k$. Since $\{u\}$ is a full closed dominating set, $\gamma_{FC}(G) = 1$. Moreover, $V(G)$ is the only full open dominating set, so $\gamma_{FO}(G) = |V(G)| = 2k+1 = b$. Next suppose that b is even. Then $b = 2k$ for some integer $k \geq 2$. Here we let $G = (P_5 \cup (k-2)K_2) + K_1$, where $P_5 : v_1, v_2, v_3, v_4, v_5$ and $\deg_G u = 2k+1$. Then $G = P_5 + K_1$ (shown in Figure 5) for $k = 2$. Again, $\{u\}$ is a full closed dominating set and so $\gamma_{FC}(G) = 1$. On the other hand, the set $\{u, v_2, v_3, v_4\} \cup V((k-2)K_2)$ is a γ_{FO}-set, implying that $\gamma_{FO}(G) = 2k$.

Case 2. $a = 2$. For $b = 3$, let $G = K_{2,2,2}$ (shown in Figure 5). Then $\gamma_{FC}(G) = 2$ and $\gamma_{FO}(G) = 3$. For $b = 2k+1$, where $k \geq 2$, let G be obtained from $K_{2,2,2}$ and $F = (k-1)K_2 + K_1$, where $\deg_F u = 2k-2$, by identifying some vertex in $K_{2,2,2}$ and the vertex u in F. Then $\gamma_{FC}(G) = 2$ and $\gamma_{FO}(G) = 3 + 2(k-1) = 2k+1 = b$.

Now suppose that b is even. Then $b = 2k$ for some integer $k \geq 2$. Let $G_1 = P_5 + \overline{K}_2$, where $P_5 : v_1, v_2, v_3, v_4, v_5$ and the remaining two vertices of G_1 are u and v; and for $k \geq 3$, let $G_2 = (k-2)K_2 + K_1$, where $\deg_{G_2} x = 2k-4$. For $k = 2$, let $G = G_1$ and for $k \geq 3$, let G be the graph obtained from G_1 and G_2 by identifying u and x. Then $\gamma_{FC}(G) = 2$, while $\gamma_{FO}(G) = 4 + 2(k-2) = 2k = b$.

Case 3. $a \geq 3$. We consider three subcases here, depending on whether $b = a$, $b = a+1$, or $b \geq a+2$.

Subcase 3.1. $b = a$. Let F_0 be a copy of the complete graph K_a with $V(F_0) = \{u_1, u_2, \ldots, u_a\}$. For each i with $1 \leq i \leq a$, let $F_i : x_i, y_i$ be a copy of K_2. Then let G be the graph obtained from the graphs F_i $(0 \leq i \leq a)$ by adding the $4a$ new edges $u_{i-1}x_i$, u_ix_i, u_iy_i, and $u_{i+1}y_i$ for all $1 \leq i \leq a$, where each subscript is one of the integers $1, 2, \ldots, a$ modulo a. The graph G is shown in Figure 8 for $a = 3, 4$. We show that $\gamma_{FO}(G) = \gamma_{FC}(G) = a$.

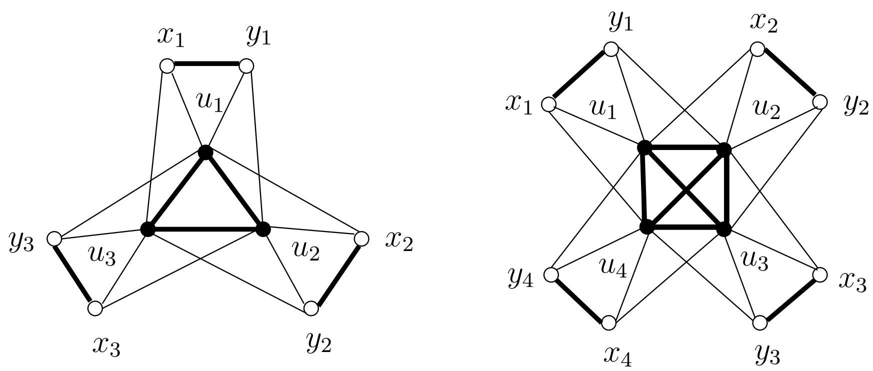

Figure 8. Graphs G with $\gamma_{FO}(G) = \gamma_{FC}(G) = a$ for $a = 3, 4$

Let $S = \{u_1, u_2, \ldots, u_a\}$. Since S is a full closed and full open dominating set of G, it follows that $\gamma_{FC}(G) \leq a$ and $\gamma_{FO}(G) \leq a$. On the other hand, each edge x_iy_i $(1 \leq i \leq k)$ in G is dominated only by x_i, y_i, or u_i. Hence every full closed dominating set of G must contain at least one vertex from each set $\{u_i, x_i, y_i\}$ for all $1 \leq i \leq a$. Thus $\gamma_{FC}(G) \geq a$. Since $\gamma_{FO}(G) \geq \gamma_{FC}(G)$, it follows that $\gamma_{FO}(G) \geq a$. Therefore, $\gamma_{FO}(G) = \gamma_{FC}(G) = a$.

Subcase 3.2. $b = a+1$. Let G be obtained from the graph G constructed in Subcase 3.1 by first subdividing the edge x_1y_1 into x_1z and zy_1 and then adding the edge zu_1. The graph G is shown in Figure 9 for $a = 3$. Since $S = \{u_1, u_2, \ldots, u_a\}$ is a γ_{FC}-set of G, it follows that $\gamma_{FC}(G) = a$. On the other hand, $S \cup \{x_1\}$ is a γ_{FO}-set and so $\gamma_{FO}(G) = a + 1$.

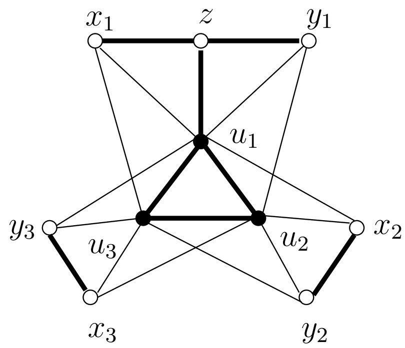

Figure 9. A graph G with $\gamma_{FO}(G) = 4$ and $\gamma_{FC}(G) = 3$

Subcase 3.3. $b \geq a + 2$. Suppose first that $b = a + 2k$, where $k \geq 1$. For each integer i with $1 \leq i \leq k$, let $H_i : v_i, w_i$ be a copy of K_2. Now let H be the graph obtained from the graph G in Subcase 3.1 and the graphs H_i $(1 \leq i \leq k)$ by adding the $2k$ edges u_1v_i, u_1w_i for all i with $1 \leq i \leq k$. For $a = 3$ and $k = 1$ (so $b = 5$), the graph H is shown in Figure 10(a).

Next, suppose that $b = a + 2k + 1$, where $k \geq 1$. Let H be the graph obtained from the graph G in Subcase 3.2 and the graphs H_i $(1 \leq i \leq k)$ by adding the $2k$ edges u_1v_i, u_1w_i for all $1 \leq i \leq k$. For $a = 3$ and $k = 1$ (so $b = 6$), the graph H is shown in Figure 10(b).

Since $S = \{u_1, u_2, \ldots, u_a\}$ is a γ_{FC}-set in H for all $b \geq a + 2$, it follows that $\gamma_{FC}(H) = a$. Next we show that $\gamma_{FO}(H) = b$. Let $V = \{v_1, v_2, \ldots, v_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$. For $b = a + 2k$, the set $S_1 = S \cup V \cup W$ is a γ_{FO}-set of H and so $\gamma_{FO}(H) = |S_1| = a + 2k = b$. For $b = a+2k+1$, the set $S_2 = S \cup \{x_1\} \cup V \cup W$ is γ_{FO}-set of H and so $\gamma_{FO}(H) = |S_2| = a+1+2k = b$. ■

Certainly, every full open dominating set of a graph G is also an open dominating set. Thus if G is a graph without isolated vertices in which every edge is in a triangle, then $\gamma_{FO}(G) \geq \gamma_t(G)$. Next we show that there is no graph G with $\gamma_{FO}(G) = \gamma_t(G)$.

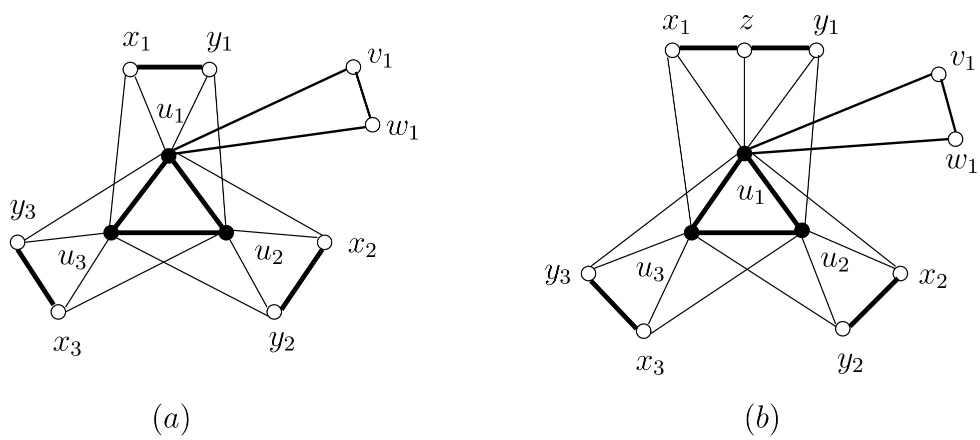

Figure 10. Graphs in Subcase 3.3 for $a = 3$ and $b = 5, 6$

Proposition 4.6. *If G is a graph without isolated vertices in which every edge is in a triangle, then $\gamma_{FO}(G) > \gamma_t(G)$.*

Proof. Assume, to the contrary, that there exists a graph G with $\gamma_t(G) = \gamma_{FO}(G)$. Let S be a γ_{FO}-set of G. Since every full open dominating set is also an open dominating set and $\gamma_t(G) = \gamma_{FO}(G)$, it follows that S is also a γ_t-set in G. Let $u \in S$. We consider two cases.

Case 1. *There exists a vertex x that is openly dominated by vertex $u \in S$ but not by any vertex in $S - \{u\}$.* This implies that x is adjacent to u, but x is not adjacent to any vertex in $S - \{u\}$. On the other hand, since S is a γ_{FO}-set of G, the edge ux is openly dominated by some vertex in $v \in S - \{u\}$. Hence ux belongs to $\langle N(v) \rangle$, implying that x is adjacent to $v \in S - \{u\}$, a contradiction.

Case 2. *Each vertex in G that is openly dominated by u is also openly dominated by some vertex in $S - \{u\}$.* Since S is a γ_t-set in G, there is a vertex v adjacent to u such that v is not adjacent to any other vertex in $S - \{u\}$. However, then, the edge uv is not openly dominated by any vertex in S, a contradiction. ■

Next we show that every pair a, b of integers with $2 \leq a < b$ is realizable as the open domination number and the full open domination number, respectively, of some graph.

Theorem 4.7. *For every pair a, b of integers with $2 \leq a < b$, there exists a graph G with $\gamma_t(G) = a$ and $\gamma_{FO}(G) = b$.*

Proof. We consider two cases.

Case 1. $b = a+1$ *or* $b = a+2$. Let G_a be the graph obtained from K_{a+1} with $V(K_{a+1}) = \{v_1, v_2, \ldots, v_{a+1}\}$ by adding, for each edge $e_{ij} = v_iv_j$, where $1 \leq i < j \leq a+1$, a new vertex w_{ij} and joining it to v_i and v_j. The graphs G_2 and G_3 are shown in Figure 11. Let $H_a = G_{a+1} - w_{12}$. The graph H_2 is shown in Figure 11. Since $\{v_1, v_2, \ldots, v_a\}$ is a γ_t-set of G_a and $\{v_1, v_2, \ldots, v_{a+1}\}$ is a γ_{FO}-set of G_a, it follows that $\gamma_t(G_a) = a$ and $\gamma_{FO}(G_a) = a+1$. Moreover, since $\{v_3, v_4, \ldots, v_{a+2}\}$ is a γ_t-set of H_a and $\{v_1, v_2, \ldots, v_{a+2}\}$ is a γ_{FO}-set of H_a, we have $\gamma_t(H_a) = a$ and $\gamma_{FO}(H_a) = a + 2$.

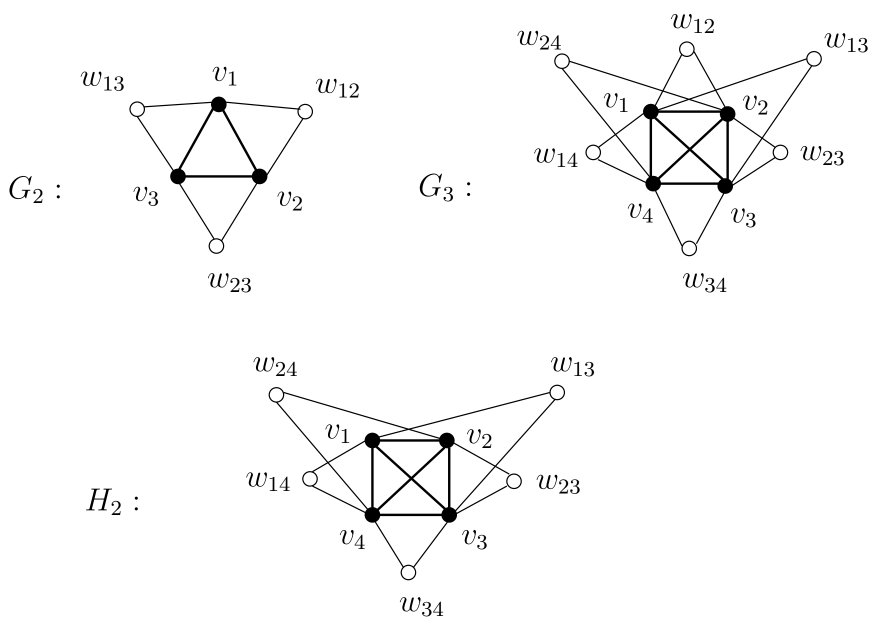

Figure 11. The graphs G_2, G_3, and H_2

Case 2. $b \geq a+3$. Suppose, first, that $b = a+2k+1$ ($k \geq 1$). Let G be the graph obtained from the graph G_a (from Case 1) and kK_2 by joining v_1 to each of the vertices of kK_2. If $b = a+2k$ ($k \geq 2$), then let G be the graph obtained from the graph H_a (of Case 1) and $(k-1)K_2$ by joining v_3 to each vertex of $(k-1)K_2$. It is routine to verify that $\gamma_t(G) = a$ and $\gamma_{FO}(G) = b$. ■

We conclude this paper with a problem.

Problem 4.8. *Determine all triples* a, b, c *of integers with* $\max\{a, b\} \leq c$, $a \geq 2$, $c \geq 3$, *and* $a < c$ *for which there exists a graph* G *with* $\gamma_t(G) = a$, $\gamma_{FC}(G) = b$, *and* $\gamma_{FO}(G) = c$.

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