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ON GRAPHS WITH A UNIQUE MINIMUM HULL SET

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Abstract

We show that for every integer $k \geq 2$ and every k graphs G_1, $G_2, \ldots, G_k$, there exists a hull graph with k hull vertices $v_1, v_2, \ldots, v_k$ such that link $L(v_i) = G_i$ for $1 \leq i \leq k$. Moreover, every pair a, b of integers with $2 \leq a \leq b$ is realizable as the hull number and geodetic number (or upper geodetic number) of a hull graph. We also show that every pair a, b of integers with $a \geq 2$ and $b \geq 0$ is realizable as the hull number and forcing geodetic number of a hull graph.

Keywords: geodetic set, geodetic number, convex hull, hull set, hull number, hull graph.

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1. Introduction

The best known metric space in graph theory is $(V(G), d)$, where $V(G)$ is the vertex set of a connected graph G and $d(u, v)$ is the distance between two vertices u and v in G (defined as the length of a shortest $u - v$ path). A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. The set (interval) $I(u, v)$ consists of all vertices lying on some $u - v$ geodesic of G, while for $S \subseteq V(G)$,

$$I(S) = \bigcup_{u,v \in S} I(u, v).$$

A set S of vertices in a connected graph G is *convex* if $I(S) = S$. The *convex hull* $[S]$ is the smallest convex set containing S. The convex hull $[S]$ of S

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can also be obtained from the sequence $\{I^k(S)\}$, $k \geq 0$, where $I^0(S) = S$, $I^1(S) = I(S)$, and $I^k(S) = I(I^{k-1}(S))$ for $k \geq 2$. From some term on, this sequence is constant. The convex hull $[S]$ is the set $I^p(S)$, where p is an integer such that $I^p(S) = I^{p+1}(S)$. A set S of vertices of G is called a *hull set* of G if $[S] = V(G)$, and a hull set of minimum cardinality is a *minimum hull set of* G. If a vertex v belongs to every hull set in a graph G, then v is called a *hull vertex.* Of course, every hull vertex belongs to every minimum hull set of G as well. The cardinality of a minimum hull set in G is its *hull number* $h(G)$. Clearly, $2 \leq h(G) \leq n$ for every connected graph G of order $n \geq 2$.

The intervals $I(u, v)$ were studied and characterized by Nebeský [12, 13]. These sets were also investigated extensively in the book by Mulder [10], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. Convexity in graphs is discussed in the book by Buckley and Harary [1] and studied by Harary and Nieminen [8]. The hull number of a graph was introduced by Everett and Seidman [7] and investigated further in [3], [6] and [11]. We refer to the book by Buckley and Harary [1] for concepts and results on distance in graphs.

As an illustration of these concepts, consider the graphs shown in Figure 1. All three graphs have hull number 2. In G_0, $\{x, y\}$ and $\{x', y'\}$ are (disjoint) minimum hull sets. For $S_1 = \{u, v\}$ in G_1, $I(S_1) = V(G_1) - \{w\}$ and $I(I(S_1)) = V(G_1)$. Therefore, $[S_1] = V(G_1)$ and so S_1 is a minimum hull set of G_1. On the other hand, S_1 is not the unique minimum hull set of G_1 since G_1 has two minimum hull sets, namely S_1 and $S_1' = \{u, w\}$. Consequently, G_1 does not have a unique minimum hull set. However, the set $\{s, t\}$ in the graph G_2 of Figure 1 is the unique minimum hull set of G_2. Therefore, for $0 \leq i \leq 2$, G_i contains exactly i hull vertices.

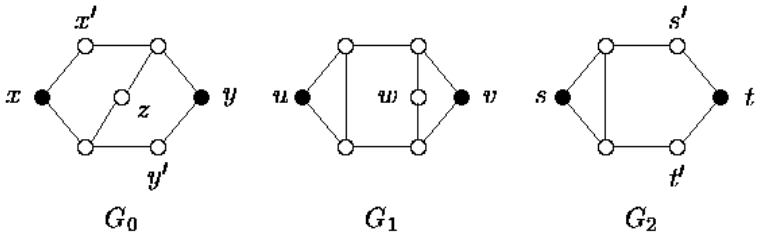

Figure 1. Three graphs with hull number 2

For a vertex v in a graph G, the *link* $L(v)$ of v is the subgraph induced by the neighbors of v. A vertex v in a graph G is called an *extreme vertex* if $L(v)$ is complete. For example, the vertex u in the graph G_1 and the vertex s in the graph G_2 of Figure 1 are extreme vertices. Certainly, if v is an extreme vertex of a graph G, then v is an end-vertex of every geodesic containing v. This observation gives the following result, which was mentioned in [7].

Theorem A. *Every hull set in a graph contains its extreme vertices. In particular, every hull set in a graph contains its end-vertices.*

By Theorem A, it follows that every extreme vertex is a hull vertex. The converse is not true, however, since the vertex t in the graph G_2 of Figure 1 is a hull vertex that is not extreme.

In [6] the forcing hull number of a graph was introduced. For a minimum hull set S of G, a subset T of S is called a *forcing subset* of S if S is the unique minimum hull set containing T. The *forcing hull number* $f(S, h)$ *of* S is the minimum cardinality of a forcing subset for S, while the *forcing hull number* $f(G, h)$ *of* G is the smallest forcing number among the minimum hull sets of G. For example, let $G = K_{2,3}$ with partite sets $\{x, y\}$ and $\{u, v, w\}$. Then $S_1 = \{x, y\}$ and $S_2 = \{u, v\}$ are minimum hull sets of G. The remaining minimum hull sets are similar to S_2. Since S_1 is the unique minimum hull set containing x, it follows that $f(S_1, h) = 1$. On the other hand, S_2 is not the unique minimum hull set containing any of its proper subsets, so $f(S_2, h) = 2$. Therefore, $f(G, h) = 1$.

If G is a graph with $f(G, h) = 0$, then G has a unique minimum hull set and conversely. Hence $f(G, h) = 0$ if and only if G contains exactly $h(G)$ hull vertices. We refer to any graph containing a unique minimum hull set as a *hull graph*. Therefore, the graph G_2 of Figure 1 is a hull graph, while G_0 and G_1 are not. It is the goal of this paper to study hull graphs with certain prescribed properties or prescribed geodetic parameters.

For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. We present a lemma whose routine proof is omitted.

Lemma 1.1. *Let S be a minimum hull set in a nontrivial connected graph G. Then*

(a) *no cut-vertex in G belongs to S, and*

(b) *for each cut-vertex v of G and every branch B of G at v, $V(B) \cap S \neq \emptyset$.*

The following corollaries are immediate consequences of Lemma 1.1.

Corollary 1.2. *Let S be a minimum hull set in a nontrivial connected graph G. If B is an end-block of G containing a cut-vertex v, then $v \notin S$ and $V(B) \cap S \neq \emptyset$.*

Corollary 1.3. *If G is a connected graph containing k end-blocks, then $h(G) \geq k$.*

2. Hull Graphs Whose Hull Vertices Have Prescribed Links

In the closing section of the paper by Everett and Seidman [7], they state that a hull graph need not contain any extreme vertices and give an example of a hull graph with hull number 3, each of whose hull vertices has degree 2 and whose two neighbors are not adjacent. In this section, we show in fact that a hull graph can have any prescribed hull number and its hull vertices can have any prescribed links.

Theorem 2.1. *For every integer $k \geq 2$ and every k graphs $G_1, G_2, \ldots, G_k$, there exists a connected hull graph with hull vertices $v_1, v_2, \ldots, v_k$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.*

Proof. We construct a graph G with the desired property. For each integer i $(1 \leq i \leq k)$, let $F_i = \overline{K}_2 + G_i$, where $V(\overline{K}_2) = \{u_i, v_i\}$. Then the graph G is constructed from the graphs F_i by adding a new vertex x and the k edges xu_i $(1 \leq i \leq k)$. Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$. Let $S = \{v_1, v_2, \ldots, v_k\}$. Since S is a hull set of G, it follows that $h(G) \leq k$ and by Corollary 1.3 $h(G) \geq k$. Therefore, $h(G) = k$. Hence S is a minimum hull set of G. Assume, to the contrary, that S' is a minimum hull set of G distinct from S. By Lemma 1.1, S' must contain exactly one vertex from each subgraph F_i $(1 \leq i \leq k)$. Since $S \neq S'$, we may assume that $v_1 \notin S'$. However, v_1 lies only those geodesics having v_1 as an end-vertex or having both end-vertices in $V(F_1)$. Thus, $v_1 \notin [S']$, which is impossible. Therefore, S is the unique minimum hull set of G, as desired. ■

The graph G constructed in the proof of Theorem 2.1 has a cut-vertex and so is not 2-connected. However, we can extend Theorem 2.1 by modifying the structure of the graph G in the proof of Theorem 2.1 to construct a 2-connected hull graph with the properties described in Theorem 2.1.

Corollary 2.2. *For every integer $k \geq 2$ and every k graphs $G_1, G_2, \ldots, G_k$, there exists a 2-connected hull graph with hull vertices $v_1, v_2, \ldots, v_k$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.*

Proof. For each integer i ($1 \leq i \leq k$), let $F_i = \overline{K}_3 + G_i$, where $V(\overline{K}_3) = \{u_i, v_i, w_i\}$. Then a 2-connected graph G is constructed from the graphs F_i by adding $2k$ edges $u_i w_i$ and $w_i u_{i+1}$ for $1 \leq i \leq k$, where the subscripts are expressed modulo k. Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$. For $k = 3$, the graph G is shown in Figure 2.

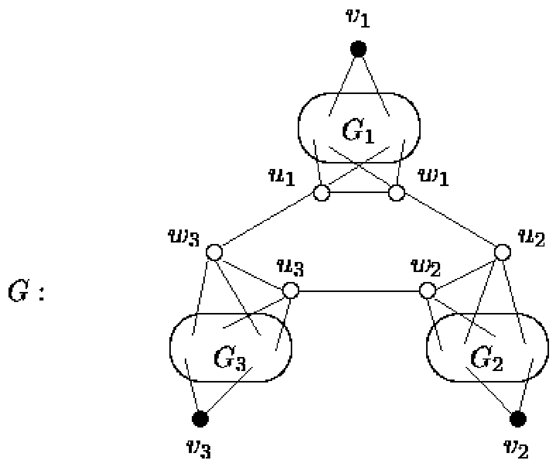

Figure 2. A 2-connected hull graph G with three hull vertices

A proof similar to that of Theorem 2.1 shows that G has the desired properties. ■

3. Hull Graphs with Prescribed Geodetic Number

In [1] a set S of vertices in a connected graph G is called a *geodetic set* if $I(S) = V(G)$. A geodetic set of minimum cardinality is a *minimum geodetic set*, and this cardinality is the *geodetic number* $g(G)$. Certainly, $2 \leq h(G) \leq g(G) \leq n$ for every connected graph G of order $n \geq 2$. For example, the set $S = \{x, y, z\}$ in the graph G_0 of Figure 1 is a geodetic set. Since no 2-element subset of $V(G_0)$ is a geodetic set of G_0, it follows that S is a minimum geodetic set of G_0 and so $g(G_0) = 3$. The geodetic number of a graph is discussed in the book by Buckley and Harary [1] and further studied in [2]; while the geodetic number of an oriented graph has been studied in [4]. It was shown in [9] that the determination of $g(G)$ is an NP-hard problem and its decision problem is NP-complete.

For every pair a, b of integers with $2 \leq a \leq b$, it was shown in [3] that there exists a connected graph G with $h(G) = a$ and $g(G) = b$. In this section, we extend this result by verifying that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the hull number and geodetic number, respectively, of some connected *hull* graph G that contains a unique minimum geodetic set as well. In the following two sections, we extend this result in a different manner by showing that under certain appropriate conditions, every two given integers are realizable as the hull number and a certain type of geodetic numbers of some hull graph.

The following result, which appeared in [2], is an analogue of Theorem A.

Theorem B. *Every geodetic set of a graph G contains its extreme vertices. In particular, every geodetic set of a graph G contains the end-vertices of G.*

Similarly, if the set S of extreme vertices of a graph G is a geodetic set of G, then S is the unique minimum geodetic set of G, but the converse is not true as the graph G_2 of Figure 1 shows. In fact, since the hull graph G constructed in the proof of Theorem 2.1 has the additional property that the unique minimum hull set S of G is also the unique minimum geodetic set of G, we have the following result.

Corollary 3.1. *For every integer $k \geq 2$ and every k graphs $G_1, G_2, \ldots, G_k$, there exists a connected hull graph G with unique minimum geodetic set $S = \{v_1, v_2, \ldots, v_k\}$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.*

It was shown in [2, 3] that for each integer $n \geq 2$, $h(K_n) = g(K_n) = n$, where $V(K_n)$ is the unique minimum hull set and unique minimum geodetic set of K_n. Moreover, for each integer $n \geq 2$ and every integer k with $2 \leq k \leq n-1$, a tree T of order n with exactly k end-vertices has $h(T) = g(T) = k$, where the set of end-vertices of T is the unique minimum hull set and unique geodetic set. Therefore, we have the following result.

Theorem 3.2. *For every pair k, n of integers with $2 \leq k \leq n$, there exists a connected hull graph G of order n with $h(G) = g(G) = k$ such that G contains a unique minimum geodetic set.*

We are now prepared to present the main result of this section.

Theorem 3.3. *For each pair a, b of integers with $2 \leq a < b$ there exists a connected hull graph G with a unique minimum geodetic set such that $h(G) = a$ and $g(G) = b$.*

Proof. For each integer i with $1 \leq i \leq b-a$, let $H_i : x_i, y_i, z_i, w_i, s_i, t_i$ be a copy of C_6. Then the graph F_i $(1 \leq i \leq b-a)$ is obtained from H_i by adding a new vertex v_i and the two edges $t_i v_i$ and $v_i z_i$. Then a graph G is formed from graphs F_i $(1 \leq i \leq b-a)$ by adding a new vertices u_j $(1 \leq j \leq a)$ and the edges (1) $u_1 x_1$ and $u_j w_{b-a}$ $(2 \leq j \leq a)$ and (2) $x_i x_{i+1}, w_i w_{i+1}$ for $(1 \leq i \leq b-a-1)$. This completes the construction of G.

We now show that the graph G has the desired properties. Let $S = \{u_1, u_2, \ldots, u_a\}$ be the set of end-vertices of G. Then $I(S) = V(G) - \{v_1, v_2, \ldots, v_{b-a}\}$. Since $I^2(S) = [S] = V(G)$, it follows that S is the unique minimum hull set of G. So G is a hull graph with $h(G) = a$. Next we show that G contains a unique minimum geodetic set of cardinality b.

First we show that $g(G) = b$. Let $S_1 = S \cup \{v_1, v_2, \ldots, v_{b-a}\}$. It is routine to verify that S_1 is a geodetic set and so $g(G) \leq b$. Let W be a geodetic set of G. Certainly, $S \subseteq W$. Let $V_i = \{v_i, y_i, z_i, w_i, s_i, t_i\}$, where $1 \leq i \leq b-a$. Since v_i does not lie on any $x-y$ geodesic in G for $x, y \notin V_i$, it follows that W contains at least one vertex from each set V_i $(1 \leq i \leq b-a)$ and so $|W| \geq a + (b-a) = b$. Therefore, $g(G) = b$.

Next we show that S_1 is the unique minimum geodetic set of G. Let S_2 be a minimum geodetic set of G. By the discussion above, $S \subseteq S_2$ and S_2 contains exactly one vertex from each set V_i $(1 \leq i \leq b-a)$. Because v_i lies only on those geodesics having v_i as one of its end-vertices or having both end-vertices belonging to V_i, it follows that $v_i \in S_2$ for $1 \leq i \leq b-a$. Thus $S_2 = S_1$, and S_1 is the unique minimum geodetic set of G. ∎

In every example we have seen thus far, if a hull graph G also contains a unique minimum geodetic set, then its minimum hull set is a subset of the minimum geodetic set of G. In fact, it may seem that this is true in general. However, this is not the case. Indeed, there are hull graphs G possessing a unique minimum geodetic set that does not contain its unique minimum hull set as a subset. For example, in the hull graph G of Figure 3, the set $S = \{u, v\}$ is the unique minimum hull set of G and the set $S' = \{u, x, y\}$ is the unique minimum geodetic set of G. Of course, $S \not\subseteq S'$.

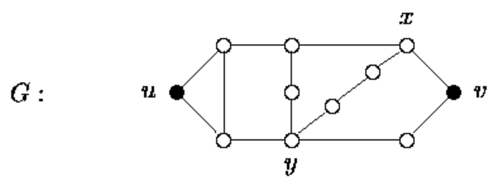

Figure 3. A hull graph

4. Hull Graphs with Prescribed Upper Geodetic Number

A geodetic set S in a connected graph G is *minimal* if no proper subset of S is a geodetic set. Of course, every minimum geodetic set is a minimal geodetic set, but the converse is not true. For example, let $G = K_{2,3}$ with partite sets $V_1 = \{x, y\}$ and $V_2 = \{u, v, w\}$. Then $\{u, v, w\}$ is a minimal geodetic set of $K_{2,3}$ but is not a minimum geodetic set of $K_{2,3}$ since $\{x, y\}$ is its unique minimum geodetic set. The *upper geodetic number* $g^+(G)$ is the maximum cardinality of a minimal geodetic set of G. So $g(K_{2,3}) = 2$ and $g^+(K_{2,3}) = 3$. Obviously, $h(G) \le g(G) \le g^+(G)$. Next we show that every pair a, b of integers with $2 \le a \le b$ is realizable as the hull number and upper geodetic number of some connected hull graph. First we state a lemma, which is an analogue of Corollary 1.2.

Lemma 4.1. *Let S be a minimal geodetic set in a nontrivial connected graph G. If B is an end-block of G containing a cut vertex v, then $v \notin S$ and $V(B) \cap S \ne \emptyset$.*

Theorem 4.2. *For each pair a, b of integers with $2 \le a \le b$ there exists a connected hull graph G such that $h(G) = a$ and $g^+(G) = b$.*

Proof. For $a = b$, any tree with exactly a end-vertices has the desired properties. So we assume that $a < b$. We consider two cases.

Case 1. $b = a + 1$. We construct a hull graph G with $h(G) = a$ and $g^+(G) = a + 1$. First assume that $a = 2$ and $b = 3$. For the hull graph G_2 of Figure 1, we have seen that $h(G_2) = 2$. Since $\{s, s', t'\}$ is a minimal geodetic set of maximum cardinality of G_2, it follows that $g^+(G_2) = 3$. So now let $a \ge 3$. Let $H = K_{2,3}$ with partite sets $\{u, v, w\}$ and $\{x, y\}$. Then the graph G is obtained from H by adding a new vertices u_i $(1 \leq i \leq a)$ and the a edges uu_1, xu_2, and wu_i for $3 \leq i \leq a$. Let $U = \{u_1, u_2, \ldots, u_a\}$ be the set of end-vertices of G. Since U is a minimum hull set of G, it follows that G is a hull graph and $h(G) = a$. On the other hand, by Lemma 4.1, G has only two distinct minimal geodetic sets, namely $S_1 = U \cup \{v\}$ and $S_2 = U \cup \{y\}$. (In fact, S_1 and S_2 are also minimum geodetic sets of G.) Therefore, $g^+(G) = a + 1$.

Case 2. $b \geq a + 2$. Let $H = K_{2,b-a+2}$ with partite sets $\{u, v\}$ and $\{v_1, v_2, \ldots, v_{b-a+2}\}$. Then the graph G is obtained from H by adding a new vertices u_i $(1 \leq i \leq a)$ and the a edges u_1v_1 and u_iv_{b-a+2} $(2 \leq i \leq a)$. Since $U = \{u_1, u_2, \ldots, u_a\}$ is a minimum hull set, G is a hull graph with $h(G) = a$. It remains to show that $g^+(G) = b$.

Let $S = U \cup \{v_2, v_3, \ldots, v_{b-a+1}\}$. Since $I(S) = V(G)$, it follows that S is a geodetic set. Next we show that S is a minimal geodetic set of G. Let $x \in S$. We show that $I(S - \{x\}) \neq V(G)$. Since every geodetic set contains the end-vertices of G, we may assume that $x \notin U$. So $x = v_i$ for some i with $2 \leq i \leq b - a + 1$. Since $I(S - \{v_i\}) = V(G) - \{v_i\}$, it follows that S is a minimal geodetic set with $|S| = b$. Therefore, $g^+(G) \geq b$. On the other hand, let W be a minimal geodetic set with $|W| = g^+(G) \geq b$. Necessarily, $U \subseteq W$. By Lemma 4.1, $v_1, v_{b-a+2} \notin W$. If $u, v \notin W$, then $W \subseteq V(G) - \{v_1, v_{b-a+2}, u, v\}$ and so $|W| \leq b$, implying that $g^+(G) \leq b$. So we may assume that at least one of u and v belongs to W. Certainly, $u, v \in I(u_1, u_2)$. Assume first that exactly one of u and v belongs to W, say u. Then $W - \{u\}$ is not a geodetic set and so there exists $x \notin I(W - \{u\})$. Consequently, $x \notin W$ and x lies on some $u - w$ geodesic, where $w \in W$. However, each such $u - w$ geodesic is a $u - u_i$ geodesic for $1 \leq i \leq a$. This implies that $x = v_1$ or $x = v_{b-a+2}$. But $v_1, v_{b-a+2} \in I(u_1, u_2) \subseteq I(W - \{u\})$, producing a contradiction. Hence both u and v belong to W. Then $\{v_2, v_3, \ldots, v_{b-a+1}\} \subseteq I(u, v)$ and $I(W - \{v_2, v_3, \ldots, v_{b-a+1}\}) = I(W)$. This implies that $v_2, v_3, \ldots, v_{b-a+1} \notin W$. Therefore, $g^+(G) = |W| = 2 + a \leq b$. ■

5. Hull Graphs with Prescribed Forcing Geodetic Number

As we have seen, the graphs with a unique minimum hull set are precisely those having forcing hull number 0. We now define a related concept. For a minimum geodetic set S of a nontrivial connected graph G, a subset T

of S is called a *forcing subset* of S if S is the unique minimum geodetic set containing T. The *forcing geodetic number* $f(S,g)$ of S is the minimum cardinality of a forcing subset for S, while the *forcing geodetic number* $f(G,g)$ of G is the smallest forcing number among all minimum geodetic sets of G. These concepts were introduced and studied in [5]. Hence if G is a graph with $f(G,g)=a$ and $g(G)=b$, then $0 \le a \le b$ and there exists a minimum geodetic set S of cardinality b containing a forcing subset T of cardinality a but no forcing subset of smaller cardinality.

For a graph G, the forcing geodetic number $f(G,g)=0$ if and only if G has a unique minimum geodetic set. Moreover, $f(G,g)=1$ if and only if G does not have a unique minimum geodetic set but some vertex of G belongs to exactly one minimum geodetic set. Next we show that every pair a,b of integers with $a \ge 2$ and $b \ge 0$ is realizable as the hull number and forcing geodetic number of some connected hull graph. This verifies a conjecture stated in [6].

Theorem 5.1. *For each pair a,b of integers with $a \ge 2$ and $b \ge 0$ there exists a connected hull graph G such that $h(G)=a$ and $f(G,g)=b$.*

Proof. For $a \ge 2$ and $b=0$, any tree T with exactly a end-vertices is a hull graph with $h(T)=a$ and $f(T,g)=0$. So we assume that $b \ge 1$. For each pair a,b of integers with $a \ge 2$ and $b \ge 1$, we construct a connected hull graph $G_{a,b}$ with $h(G_{a,b})=a$ and $f(G_{a,b},g)=b$.

First we assume that $a \ge 2$ and $b=1$. To define the graph $G_{a,1}$, we begin with four paths $P(i)$, $0 \le i \le 3$. Let $P(0): u_1,u_2,\dots,u_5$, let $P(i): v_{i1},v_{i2},\dots v_{i,2i+3}$ for $i=1,2$, and let $P(3): w_1,w_2,\dots,w_9$. Then the graph $G_{a,1}$ is obtained from the graphs $P(i)$, $0 \le i \le 3$, by adding a new vertices $x_1,x_2,\dots,x_a$ and the edges (1) x_1u_3 and x_iw_5 for $2 \le i \le a$, (2) $u_1v_{11}, v_{11}v_{21}, v_{21}w_1$, and (3) $u_5v_{15}, v_{15}v_{27}, v_{27}w_9$. Since the set $X=\{x_1,x_2,\dots,x_a\}$ of end-vertices of $G_{a,1}$ is a minimum hull set, it follows that G is a hull graph with $h(G_{a,1})=a$. It remains to show that $f(G_{a,1},g)=1$.

We first show that $g(G_{a,1})=a+2$. Let $S=X\cup\{v_{13},v_{24}\}$. Since $I(S)=V(G_{a,1})$, it follows that $g(G_{a,1}) \le a+2$. On the other hand, for every $v \in V(G_{a,1})-X$, the set $X\cup\{v\}$ is not a geodetic set of $G_{a,1}$ and so $g(G_{a,1}) \ge a+2$. Therefore, $g(G)=a+2$. We now show that $f(G_{a,1},g)=1$. Since $S'=X\cup\{v_{21},v_{27}\}$ is a geodetic set of $G_{a,1}$ distinct from S, it follows that $f(G_{a,1},g) \ge 1$. On the other hand, S is the unique minimum geodetic set containing v_{13} and so $f(S,g)=1$. Therefore, $f(G_{a,1},g)=1$.

For the remainder of the proof we assume that $a, b \geq 2$. The structure of $G_{a,1}$ can be modified to produce a graph $G_{a,b}$ with $h(G_{a,b}) = a$ and $f(G_{a,b}, g) = b$. Let $P(0) : u_1, u_2, \ldots, u_5$ and $P(b+2) : w_1, w_2, \ldots, w_{2b+7}$ be paths, and for $1 \leq i \leq b+1$, let $P(i) : v_{i1}, v_{i2}, \ldots v_{i,2i+3}$ be paths. Then the graph $G_{a,b}$ is obtained from the $b+3$ paths $P(i)$, $0 \leq i \leq b+2$, by adding a new vertices $x_1, x_2, \ldots, x_a$ and the edges (1) x_1u_3 and $w_{b+4}x_i$ for $2 \leq i \leq a$, (2) u_1v_{11}, $v_{b+1,1}w_1$, and $v_{i1}v_{i+1,1}$ for $1 \leq i \leq b$, and (3) u_5v_{15}, $v_{b+1,2b+5}w_{2b+7}$, and $v_{i,2i+3}v_{i+1,2i+5}$ for $1 \leq i \leq b$. Thus for each $b \geq 2$, the graph $G_{a,b}$ is a hull graph with unique minimum hull set $X = \{x_1, x_2, \ldots, x_a\}$. Thus $h(G_{a,b}) = a$. We show only that $f(G_{a,2}, g) = 2$ as the proofs that $f(G_{a,b}, g) = b$ for $b \geq 3$ are similar and are therefore omitted.

Since $S = X \cup \{v_{13}, v_{24}, v_{35}\}$ is a minimum geodetic set of G_2, it follows that $g(G_{a,2}) = a + 3$. We now show that $f(G_{a,2}, g) = 2$. Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i,2i+3}\}$ for $1 \leq i \leq 3$. If S' is a minimum geodetic set of G, then S' has one of the following three forms: (1) $S' = X \cup \{v_1, v_2, v_3\}$, (2) $S' = X \cup \{v_{21}, v_{27}, v_3\}$, and (3) $S' = X \cup \{v_{31}, v_{39}, v_1\}$, where $I(\{v_1, v_2, v_3\}) = I(\{v_{21}, v_{27}, v_3\}) = I(\{v_{31}, v_{39}, v_1\}) = V_1 \cup V_2 \cup V_3$ and $v_i \in V_i$ for $1 \leq i \leq 3$. Thus for each $y \in S'$, there exists no unique minimum geodetic set containing y; while the 2-element set $\{v_{13}, v_{24}\}$ is only a subset of one minimum geodetic set, namely S. Therefore, $f(G_{a,2}, g) = 2$. Similarly, $f(G_{a,b}, g) = b$ for all $b \geq 3$. ∎

In [6] it was shown that for infinitely many nonnegative integers a, there exist infinitely many integers b with $b \geq a$ such that there exists a connected graph G with $f(G, h) = a$ and $f(G, g) = b$. Also, for infinitely many nonnegative integers c, there exist infinitely many integers d with $d \geq c$ such that there exists a connected graph H with $f(H, g) = c$ and $f(H, h) = d$. Whether such graphs exist with prescribed hull numbers as well is an open question.

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