

## THE DECOMPOSABILITY OF ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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### Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. If  $\mathcal{P}_1, \dots, \mathcal{P}_n$  are properties of graphs, then a  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -decomposition of a graph  $G$  is a partition  $E_1, \dots, E_n$  of  $E(G)$  such that  $G[E_i]$ , the subgraph of  $G$  induced by  $E_i$ , is in  $\mathcal{P}_i$ , for  $i = 1, \dots, n$ . We define  $\mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_n$  as the property  $\{G \in \mathcal{I} : G \text{ has a } (\mathcal{P}_1, \dots, \mathcal{P}_n)\text{-decomposition}\}$ . A property  $\mathcal{P}$  is said to be decomposable if there exist non-trivial hereditary properties  $\mathcal{P}_1$  and  $\mathcal{P}_2$  such that  $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ . We study the decomposability of the well-known properties of graphs  $\mathcal{I}_k, \mathcal{O}_k, \mathcal{W}_k, \mathcal{T}_k, \mathcal{S}_k, \mathcal{D}_k$  and  $\mathcal{O}^p$ .

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## 1 Introduction

Following [2] we denote the class of all finite simple graphs by  $\mathcal{I}$ . A *property* of graphs is a non-empty isomorphism-closed subclass of  $\mathcal{I}$ . The fact that  $H$  is a subgraph of  $G$  is denoted by  $H \subseteq G$  and the disjoint union of two graphs  $G$  and  $H$  is denoted by  $G \cup H$ . A property  $\mathcal{P}$  is called *hereditary* if  $G \in \mathcal{P}$  and  $H \subseteq G$  implies  $H \in \mathcal{P}$ ;  $\mathcal{P}$  is called *additive* if  $G \cup H \in \mathcal{P}$  whenever  $G \in \mathcal{P}$  and  $H \in \mathcal{P}$ . Throughout this paper the term property (of graphs) is used to refer to an additive hereditary property (of graphs).

**Example.** Some well-known properties are given in the list below.

$$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\},$$

$$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},$$

$$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices}\},$$

$$\mathcal{W}_k = \{G \in \mathcal{I} : \text{the length of any path in } G \text{ is at most } k\},$$

$$\mathcal{T}_k = \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},$$

$$\mathcal{S}_k = \{G \in \mathcal{I} : \text{the maximum degree of } G \text{ is at most } k\},$$

$$\mathcal{D}_k = \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e. every subgraph of } G \text{ has a vertex of degree at most } k\}.$$

We remark that  $\mathcal{D}_1 = \mathcal{T}_1$  is the class of all forests so that

$$\mathcal{SF} = \mathcal{D}_1 \cap \mathcal{W}_2 = \{G \in \mathcal{I} : \text{every component of } G \text{ is a star}\}.$$

The properties  $\mathcal{I}$  and  $\mathcal{O}$  are defined to be the *trivial properties* and an edgeless graph is called a *trivial graph*. We use the phrase *G has property P* to denote the fact that  $G \in \mathcal{P}$ .

## 2 Decomposability

Let  $\mathcal{P}_1, \dots, \mathcal{P}_n$  be properties of graphs. A  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -*decomposition* of a graph  $G$  is a partition  $E_1, \dots, E_n$  of  $E(G)$  such that  $G[E_i]$ , the subgraph of  $G$  induced by  $E_i$ , has property  $\mathcal{P}_i$ , for  $i = 1, \dots, n$ . We denote by  $\mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_n$  the property  $\{G \in \mathcal{I} : G \text{ has a } (\mathcal{P}_1, \dots, \mathcal{P}_n)\text{-decomposition}\}$ . If  $G \in \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_n$  we also write  $G = G_1 \oplus \dots \oplus G_n$  where  $G_i = G[E_i]$  for  $i = 1, \dots, n$ . If  $\mathcal{P}_1 = \dots = \mathcal{P}_n = \mathcal{P}$ , then the property  $\mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_n$  is also denoted by  $n\mathcal{P}$ . Note that  $\mathcal{O}$  is the identity element for  $\oplus$  in the sense that  $\mathcal{P} \oplus \mathcal{O} = \mathcal{P}$  for every property  $\mathcal{P}$ . It is easy to see that if  $\mathcal{P}_i$  is additive and hereditary for every  $i$  then  $\mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_n$  is also additive and hereditary.

If  $\mathbb{L}$  is a set of hereditary properties and  $\mathcal{P} \in \mathbb{L}$  then  $\mathcal{P}$  is said to be *decomposable in  $\mathbb{L}$*  if there exist non-trivial hereditary properties  $\mathcal{P}_1$  and  $\mathcal{P}_2$  in  $\mathbb{L}$  such that  $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ ; otherwise  $\mathcal{P}$  is said to be *indecomposable in  $\mathbb{L}$* . Throughout this paper we use for  $\mathbb{L}$  the lattice  $\mathbb{L}_{\subseteq}^a$  of all additive hereditary properties of graphs — see [2] for more details on this lattice.

The property  $\mathcal{P} \circ \mathcal{Q}$  is the vertex-analogue of  $\mathcal{P} \oplus \mathcal{Q}$ . For the sake of completeness we give the necessary definitions: For given properties  $\mathcal{P}_1, \dots, \mathcal{P}_n$ , a *vertex  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition* of a graph  $G$  is a partition  $V_1, \dots, V_n$  of  $V(G)$  such that for each  $i = 1, \dots, n$  the induced subgraph  $G[V_i]$  has property  $\mathcal{P}_i$ .

(In this context it is convenient to regard the empty set  $\emptyset$  as a set inducing a subgraph with every property  $\mathcal{P}$ .) The *product*  $\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$  of the properties  $\mathcal{P}_1, \dots, \mathcal{P}_n$  is now defined as the set of all graphs having a vertex  $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition; each  $\mathcal{P}_i$  is called a *factor* of this product. If  $\mathcal{P}_1 = \dots = \mathcal{P}_n = \mathcal{P}$ , then we write  $\mathcal{P}^n = \mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$ . As an example we note that  $\mathcal{O}^k$  denotes the class of all  $k$ -colourable graphs.

These operations come together in the following distributive law which is proved in [3].

**Lemma 21.** *Let  $\mathcal{P}$ ,  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  be additive and hereditary properties of graphs. Then  $\mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2) = (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2)$ . ■*

A property  $\mathcal{R}$  is *reducible* if there are properties  $\mathcal{P}$  and  $\mathcal{Q}$  such that  $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ ; otherwise it is *irreducible*. Reducible properties play an important role in the lattice  $\mathbb{L}_{\subseteq}^a$  because of following Unique Factorization Theorem of Mihók, Semanišin and Vaský which appears in [5].

**Theorem 22.** *Every reducible property  $\mathcal{P} \neq \mathcal{I}$  is uniquely factorizable into irreducible factors (up to the order of the factors). ■*

In the light of this result it seems imperative to study the decomposability of properties of graphs. It is our aim to study the decomposability of the well-known properties of graphs  $\mathcal{I}_k$ ,  $\mathcal{O}_k$ ,  $\mathcal{W}_k$ ,  $\mathcal{T}_k$ ,  $\mathcal{S}_k$ ,  $\mathcal{D}_k$  and  $\mathcal{O}^p$ .

### 3 The Indecomposability of $\mathcal{I}_k$ , $\mathcal{O}_k$ , $\mathcal{W}_k$ , and $\mathcal{T}_k$

The indecomposability of  $\mathcal{I}_k$  is easy to show using the following well-known result of Nešetřil and Rödl — see [6]. In it we use the notation  $\omega(G)$  to denote the clique number of a graph  $G$ .

**Theorem 31.** *For every graph  $G$  there is a graph  $H$  with  $\omega(G) = \omega(H)$  such that for every partition  $E_1, E_2$  of  $E(H)$  there is an induced subgraph  $G'$  of  $H$  with  $G'$  isomorphic to  $G$  and  $E(G') \subseteq E_1$  or  $E(G') \subseteq E_2$ . ■*

**Theorem 32.** *For every positive integer  $k$  the property  $\mathcal{I}_k$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$ .*

**Proof.** Suppose to the contrary that  $\mathcal{I}_k = \mathcal{P} \oplus \mathcal{Q}$  with both  $\mathcal{P}$  and  $\mathcal{Q}$  non-trivial. Applying Theorem 31 to  $G = K_{k+2} - e \in \mathcal{I}_k$  it follows that there is a graph  $H \in \mathcal{I}_k$  so that  $H$  has a  $(\mathcal{P}, \mathcal{Q})$ -decomposition and by considering

any  $(\mathcal{P}, \mathcal{Q})$ -decomposition of  $H$  we see that  $K_{k+2} - e \in \mathcal{P}$  or  $K_{k+2} - e \in \mathcal{Q}$ . But then  $K_{k+2} \in \mathcal{P} \oplus \mathcal{Q} = \mathcal{I}_k$ , a contradiction. ■

The following lemma will prove to be useful to show the indecomposability of some properties.

**Lemma 33.** *If  $\mathcal{P}$  is decomposable in  $\mathbb{L}_{\subseteq}^a$  then  $2\mathcal{S}_1 \subseteq \mathcal{P}$ .*

**Proof.** If  $\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}$  with  $\mathcal{Q}$  and  $\mathcal{R}$  non-trivial then  $\mathcal{S}_1 \subseteq \mathcal{Q}$  and  $\mathcal{S}_1 \subseteq \mathcal{R}$  since  $\mathcal{S}_1$  is the smallest non-trivial property in  $\mathbb{L}_{\subseteq}^a$ . It follows that  $2\mathcal{S}_1 \subseteq \mathcal{Q} \oplus \mathcal{R}$ . ■

**Corollary 34.** *For every positive integer  $k$  the properties  $\mathcal{O}_k$  and  $\mathcal{W}_k$  are indecomposable in  $\mathbb{L}_{\subseteq}^a$ .*

**Proof.** The path  $P_{k+2}$  of order  $k+2$  is in  $2\mathcal{S}_1$  but not in  $\mathcal{O}_k$ , hence  $\mathcal{O}_k$  is indecomposable by Lemma 33. Since  $P_{k+2}$  is not in  $\mathcal{W}_k$  either,  $\mathcal{W}_k$  is also indecomposable. ■

**Theorem 35.** *If  $\mathcal{SF} \subseteq \mathcal{P}$  and  $\overline{\mathcal{P}} \neq \emptyset$  is closed under taking subdivisions of graphs, then  $\mathcal{P}$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$ .*

**Proof.** Suppose that  $\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}$  with  $\mathcal{Q}$  and  $\mathcal{R}$  non-trivial. Let  $G$  be any graph in  $\overline{\mathcal{P}}$  and let  $G'$  be the graph obtained from  $G$  by subdividing every edge of  $G$  twice. Clearly,  $G' \in \mathcal{SF} \oplus \mathcal{S}_1$ . Since  $G' \notin \mathcal{P}$  it follows that  $\mathcal{SF} \not\subseteq \mathcal{Q}$  and  $\mathcal{SF} \not\subseteq \mathcal{R}$ . Therefore there exist positive integers  $a$  and  $b$  such that  $K_{1,a} \notin \mathcal{Q}$  and  $K_{1,b} \notin \mathcal{R}$ . But then  $K_{1,a+b-1} \notin \mathcal{Q} \oplus \mathcal{R}$ , contradicting the assumption that  $\mathcal{SF} \subseteq \mathcal{P}$ . ■

**Corollary 36.** *For every positive integer  $k$  the property  $\mathcal{T}_k$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$ .* ■

## 4 The Decomposability of $\mathcal{S}_k$

**Theorem 41.** *Let  $p$  and  $q$  be any positive integers. Then*

- (1)  $\mathcal{S}_p \oplus \mathcal{S}_q \subseteq \mathcal{S}_{p+q}$ .
- (2)  $\mathcal{S}_p \subseteq (p+1)\mathcal{S}_1$ .
- (3)  $\mathcal{S}_{p+q} \subseteq \mathcal{S}_{p+1} \oplus \mathcal{S}_q$ .
- (4)  $\mathcal{S}_{2p} = p\mathcal{S}_2$ .

**Proof.** (1) If  $G \notin \mathcal{S}_{p+q}$  then  $G$  has a vertex of degree at least  $p + q + 1$ , hence  $K_{1,p+q+1} \subseteq G$ . But since  $K_{1,p+q+1} \notin \mathcal{S}_p \oplus \mathcal{S}_q$ , it then follows that  $G \notin \mathcal{S}_p \oplus \mathcal{S}_q$ .

(2) This is Vizing's fundamental result on edge colourings of graphs.

(3) From (2) and (1) it follows that  $\mathcal{S}_{p+q} \subseteq (p + q + 1)\mathcal{S}_1 \subseteq \mathcal{S}_{p+1} \oplus \mathcal{S}_q$ .

(4) The inclusion  $p\mathcal{S}_2 \subseteq \mathcal{S}_{2p}$  follows from (1). For the other inclusion, suppose that  $G \in \mathcal{S}_{2p}$ . Then  $G$  is a subgraph of a  $2p$ -regular graph  $H$ . Since  $H$  is regular of even degree,  $H$  is 2-factorable by Petersen's Theorem. This 2-factorization of  $H$  clearly is a  $(p\mathcal{S}_2)$ -decomposition of  $H$  and therefore induces a  $(p\mathcal{S}_2)$ -decomposition of  $G$ . Therefore  $G \in p\mathcal{S}_2$ . ■

We are now ready to determine exactly when equality holds in part (1) of Theorem 41.

**Corollary 42.** *Let  $p$  and  $q$  be any even positive integers. Then  $\mathcal{S}_p \oplus \mathcal{S}_q = \mathcal{S}_{p+q}$ .*

**Proof.** An easy calculation using part (4) of Theorem 41. ■

**Theorem 43.** *Let  $p$  and  $q$  be positive integers with  $q$  odd. Then  $\mathcal{S}_{p+q} \not\subseteq \mathcal{S}_p \oplus \mathcal{S}_q$ .*

**Proof.** Suppose first that  $p$  is odd too. Since  $K_{p+q+1} \in \mathcal{S}_{p+q}$  it is sufficient to show that  $K_{p+q+1} \notin \mathcal{S}_p \oplus \mathcal{S}_q$ : Suppose, to the contrary, that  $E_1, E_2$  is an  $(\mathcal{S}_p, \mathcal{S}_q)$ -decomposition of  $E(K_{p+q+1})$ . Then  $E_1$  has at most  $\frac{1}{2}[p(p+q+1) - 1]$  edges, since any graph of maximum degree  $p$  on  $n$  vertices has at most  $\frac{1}{2}pn$  edges and if  $n$  is odd this inequality is strict. Similarly,  $E_2$  has at most  $\frac{1}{2}[q(p + q + 1) - 1]$  edges. But  $K_{p+q+1}$  has  $\frac{1}{2}(p + q)(p + q + 1) > \frac{1}{2}[p(p + q + 1) - 1] + \frac{1}{2}[q(p + q + 1) - 1]$  edges, a contradiction.

Next let  $p = 2$ . Let  $F = K_{q+1,q+1}$  with (vertex) partite sets  $V = \{v_1, \dots, v_{q+1}\}$  and  $W = \{w_1, \dots, w_{q+1}\}$ . Let  $H$  be the graph obtained from  $F$  by adding vertices  $x$  and  $y$  and edges  $\{xv_i : i = 1, \dots, q + 1\}$ ,  $\{w_iw_{i+1} : i = 1, 3, \dots, q\}$  and  $xy$ . Suppose now that  $E_1, E_2$  is any  $(\mathcal{S}_2, \mathcal{S}_q)$ -decomposition of  $E(H)$ . Since every vertex other than  $y$  has degree  $q + 2$  in  $H$ , every such vertex must have degree 2 in  $H[E_1]$ . Therefore  $xy \notin E_1$ , otherwise  $H[E_1]$  would have an odd number of odd vertices. Now let  $G$  be the graph obtained from  $q + 1$  copies of  $H$  by identifying the vertices of degree one. Then  $G \in \mathcal{S}_{2+q}$  but, by the above argument,  $G \notin \mathcal{S}_2 \oplus \mathcal{S}_q$ . Hence  $\mathcal{S}_{2+q} \not\subseteq \mathcal{S}_2 \oplus \mathcal{S}_q$  for every odd positive integer  $q$ .

Finally, let  $p = 2k$  with  $k \geq 2$  and suppose that  $\mathcal{S}_{2k+q} \subseteq \mathcal{S}_{2k} \oplus \mathcal{S}_q$ . Then  $\mathcal{S}_{2+(2(k-1)+q)} = \mathcal{S}_{2k+q} \subseteq \mathcal{S}_{2k} \oplus \mathcal{S}_q = \mathcal{S}_2 \oplus (\mathcal{S}_{2(k-1)} \oplus \mathcal{S}_q) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_{2(k-1)+q}$ , contradicting the statement proven in the previous paragraph. ■

**Lemma 44.** *If  $\mathcal{S}_k = \mathcal{P} \oplus \mathcal{Q}$  with  $\mathcal{P} \in \mathbb{L}_{\subseteq}^a$  and  $\mathcal{Q} \in \mathbb{L}_{\subseteq}^a$  both non-trivial, then  $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$  for some positive integers  $p$  and  $q$  with  $p + q = k$ .*

**Proof.** If  $G \in \mathcal{P}$  has a vertex  $v$  of degree  $a$  and  $H \in \mathcal{Q}$  has a vertex  $w$  of degree  $b$  then there is a graph in  $\mathcal{P} \oplus \mathcal{Q}$  with a vertex of degree  $a + b$ , for example the graph obtained from  $G \cup H$  by identifying  $v$  and  $w$ . Therefore there exist positive integers  $p$  and  $q$  such that  $\mathcal{P} \subseteq \mathcal{S}_p$ ,  $\mathcal{Q} \subseteq \mathcal{S}_q$  and  $p + q \leq k$ . If  $p + q < k$  then  $K_{1,k} \in \mathcal{S}_k$  but  $K_{1,k} \notin \mathcal{P} \oplus \mathcal{Q}$ ; therefore  $p + q = k$ . Now  $\mathcal{S}_k = \mathcal{P} \oplus \mathcal{Q} \subseteq \mathcal{S}_p \oplus \mathcal{S}_q \subseteq \mathcal{S}_k$ , hence  $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$ . ■

We are now ready to prove the main result of this section.

**Theorem 45.**  *$\mathcal{S}_k$  is decomposable in  $\mathbb{L}_{\subseteq}^a$  if and only if  $k = 2n$  for some  $n \geq 2$ .*

**Proof.** Suppose  $\mathcal{S}_k$  is decomposable and  $k$  is odd or  $k = 2$ . Then, by Lemma 44,  $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$ , where  $p + q = k$ . In either case  $p$  or  $q$  is odd and Theorem 43 is contradicted.

The other direction follows from part (4) of Theorem 41. ■

## 5 The Decomposability of $\mathcal{D}_k$

**Theorem 51.**  *$\mathcal{D}_1$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$ .*

**Proof.** The cycle  $C_4$  is in  $2\mathcal{S}_1$  but not in  $\mathcal{D}_1$ , hence  $\mathcal{D}_1$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$  by Lemma 33. ■

The indecomposability of  $\mathcal{D}_2$  is also known — see [4]. We state their result as

**Theorem 52.**  *$\mathcal{D}_2$  is indecomposable.* ■

In order to prove that  $\mathcal{D}_k$  is indecomposable for each  $k \geq 3$ , we first discuss some decomposable bounds for  $\mathcal{D}_k$  which are useful. The first bound is due to Borowiecki and Hałuszczak — see [1].

**Theorem 53.** *For all positive integers  $a$  and  $b$  the inclusion  $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$  holds.* ■

In order to show that  $\mathcal{D}_a \oplus \mathcal{D}_b$  is not an upper bound for  $\mathcal{D}_{a+b+1}$ , we need the following lemma.

**Lemma 54.** *For all positive integers  $a, b$  and  $n$  there is a positive integer  $t$  such that for every  $(\mathcal{D}_a, \mathcal{D}_b)$ -decomposition  $E_1, E_2$  of  $K_{a+b,t}$ , the inclusions  $K_{a,n} \subseteq K_{a+b,t}[E_1]$  and  $K_{b,n} \subseteq K_{a+b,t}[E_2]$  hold.*

**Proof.** In order to simplify the arguments we assume throughout that  $n > a + b$ . For  $a = b = 1$  we can take  $t = 2n + 2$ . Assume now inductively that the statement is true for  $a, b, n$  and  $t$  and consider any  $(\mathcal{D}_a, \mathcal{D}_{b+1})$ -partition  $E_1, E_2$  of  $E(G)$  where  $G = K_{a+b+1,2t}$ . Let  $V$  be the (vertex) partite set of order  $a + b + 1$  of  $G$ ,  $U = V(G) - V$  and let  $v$  be any vertex of  $V$ . We consider two cases:

(1) At least  $t$  edges incident with  $v$  are in  $E_2$ : In this case we apply the induction hypothesis to the subgraph of  $G$  induced by  $(V - v) \cup \{u \in U : vu \in E_2\}$  to obtain  $K_{a,n} \subseteq G[E_1]$  and  $K_{b,n} \subseteq G[E_2]$ . Together with the  $t$  edges in  $E_2$  incident with  $v$  we then also have  $K_{b+1,n} \subseteq G[E_2]$ . (Note that the  $b$  independent vertices of the  $K_{b,n}$  in  $G[E_2]$  is necessarily a subset of  $V$  since  $n > b$ .)

(2) At least  $t$  edges incident with  $v$  are in  $E_1$ : By the same arguments as in case (1) we now obtain that  $K_{a+1,n} \subseteq G[E_1]$ . Since  $n > a$  we have a contradiction, hence this case is impossible. ■

**Theorem 55.** *Let  $a$  and  $b$  be positive integers. Then  $\mathcal{D}_{a+b+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{D}_b$ .*

**Proof.** Clearly  $K_{a+b+1,t} \in \mathcal{D}_{a+b+1}$  for every positive integer  $t$ . Further, by Lemma 54, every  $(\mathcal{D}_a, \mathcal{D}_{b+1})$ -decomposition  $E_1, E_2$  of  $K_{a+b+1,t}$  has  $K_{b+1,b+1} \subseteq K_{a+b+1,t}[E_2]$  for  $t$  large enough. Since a  $(\mathcal{D}_a, \mathcal{D}_b)$ -decomposition is also a  $(\mathcal{D}_a, \mathcal{D}_{b+1})$ -decomposition,  $K_{a+b+1,t} \notin \mathcal{D}_a \oplus \mathcal{D}_b$ . ■

If  $b = 1$  we can demonstrate the sharpness of the inclusion in Theorem 53 in a stronger sense.

**Theorem 56.** *Let  $a$  be a positive integer. If  $\mathcal{D}_1 \not\subseteq \mathcal{P}$  then  $\mathcal{D}_{a+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{P}$ .*

**Proof.** If  $\mathcal{D}_1 \not\subseteq \mathcal{P}$ , then there is a tree  $T$  with  $T \in \mathcal{D}_1$  and  $T \notin \mathcal{P}$ ; suppose  $T$  is of size  $m$ . We construct a sequence  $G_0, G_1, \dots, G_m$  of graphs in  $\mathcal{D}_{a+1}$  as follows:  $G_0 = K_1$  and  $G_{i+1}$  is obtained from  $a$  copies of  $G_i$  by adding, for every set  $V$  consisting of one vertex from each copy of  $G_i$ ,  $a$  copies of  $T$  together with all edges between  $V$  and these copies of  $T$ .

We now prove by induction on  $i$  that for every  $(\mathcal{D}_a, \mathcal{P})$ -decomposition  $E_1, E_2$  of  $G_i$  the graph  $G_i[E_2]$  contains every tree of size  $i$ . It then follows that  $G_m \notin \mathcal{D}_a \oplus \mathcal{P}$ .

For  $i = 0$  this is trivial. Suppose therefore the statement is true for  $i - 1$  and consider any  $(\mathcal{D}_a, \mathcal{P})$ -decomposition  $E_1, E_2$  of  $E(G_i)$  and any tree  $T'$  of size  $i$ . Let  $v$  be a leaf of  $T'$  and let  $u$  be the neighbour of  $v$ . Every copy of  $G_{i-1}$  contains a copy of  $T' - v$  that is in  $E_2$ . Let  $U = \{u_1, u_2, \dots, u_a\}$  be the set of vertices corresponding to  $u$  in each copy of  $G_{i-1}$ . Now consider the subgraph of  $G_i$  isomorphic to  $aK_1 + aT$  consisting of  $U$  together with the corresponding  $a$  copies of  $T$  added in the construction of  $G_i$ . Since  $T \notin \mathcal{P}$ , every copy of  $T$  has an edge in  $E_1$ . The end-vertices of these  $a$  edges together with  $U$  induce a subgraph  $H$  isomorphic to  $aK_2 + aK_1 \notin \mathcal{D}_a$ . Therefore some edge of  $H$  incident with an element of  $U$  must be in  $E_2$ , hence  $T' \subseteq G_i[E_2]$ . ■

**Corollary 57.** *If  $\mathcal{P} \subset \mathcal{D}_1$  then  $\mathcal{D}_{a+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{P}$ .* ■

The next bound is also due to Borowiecki and Hałuszczak — see [1].

**Theorem 58.** *Let  $n_1, \dots, n_k$  be positive integers. Then  $\mathcal{D}_{n_1} \oplus \dots \oplus \mathcal{D}_{n_k} \subseteq \mathcal{D}_{2 \sum_{i=1}^k n_i - 1}$ .* ■

The sharpness of the inclusion of Theorem 58 is demonstrated in Theorem 510.

**Lemma 59.** *Suppose  $\mathcal{P}$  and  $\mathcal{Q}$  are properties,  $G \in \mathcal{P}$  and  $H \in \mathcal{Q}$  are graphs and  $a, b, c, d, k$  and  $n$  are positive integers such that the following holds:*

- (1)  $G$  and  $H$  both have order  $n$ ,
- (2)  $G$  has  $k$  independent vertices of degree  $a$  and the other  $n - k$  vertices have degree  $c$ ,
- (3)  $H$  has  $n - k$  independent vertices of degree  $b$  and the other  $k$  vertices have degree  $d$ .

*Then  $\mathcal{P} \oplus \mathcal{Q} \not\subseteq \mathcal{D}_m$  where  $m = \min\{a + d, b + c\} - 1$ .*

**Proof.** Let  $G_i$  be a copy of  $G$  and  $H_i$  be a copy of  $H$  and suppose  $U_i$  is the set of  $k$  vertices of degree  $a$  in  $G_i$ ,  $V_i$  is the set of  $k$  vertices of degree  $d$  in  $H_i$ ,  $W_i = V(G_i) - U_i$  and  $X_i = V(H_i) - V_i$ ,  $i = 1, 2$ . Let  $F$  be the graph obtained from these four graphs by identifying the vertices of  $U_1$  one



by one with the vertices of  $V_2$ , those of  $W_2$  with  $X_2$ , those of  $V_1$  with  $U_2$  and those of  $X_1$  with  $W_1$ . Then  $F \in \mathcal{P} \oplus \mathcal{Q}$  and, since  $U_1, U_2, X_1$  and  $X_2$  are independent, every vertex in  $F$  has degree  $a + d$  or  $b + c$  so that  $F \notin \mathcal{D}_m$ . ■

If  $G$  is a graph with vertex set  $\{v_1, \dots, v_n\}$  then we denote by  $G[k_1, \dots, k_n]$  the graph with vertex set  $\cup_{i=1}^n \{(v_i, j) : j = 1, \dots, k_i\}$  and edge set  $\{(v_i, j)(v_m, j) : v_i v_m \in E(G)\}$ . If  $k_1 = \dots = k_n = k$  we write  $G[k]$  for  $G[k_1, \dots, k_n]$ .

**Theorem 510.** *For all positive integers  $a$  and  $b$  we have that  $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_{2a+2b-2}$ .*

**Proof.** Let  $G_p$  have vertex set  $\{v_0, v_1, \dots, v_{p+1}, u_1, u_2, \dots, u_p\}$  and edge set  $\{v_i v_{i+1} : i = 0, 1, \dots, p\} \cup \{u_i v_i : i = 1, 2, \dots, p\}$ . Let  $G_p[k, m] = G_p[k, k, \dots, k, m, m, \dots, m]$  (with  $p + 2$   $k$ 's and  $p$   $m$ 's). Note that  $G_p[k, m]$  has  $pk$  vertices of degree  $2k + m$  and  $2k + mp$  independent vertices of degree  $k$ . Now take  $x = 2(a^2 + b^2 - a) - (a + b - 2) > 0$ ,  $y = 2(a^2 + b^2 - b) - (a + b - 2) > 0$  and let  $G = G_x[a, b - 1] \cup (a + b - 2)G_1[a, b - 1] \in \mathcal{D}_a$  and  $H = G_y[b, a - 1] \cup (a + b - 2)G_1[b, a - 1] \in \mathcal{D}_b$ .

$G$  has  $a(x + a + b - 2) = 2a(a^2 + b^2 - a)$  vertices of degree  $2a + b - 1$  and  $2a(1 + a + b - 2) + (b - 1)(x + a + b - 2) = 2b(a^2 + b^2 - b)$  independent vertices of degree  $a$ . By symmetry,  $H$  has  $2b(a^2 + b^2 - b)$  vertices of degree  $2b + a - 1$  and  $2a(a^2 + b^2 - a)$  independent vertices of degree  $b$ . Therefore  $G$  and  $H$  satisfy the conditions of Lemma 59, with  $c = 2a + b - 1$ ,  $d = 2b + a - 1$ ,  $k = 2b(a^2 + b^2 - b)$  and  $n = 2(a^2 + b^2)(a + b - 1)$ , hence  $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_m$  where  $m = 2a + 2b - 2$ . ■

**Theorem 511.** *For every positive integer  $k$  the property  $\mathcal{D}_k$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$ .*

**Proof.** Suppose that  $\mathcal{D}_k = \mathcal{P} \oplus \mathcal{Q}$  with  $\mathcal{P}$  and  $\mathcal{Q}$  non-trivial. Note that if  $G \in \mathcal{P}$  and  $H \in \mathcal{Q}$  then  $G \times H \in \mathcal{P} \oplus \mathcal{Q}$  and  $\delta(G \times H) = \delta(G) + \delta(H)$ . Therefore there exist positive integers  $a$  and  $b$  with  $a + b = k$  such that  $\delta(G) \leq a$  and  $\delta(H) \leq b$  for all  $G \in \mathcal{P}$  and  $H \in \mathcal{Q}$ . It follows that  $\mathcal{P} \subseteq \mathcal{D}_a$  and  $\mathcal{Q} \subseteq \mathcal{D}_b$ .

Let  $P_n[l, m]$  be the graph  $P_n[l, m, m, \dots, m]$  where we take  $V(P_n)$  to be  $\{v_1, v_2, \dots, v_n\}$  and  $E(P_n)$  to be  $\{v_i v_{i+1} : i = 1, 2, \dots, n - 1\}$ . We now show that  $P_n[a + b, a] \in \mathcal{P}$  and  $P_n[a + b, b] \in \mathcal{Q}$  for every  $n$ : We do this by finding a graph  $G_n \in \mathcal{D}_k$  such that for every  $(\mathcal{D}_a, \mathcal{D}_b)$ -partition  $E_1, E_2$  of  $E(G_n)$ ,  $P_n[a + b, a] \subseteq G_n[E_1]$  and  $P_n[a + b, b] \subseteq G_n[E_2]$ . For  $n = 2$  we can take  $G_2 = K_{a+b,t}$  with  $t$  large enough, by Lemma 54. Assume therefore that

$G_n$  has been found and consider a  $(\mathcal{D}_a, \mathcal{D}_b)$ -partition  $E_1, E_2$  of  $E(G_{n+1})$ , where  $G_{n+1}$  is obtained from  $G_n$  by adding, for every subset of  $a+b$  vertices of  $V(G_n)$ ,  $t$  vertices together with all edges connecting the new vertices with these  $a+b$  vertices. We know that  $P_n[a+b, a] \subseteq G_n[E_1]$ . By applying Lemma 54 to the appropriate set of  $a+b$  independent vertices and the  $t$  new vertices adjacent to them it now follows that  $P_{n+1}[a+b, a] \subseteq E_1$ . Similarly,  $P_{n+1}[a+b, b] \subseteq E_2$ .

In particular,  $P_4[a] \in \mathcal{P}$  and  $P_4[b] \in \mathcal{Q}$ . Then  $G = bP_4[a]$  and  $H = aP_4[b]$  satisfy the conditions of Lemma 59 with  $c = 2a$ ,  $d = 2b$ ,  $n = 4ab$  and  $k = 2ab$ . Therefore  $\mathcal{P} \oplus \mathcal{Q} \not\subseteq \mathcal{D}_m$  where  $m = \min\{a+2b, b+2a\} - 1 \geq a+b$ , a contradiction. ■

## 6 The Decomposability of $\mathcal{P}^k$

**Corollary 61.** *If  $\mathcal{P} \neq \mathcal{O}$  is a property of graphs and  $k \geq 2$  then  $\mathcal{P}^k$  is decomposable.*

**Proof.** It follows easily from Lemma 21 that  $\mathcal{P}^k = \mathcal{P} \oplus \mathcal{O}^k$ . ■

In the rest of this section we consider the decomposability of the property  $\mathcal{O}^k$ . First we define two parameters for a property  $\mathcal{P}$ : Let  $\chi(\mathcal{P})$  denote the least  $k$  such that  $\mathcal{P} \subseteq \mathcal{O}^k$  and let  $\chi^*(\mathcal{P})$  denote the largest  $k$  such that  $\mathcal{O}^k \subseteq \mathcal{P}$ . The next lemma is useful in the proof of the main result of this section.

**Lemma 62.** *Let  $\mathcal{P}$  and  $\mathcal{Q}$  be properties of graphs. Then  $\chi^*(\mathcal{P} \oplus \mathcal{Q}) \leq \chi^*(\mathcal{P})\chi(\mathcal{Q}) \leq \chi(\mathcal{P} \oplus \mathcal{Q})$ .*

**Proof.** Note that the result holds if  $\chi(\mathcal{Q})$  is infinite. We may therefore suppose that  $a = \chi^*(\mathcal{P})$  and  $b = \chi(\mathcal{Q})$  are finite. For the first inequality, let  $t$  be such that  $K_{a+1}[t] \notin \mathcal{P}$ . In order to show that  $\chi^*(\mathcal{P} \oplus \mathcal{Q}) \leq \chi^*(\mathcal{P})\chi(\mathcal{Q})$ , it is sufficient to show that  $G = K_{ab+1}[bt] \notin \mathcal{P} \oplus \mathcal{Q}$ . Suppose, to the contrary, that  $E_1, E_2$  is a  $(\mathcal{P}, \mathcal{Q})$ -decomposition of  $E(G)$ . Let  $U_1, \dots, U_{ab+1}$  be the partite sets of  $G$ , let  $V_1, \dots, V_b$  be a  $b$ -colouring of  $G[E_2]$  and set  $V_{i,j} = V_i \cap U_j$ . For every  $j$  there must be an  $i$  such that  $|V_{i,j}| \geq t$  since  $|U_j| = bt$  and  $U_j$  is a union of only  $b$  of the  $V_{i,j}$ 's. Therefore there are at least  $ab+1$   $V_{i,j}$ 's with  $|V_{i,j}| \geq t$ . On the other hand, for every  $i$  there are at most  $a$   $j$ 's such that  $|V_{i,j}| \geq t$  since every edge of  $G[V_i]$  is in  $E_1$  and  $K_{a+1}[t] \notin \mathcal{P}$ . Therefore there are at most  $ab$   $V_{i,j}$ 's with  $|V_{i,j}| \geq t$ , a contradiction.

For the second inequality, suppose  $G \in \mathcal{Q}$  has chromatic number  $b$ . Then the graph  $H = G + G + \cdots + G$  ( $a$   $G$ 's) has chromatic number  $ab$  and is in  $\mathcal{P} \oplus \mathcal{Q}$ . ■

We are now ready to discuss the decomposability of  $\mathcal{O}^p$ .

**Theorem 63.**  $\mathcal{O}^p$  is indecomposable in  $\mathbb{L}_{\subseteq}^a$  if and only if  $p$  is prime.

**Proof.** If  $p$  is not prime, say  $p = ab$ , then it is easy to see from Lemma 21 that  $\mathcal{O}^p = \mathcal{O}^a \oplus \mathcal{O}^b$ .

Conversely, suppose that  $\mathcal{O}^p = \mathcal{P} \oplus \mathcal{Q}$  with  $\mathcal{P}$  and  $\mathcal{Q}$  non-trivial properties of graphs. Then  $\chi(\mathcal{P}), \chi(\mathcal{Q}), p \geq 2$  and, since  $\chi(\mathcal{O}^p) = \chi^*(\mathcal{O}^p) = p$ , it follows from Lemma 62 that  $p = \chi^*(\mathcal{P})\chi(\mathcal{Q}) = \chi(\mathcal{P})\chi^*(\mathcal{Q})$ . If  $\chi^*(\mathcal{P}) > 1$  or  $\chi^*(\mathcal{Q}) > 1$  we are done. Suppose therefore that  $\chi^*(\mathcal{P}) = \chi^*(\mathcal{Q}) = 1$ . Then  $\chi(\mathcal{P}) = \chi(\mathcal{Q}) = p$ , hence there exist graphs  $F \in \mathcal{P}$  and  $H \in \mathcal{Q}$  which both have chromatic number  $p$ . Since  $F + K_1 \notin \mathcal{O}^p$  and  $F + K_1 \in \mathcal{P} \oplus \mathcal{S}\mathcal{F}$  it follows that  $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{Q}$ . Similarly,  $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{P}$ . But then  $\mathcal{S}\mathcal{F} \not\subseteq \mathcal{P} \oplus \mathcal{Q} = \mathcal{O}^p$ , a contradiction, since  $p \geq 2$ . ■

## References

- [1] M. Borowiecki and M. Hałuszczak, Decompositions of some classes of graphs, Report No. IM-3-99, Institute of Mathematics, Technical University of Zielona Góra, 1999.
- [2] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.
- [3] S.A. Burr, M.S. Jacobson, P. Mihók and G. Semanišin, *Generalized Ramsey theory and decomposable properties of graphs*, Discuss. Math. Graph Theory **19** (1999) 199–217.
- [4] M. Hałuszczak and P. Vateha, *On the completeness of decomposable properties of graphs*, Discuss. Math. Graph Theory **19** (1999) 229–236.
- [5] P. Mihók, G. Semanišin and R. Vasky, *Additive and hereditary properties of graphs are uniquely factorizable into irreducible factors*, J. Graph Theory **33** (2000) 44–53.
- [6] J. Nešetřil and V. Rödl, *Simple proof of the existence of restricted Ramsey graphs by means of a partite construction*, Combinatorica **1** (1981) 199–202.

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