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A NOTE ON PERIODICITY OF THE 2-DISTANCE OPERATOR

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To the memory of Ivan Havel

Abstract

The paper solves one problem by E. Prisner concerning the 2-distance operator T_2. This is an operator on the class C_f of all finite undirected graphs. If G is a graph from C_f, then $T_2(G)$ is the graph with the same vertex set as G in which two vertices are adjacent if and only if their distance in G is 2. E. Prisner asks whether the periodicity ≥ 3 is possible for T_2. In this paper an affirmative answer is given. A result concerning the periodicity 2 is added.

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In this paper we consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$, its edge set by $E(G)$. The symbol $\overline{G}$ denotes the complement of G, i.e., the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G.

Let ϕ be a graph operator defined on the class C_f of all finite undirected graphs. For every positive integer r we define the power ϕ^r so that $\phi^1 = \phi$ and for $r \geq 2$ the operator ϕ^r is such that $\phi^r(G) = \phi(\phi^{r-1}(G))$ for each $G \in C_f$. A graph $G \in C_f$ is called *ϕ-periodic*, if there exists a positive integer r such that $\phi^r(G) \cong G$. The minimum number r with this property is the *periodicity* of the graph G in the operator ϕ.

For an integer $k \geq 2$ the operator T_k on C_f is defined in such a way that for any graph $G \in C_f$ the graph $T_k(G)$ has the same vertex set as G

and two distinct vertices are adjacent in $T_k(G)$ if and only if their distance in G is k. The operator T_k is called the *k-distance operator.*

In [2], page 170, E. Prisner asks the following problem:

Is period ≥ 3 possible for T_2 ?

An affirmative answer is given by the following theorem.

Theorem. *Let r be an even positive integer. Then there exists a graph G_r whose periodicity in the operator T_2 is r.*

Proof. Let $q = 2^r + 1$. Let $V_0, V_1, \ldots, V_{q-1}$ be pairwise disjoint sets of vertices. Let t be an integer, $t \geq 2$ and let $|V_i| = t^i$ for $i = 0, 1, \ldots, q-1$. The vertex set of G_r is $V(G_r) = \bigcup_{i=0}^{q-1} V_i$. All sets $V_0, V_1, \ldots, V_{q-1}$ are independent in G_r. Let $x \in V_i$, $y \in V_j$ for some i and j from $\{0, 1, \ldots, q-1\}$. These vertices are adjacent in G_r if and only if $j \equiv i + 1 \pmod q$ or $j \equiv i - 1 \pmod q$. This implies that all sets $V_0, V_1, \ldots, V_{q-1}$ induce complete subgraphs in the graph $T_2(G_r)$. If $x \in V_i$, $y \in V_j$, then x, y are adjacent in $T_2(G_r)$ if and only if $j \equiv i + 2 (\mathrm{mod}\ q)$ or $j \equiv i - 2 (\mathrm{mod}\ q)$. From these facts by induction we obtain that $T_2^m(G)$ for $m \geq 2$ has the following structure. If m is even, then all sets $V_0, V_1, \ldots, V_{q-1}$ are independent; if m is odd, then they induce complete subgraphs; if $x \in V_i$, $y \in V_j$, then x, y are adjacent if and only if $j \equiv i + 2^m (\mathrm{mod}\ q)$ or $j \equiv i - 2^m (\mathrm{mod}\ q)$ in both the cases. This implies that $T_2^r(G_r) \cong G_r$. Now it remains to show that $T_2^m(G)$ is not isomorphic to G_r for $1 \leq m < r$. We do it using the independence number $\alpha(G)$. The greatest independent set in G_r is $\bigcup_{i=1}^{\frac{1}{2}(q-1)} V_{2i}$ and thus $\alpha(G_r) = \sum_{i=1}^{\frac{1}{2}(q-1)} t^{2i} = t^2(t^{q-1} - l)/(t^2 - 1)$. If m is odd, then $\alpha(T_2^m(G)) = \frac{1}{2}(q-1)$. If m is even, $2 \leq m \leq r - 2$, then the set $V_0 \cup V_{q-2} \cup V_{q-1}$ is independent in $T_2^m(G_r)$ and thus $\alpha(T_2^m(G_r)) \geq |V_0 \cup V_{q-2} \cup V_{q-1}| = 1 + t^{q-2} + t^{q-1} > t^2(t^{q-1} - 1)/(t^2 - 1)/(t^2 - 1) = \alpha(G_r)$; this inequality may be easily proved. Therefore no graph $T_2^m(G_r)$ for $1 \leq m \leq r - 1$ is isomorphic to G_r and thus the periodicity of G_r in T_2 is r. ∎

We shall remark also the periodicity 2. In [1] F. Harary, C. Hoede and D. Kadlacek have proved that if a graph G is self-complementary, i.e., $\overline{G} \cong G$, then $T_2(G) \cong G$ and thus the periodicity of G in T_2 is 1. A slight generalization of the result is the following proposition. The diameter of G is denoted by diam G.

Proposition 1. *Let G be a graph such that* $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$ *and $\overline{G}$ is not isomorphic to G. Then G is T_2-periodic with the periodicity* 2.

Proof. If two vertices x, y are adjacent in G, then their distance in G is 1 and they are not adjacent in $T_2(G)$. If they are not adjacent in G, then their distance in G is 2 and realizes diam G. Moreover, x and y are adjacent in $T_2(G)$. Hence $T_2(G) = \overline{G}$. As also diam $\overline{G} = 2$, we have $T_2^2(G) = T_2(T_2(G)) = T_2(\overline{G}) = G$. ∎

We shall create a class of graph which have the property that diam $G =$ diam $\overline{G} = 2$.

Let H_1, H_2, H_3, H_4, H_5 be pairwise disjoint graphs. The graph $G(H_1, H_2, H_3, H_4, H_5)$ contains mentioned graphs as subgraphs and has new edges xy created in the following way. If $x \in V(H_i)$, $y \in V(H_j)$, then x and y are adjacent in G if and only if $j \equiv i + 1 (\text{mod } 5)$ or $j \equiv i + 4 (\text{mod } 5)$. The simplest is the graph $G(K_1, K_1, K_1, K_1, K_1) = C_5$.

Proposition 2. *For any five graphs H_1, H_2, H_3, H_4, H_5 the graph $G(H_1, H_2, H_3, H_4, H_5)$ has the diameter 2 and so has its complement.*

Proof. Let x, y be two vertices of $G(H_1, H_2, H_3, H_4, H_5)$. Let i, j be such numbers from $\{1, 2, 3, 4, 5\}$ that $x \in V(H_i)$, $y \in V(H_j)$.

If $i = j$, then both x, y are in the graph H_i. If they are adjacent in G, then their distance is 1. If they are not adjacent, then there exists a path of length 2 connecting them; its inner vertex is in $V(H_{j+1}) \cup V(H_{i+4})$, the subscripts being taken modulo 5. If $j \equiv i + 1 (\text{mod } 5)$ or $j \equiv i + 4 (\text{mod } 5)$, then x, y are adjacent in G and their distance is 1.

If $j \equiv i + 2 (\text{mod } 5)$ or $j \equiv i + 3 (\text{mod } 5)$ then x, y are not adjacent, but there exists a path of length 2 connecting them; its inner vertex is in $V(H_{i+1}) \cup V(H_{i+4})$. Therefore diam $G = 2$. The complement of $G(H_1, H_2, H_3, H_4, H_5)$ is isomorphic to $G(\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4, \overline{H}_5)$ and thus also diam $\overline{G} = 2$. ∎

References

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[2] E. Prisner, Graph Dynamics (Longman House, Burnt Mill, Harlow, 1995).

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