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KERNELS IN THE CLOSURE OF COLOURED DIGRAPHS

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Abstract

Let D be a digraph with $V(D)$ and $A(D)$ the sets of vertices and arcs of D, respectively. A kernel of D is a set $I \subset V(D)$ such that no arc of D joins two vertices of I and for each $x \in V(D) \setminus I$ there is a vertex $y \in I$ such that $(x, y) \in A(D)$. A digraph is kernel-perfect if every non-empty induced subdigraph of D has a kernel. If D is edge coloured, we define the closure $\xi(D)$ of D the multidigraph with $V(\xi(D)) = V(D)$ and $A(\xi(D)) = \bigcup_i \{(u, v)$ with colour i: there exists a monochromatic path of colour i from the vertex u to the vertex v contained in $D\}$.

Let T_3 and C_3 denote the transitive tournament of order 3 and the 3-cycle, respectively, both of whose arcs are coloured with 3 different colours. In this paper, we survey sufficient conditions for the existence of kernels in the closure of edge coloured digraphs, also we prove that if D is obtained from an edge coloured tournament by deleting one arc and D does not contain T_3 or C_3, then $\xi(D)$ is a kernel-perfect digraph.

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1 Introduction

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. An arc $(u,v) \in A(D)$ is called *asymmetrical* in D if $(v,u) \notin A(D)$. An arc $(v,u) \in A(D)$ is called *symmetrical* in D if $(u,v) \in A(D)$. The asymmetrical part of D, denoted by $\mathrm{Asym}(D)$, is the sapanning subdigraph of D whose arcs are the asymmetrical arcs of D. The symmetrical part of D, denoted by $\mathrm{Sym}(D)$, is the spanning subdigraph of D whose arcs are the symmetrical arcs of D. A digraph D is called asymmetrical if $D = \mathrm{Asym}(D)$.

If S is a nonempty subset of $V(D)$, then the subdigraph $D[S]$ induced by S is the digraph having vertex set S and whose arcs are all those arcs of D joining vertices of S.

A set $I \subseteq V(D)$ is *independent* in D if $A(D[I]) = \emptyset$. The set of all the independent sets in D is denoted by $\mathrm{ind}(D)$, i.e., $\mathrm{ind}(D) = \{I \subseteq V(D)\colon I$ is independent in $D\}$.

A set $I \subseteq V(D)$ is *absorbent* in D if for each vertex $x \in V(D)\backslash I$, there exists a vertex $y \in I$ such that $(x,y) \in A(D)$. The set of all the absorbent sets in D is denoted by $\mathrm{abs}(D)$, i.e., $\mathrm{abs}(D) = \{I \subseteq V(D)\colon I$ is absorbent in $D\}$.

A set $I \subseteq V(D)$ is a *kernel* of D if I is an independent and absorbent set of vertices in D. The set of all the kernels of D is denoted by $\ker(D)$, i.e., $\ker(D) = \mathrm{ind}(D) \cap \mathrm{abs}(D)$.

A digraph is called a *kernel-perfect* digraph or *KP-digraph* when every induced subdigraph of D has a kernel; i.e., for every nonempty set of vertices $N \subseteq V(D)$, $\ker(D[N]) \neq \emptyset$.

A digraph D is called *complete* if for every two differents vertices $u, v \in V(D), (u,v) \in A(D)$ or $(v,u) \in A(D)$. A *tournament* is a complete asymmetrical digraph.

If γ is a directed cycle and $x, y \in V(\gamma)$, we denote by (x, γ, y) the directed path from x to y contained in γ.

We call the digraph D an *m-coloured* digraph if the arcs of D are coloured with m distinct colours. A directed path or a directed cycle is called *monochromatic* if all of its arcs are of the same colour. A directed cycle is *quasi-monochromatic* if, with at most one exception, all of its arcs are coloured alike.

The *closure* of D, denoted by $\zeta(D)$, is the m-coloured digraph defined as follows:

$V(\zeta(D)) = V(D)$, and $A(\zeta(D)) = \cup_i\{(u,v)$ with colour i: there exists a monochromatic path of colour i from the vertex u to the vertex v contained in $D\}$.

Notice that for any digraph $D, \zeta(D) = \zeta(\zeta(D))$ and $A(D) \subseteq A(\zeta(D))$.

Let T_3 and C_3 denote the transitive tournament of order 3 and the 3-cycle, respectively, both of whose arcs are coloured with three different colours.

2 Antecedentes

In [6], Sands et al., have proved that for any 2-coloured digraph $D, \zeta(D)$ is a KP-digraph. Particularly they proved that every 2-coloured tournament T has a vertex $v \in V(T)$ such that $\{v\} \in \ker(\zeta(D))$.

In [5] Shen Minggang proved that if T is an m-coloured tournament which does not contain C_3 or T_3, then $\zeta(T)$ has a kernel. He also proved that this situation is best possible for $m \geq 5$. In fact, he proved that for $m \geq 5$ there exists an m-coloured tournament T which does not contain C_3 and $\zeta(T)$ has no kernel. Also for $m \geq 5$ there exists an m-coloured tournament T' which does not contain T_3 and $\zeta(D)$ has no kernel.

The question for $m = 3$ (If T is a 3-coloured tournament which does not contain C_3 then $\zeta(T)$ has a kernel) and the respective question for $m = 4$ are still open.

Let D be a digraph, an m-coloration of D is called *$\{C_3, C_4\}$-free m-coloration* of D if it is an m-coloration of D such that every directed cycle of length at most 4 is quasi-monochromatic. Denote by $\mathcal{F}$ the class of digraphs such that the closure of every $\{C_3, C_4\}$-free m-coloration has a kernel; in [2] it is proved that every tournament belongs to $\mathcal{F}$ and in [3] it is proved that the digraph obtained from a tournament by the deletion of a single arc belongs to $\mathcal{F}$.

Let D be a digraph, an m-coloration of D is named a *$\{C_3, T_3\}$-free m-coloration* of D if it is an m-coloration of D such that every tournament of order 3 is quasi-monochromatic. Denote by $\mathcal{E}$ the class of digraphs such that the closure of every $\{C_3, T_3\}$-free coloration has a kernel. In [5] it is proved that any tournament belongs to $\mathcal{E}$. In [4] it is proved that if D is a digraph of the class $\mathcal{E}$ such that its underlying graph is hamiltonian, then the complement of its underlying graph has at most one nontrivial connected component and this component is K_3 or a star.

In this paper, we prove that the digraph obtained from a tournament by the deletion of a single arc belongs to $\mathcal{E}$. In fact, it is proved that the closure of any $\{C_3, T_3\}$-free m-coloration of a digraph obtained from a tournament by the deletion of a single arc is kernel-perfect.

Theorem 21 (Minggang [5]). *If T is an m-coloured tournament without C_3 or T_3, then there is a vertex $v \in V(T)$ such that $\{v\} \in \ker(\zeta(T))$.*

Theorem 22 (Berge and Duchet [1]). *A complete digraph is a KP-digraph if and only if every directed cycle has a symmetrical arc.*

3 Kernels in the Closure of m-Coloured Digraphs

The main result of this section is Theorem 3.3. To prove it we need Theorem 3.1 which has been proved in [3] (Lemma l and point (2) of the proof of Theorem 2).

Theorem 3.1 [3]. *Let D be an m-coloured digraph resulting from the deletion of the single arc (x, y) from some m-coloured tournament. If for every complete subdigraph H of $D \ker(\zeta(H)) \neq \emptyset$, then at least one of the two following assertions holds:*

(i) *$\zeta(D)$ has a kernel.*

(ii) *There exists a directed cycle $\gamma \subseteq \mathrm{Asym}(\zeta(D))$ such that $\{x, y\} \subseteq V(\gamma)$, and for every two non consecutive vertices u, v with $\{u, v\} \neq \{x, y\}$ we have $\{(u, v), (v, u)\} \subseteq A(\zeta(D))$.*

Theorem 3.2. *Let D be an m-coloured digraph resulting from the deletion of the single arc (x, y) from some m-coloured tournament. If D does not contain C_3 or T_3, then $\zeta(D)$ does not contain any asymmetrical cycle γ such that $x, y \in V(\gamma)$ and that between any two nonconsecutive vertices different from $\{x, y\}$ of γ there are symmetrical arcs.*

Proof. We proceed by contradiction. Assume there is an asymmetrical cycle γ in $\zeta(D)$ such that $x, y \in V(\gamma)$ and that between any two nonconsecutive vertices of γ there are symmetrical arcs.

We prove several propositions in order to reach the contradiction.

For $z \in V(\gamma)$, denote by $z-$ and $z+$ its predecessor and its successor in γ, respectively. Notice that, as γ is asymmetrical and between any two nonconsecutive vertices of γ there are symmetrical arcs, $z-$ is the only vertex of γ such that $(z, z-) \notin A(\zeta(D))$ and $z+$ is the only vertex of γ such that $(z+, z) \notin A(\zeta(D))$.

(1) If $z \in V(\gamma)$, then $(z+, z-) \in A(\zeta(D))$.

Since z is the only vertex in γ which is not absorbed by $z-$ in $\zeta(D)$. For $z \in V(\gamma)$, denote by $P(z)$ the shortest monochromatic path from $z+$ to $z-$ in D.

(2) If $z \in V(\gamma)$ is adjacent in D to every vertex of $P(z)$ then $(z-, z) \in A(D)$ and $(z, z+) \in A(D)$ are of the same colour.

$(z-, z) \in A(D)$ because z is adjacent to $z-$ in D, but $(z, z-) \notin A(D)$, as $A(D) \subseteq A(\zeta(D))$ and $(z, z-) \notin A(\zeta(D))$. Also $(z, z+) \in A(D)$ because z is adjacent to $z+$ in D, but $(z+, z) \notin A(D)$, as $A(D) \subseteq A(\zeta(D))$ and $(z+, z) \notin A(\zeta(D))$. Now, to show that the arcs $(z-, z) \in A(D)$ and $(z, z+) \in A(D)$ are of the same colour we proceed by assuming (without loss of generality) that $(z, z+) \in A(D)$ is blue and concluding that $(z-, z) \in A(D)$ is also blue.

(2a) $P(z)$ is not blue.

It is not blue, because that would give a monochromatic blue path from z to $z-$ in D, namely the one that begins with the blue arc $(z, z+) \in A(D)$ and continues along the monochromatic path $P(z)$ to reach $z-$.

Without loss of generality, assume $P(z)$ is green, and let $P(z) = (v_0 = z+, v_1, v_2, \ldots, v_t = z-)$.

(2b) For $1 \leq i \leq t, (z+, v_i) \in A(\zeta(D))$ is green and $(v_i, z-) \in A(\zeta(D))$ is green.

We just split the monochromatic green path P in D at the vertex v_i.

(2c) For $1 \leq i \leq t$, the arc between z and v_i in D in not green.

Recall that z is adjacent to every vertex of P. If $(z, v_i) \in A(D)$ is green, then there would be a monochromatic green path from z to $z-$ in D, namely the one that begins with the arc (z, v_i) and continues along the monochromatic green path obtained in (2b) from v_i to $z-$. If $(v_i, z) \in A(D)$ is green, then there would be a monochromatic green path from $z+$ to z in D, namely the one that begins with the monochromatic green path obtained in (2b) from $z+$ to v_i and continues with the arc (v_i, z).

(2d) For $1 \leq i \leq t$, if the arc between z and v_{i-1} in D is blue, then the arc between z and v_i in D is also blue.

By hypothesis, D does not have C_3 or T_3. As $(v_{i-1}, v_i) \in A(D)$ is green (it is an arc of the monochromatic green path $P(z)$), and we are assuming that the arc between z and v_{i-1} in D is blue, the arc between z and v_i in D must be blue or green. But it is not green, by (2c).

(2e) $(z-, z) \in A(D)$ is blue.

The arc between z and v_0 in D is blue, as $v_0 = z+$ and we assumed that $(z, z+) \in A(D)$ is blue. It follows from (2d) that the arc between z and v_t in D is blue. Now recall that $v_t = z-$.

We conclude the proof of (2).

(3) If $s \in V(\gamma)\backslash\{x, y\}$ then $(s-, s) \in A(D)$ and $(s, s+) \in A(D)$ are of the same colour.

As $s \neq x$ and $s \neq y$, we have that s is adjacent in D to every vertex of $P(s)$, and the conclusion follows from (2).

(4) The paths (x, γ, y) and (y, γ, x) are monochromatic in D.

This follows from (3).

(5) x and y are not consecutive in γ.

Suppose x and y are consecutive in γ. As the paths (x, γ, y) and (y, γ, x) are monochromatic in $D, (x, y) \in A(\zeta(D))$ and $(y, x) \in A(\zeta(D)), (x, y)$ would be a symmetrical arc of γ. A contradiction.

(6) The monochromatic paths (x, γ, y) and (y, γ, x) are not of the same colour.

Otherwise, γ would be monochromatic in D, there would be monochromatic paths between any two vertices of γ, and γ would not be asymmetrical in $\zeta(D)$.

Without loss of generality, assume (x, γ, y) is red in D and (y, γ, x) is blue in D.

(7) $P(x)$ is not red and is not blue. (Recall that $P(x)$ is the shortest monochromatic path from $x+$ to $x-$ in D).

If $P(x)$ was red, then there would be a monochromatic red path from x to $x-$ in D, namely the one that begins with the red arc $(x, x+) \in A(x, \gamma, y)$ and continues along $P(x)$ to reach $x-$. If $P(x)$ was blue, then there would be a monochromatic blue path from $x+$ to x in D, namely the one that begins with $P(x)$ and continues with the blue arc $(x-, x) \in A(y, \gamma, x)$.

(8) $y \in V(P(x))$.

Otherwise, x would be adjacent to every vertex of $P(x)$ in D, and by (2) we conclude that $(x-, x) \in A(D)$ and $(x, x+) \in A(D)$ are of the same colour, contradicting that $(x-, x) \in A(y, \gamma, x)$ is blue and $(x, x+) \in A(x, \gamma, y)$ is red.

Let $P(x) = (x+ = a_0, a_1, a_2, \ldots, a_r, y = a_{r+1}, \ldots, x-)$.

(9) $P(y)$ is not red and is not blue. (Recall that $P(y)$ is the shortest monochromatic path from $y+$ to $y-$ in D).

If $P(y)$ was blue, then there would be a monochromatic red path from y to $y-$ in D, namely the one that begins with the blue arc $(y, y+) \in A(y, \gamma, x)$ and continues along $P(y)$ to reach $y-$. If $P(y)$ was red, then there would be a monochromatic red path from $y+$ to y in D, namely the one that begins with $P(y)$ and continues with the red arc $(y-, y) \in A(x, \gamma, y)$.

(10) $x \in V(P(y))$.

Otherwise, y would be adjacent to every vertex of $P(y)$ in D, and by (2) we conclude that $(y-, y) \in A(D)$ and $(y, y+) \in A(D)$ are of the same colour, contradinting that $(y-, y) \in A(x, \gamma, y)$ is red and $(y, y+) \in A(y, \gamma, x)$ is blue.

Let $P(y) = (y+ = b_0, b_1, b_2, \ldots, b_s, x = b_{s+1}, \ldots, y-)$.

We consider the two possible cases and we will reach a contradiction in each one.

Case A. $P(x)$ and $P(y)$ are of the same colour. Assume (without loss of generality) they are green (see Figure 1). The contradiction, in this case, will arise at the colour of the arc between $x+$ and $y+$ in D.

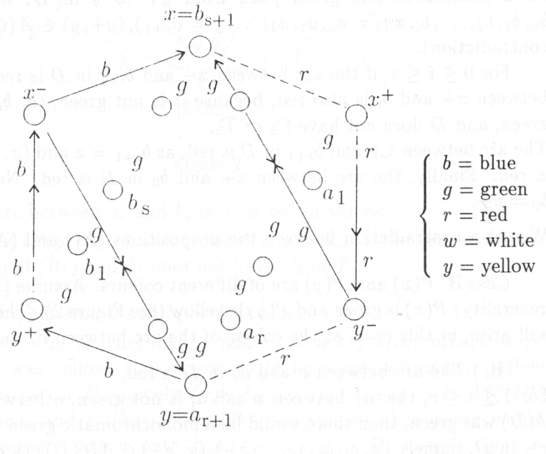

Figure 1

(A_1) The arc between $y+$ and $x+$ in D is blue.
For $0 \leq i \leq r$, the arc between $y+$ and a_i in D is not green, otherwise, if $(y+, a_i) \in A(D)$ was green, then there would be a monochromatic green path from $y+$ to y in D, namely $(y+, a_i, a_{i+1}, \ldots, y = a_{r+1}), (y+, y) \in A(\zeta(D))$ (a contradiction); and if $(a_i, y+) \in A(D)$ was green, then there would be a monochromatic green path from $x+$ to x in D, namely $(x+ = a_0, a_1, a_2, \ldots, a_i, y+ = b_0, b_1, b_2, \ldots, x = b_{s+1}), (x+, x) \in A(\zeta(D))$ (another contradiction).

For $0 \leq i \leq r$, if the arc between $y+$ and a_{i+1} in D is blue, then the arc between $y+$ and a_i is also blue, because it is not green, $(a_i, a_{i+1}) \in A(D)$ is green, and D does not have C_3 or T_3.

The arc between $y+$ and a_{r+1} in D is blue, as $a_{r+1} = y$ and $(y, y+) \in A(y, \gamma, x)$ is blue. Finally, the arc between $y+$ and a_0 in D is blue. Now, recall that $a_0 = x+$.

(A_2) The arc between $x+$ and $y+$ in D is red.
For $0 \leq i \leq s$, the arc between $x+$ and b_i in D is not green, otherwise, if $(x+, b_i) \in A(D)$ was green, then there would be a monochromatic green path from $x+$ to x in D, namely $(x+, b_i, b_{i+1}, \ldots, x = b_{s+1}), (x+, x) \in A(\zeta(D))$ (a contradicition), and if $(b_i, x+) \in A(D)$ was green, then there would be a monochromatic green path from $y+$ to y in D, namely $(y+ = b_0, b_1, b_2, \ldots, b_i, x+ = a_o, a_1, a_2, \ldots, y = a_{r+1}), (y+, y) \in A(\zeta(D))$ (another contradiction).

For $0 \leq i \leq s$, if the arc between $x+$ and b_{i+1} in D is red, then the arc between $x+$ and b_i is also red, because it is not green, $(b_i, b_{i+1}) \in A(D)$ is green, and D does not have C_3 or T_3.
The arc between $x+$ and b_{s+1} in D is red, as $b_{s+1} = x$ and $(x, x+) \in (x, \gamma, y)$ is red. Finally, the arc between $x+$ and b_0 in D is red. Now, recall that $b_0 = y+$.

We get a contradiction between the propositions (A_1) and (A_2).

Case B. $P(x)$ and $P(y)$ are of different colours. Assume (without loss of generality) $P(x)$ is green and $P(y)$ is yellow (see Figure 2). The contradiction will arise, in this case, at the colour of the arc between a_r and b_s in D.

(B_1) The arc between x and a_r in D is red.
For $1 \leq i \leq r$, the arc between x and a_i is not green, otherwise, if $(x, a_i) \in A(D)$ was green, then there would be a monochromatic green path from x to $x-$ in D, namely $(x, a_i, a_{i+1}, \ldots, x-), (x, x-) \in A(\zeta(D))$ (a contradiction), and if $(a_i, x) \in A(D)$ was green, then there would be a monochromatic

green path from $x+$ to x in D, namely $(x+ = a_0, a_1, a_2, \ldots, a_i, x), (x+, x) \in A(\zeta(D))$ (another contradiction).

For $1 \leq i \leq r$, if the arc between x and a_{i-1} in D is red then the arc between x and a_i in D is also red, because is not green, $(a_{i-1}, a_i) \in A(D)$ is green, and D does not have C_3 or T_3.

The arc between x and a_0 in D is red, as $a_0 = x+$ and $(x, x+) \in A(x, \gamma, y)$ is red. Finally, the arc between x and a_r in D is red.

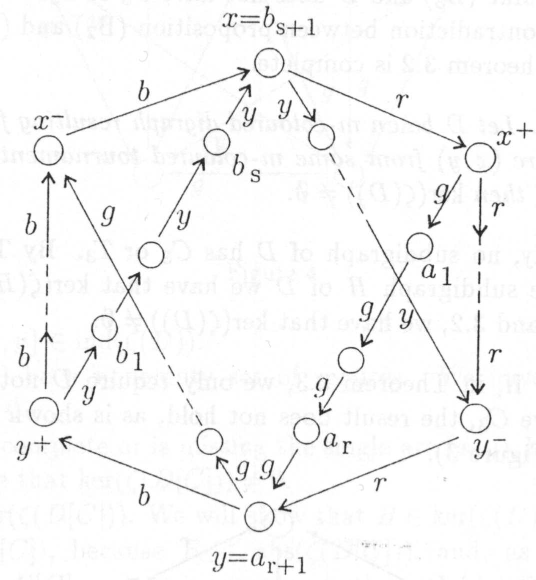

Figure 2

(B$_2$) The arc between a_r and b_s in D is red or yellow.
Since $(b_s, x) \in A(P(y))$ is yellow (recall that $x = b_{s+1}$), the arc between x and a_r in D is red (B$_1$) and D does not have C_3 or T_3.

(B$_3$) The arc between y and b_s in D is blue.
For $1 \leq i \leq s$, the arc between y and b_i is not yellow, otherwise, if $(y, b_i) \in A(D)$ was yellow, then there would be a monochromatic yellow path from y to $y-$ in D, namely $(y, b_i, b_{i+1}, \ldots, y-), (y, y-) \in A(\zeta(D))$ (a contradiction), and if $(b_i, y) \in A(D)$ was yellow, then there would be a monochromatic yellow path from $y+$ to y in D, namely $(y+ = b_0, b_1, b_2, \ldots, b_i, y), (y+, y) \in A(\zeta(D))$.

For $1 \leq i \leq s$, if the arc between y and b_{i-1} in D is blue, then the arc between y and b_i in D is also blue, because it is not yellow, $(b_{i-1}, b_i) \in A(D)$

is yellow, and D does not have C_3 or T_3. The arc between y and b_0 in D is blue, as $b_0 = y+$ and $(y, y+) \in A(y, \gamma, x)$ is blue. Finally, the arc between y and b_s in D is blue.

(B_4) The arc between a_r and b_s in D is green or blue.
Since $(a_r, y) \in A(P(x))$ is green (recall that $y = a_{r+1}$), the arc between y and b_s in D is blue (B_3) and D does not have C_3 or T_3.

We get a contradiction between proposition (B_2) and (B_4).
The proof of Theorem 3.2 is complete. ■

Theorem 3.3. *Let D be an m-coloured digraph resulting from the deletion of the single arc (x, y) from some m-coloured tournament. If D does not have C_3 or T_3, then* $\ker(\zeta(D)) \neq \emptyset$.

Proof. Clearly, no subdigraph of D has C_3 or T_3. By Theorem 2.1, for every complete subdigraph H of D we have that $\ker(\zeta(H)) \neq \emptyset$. So, by Theorems 3.1 and 3.2, we have that $\ker(\zeta(D)) \neq \emptyset$. ■

Remark 31. If, in Theorem 3.3, we only require D not to have T_3 and allow it to have C_3, the result does not hold, as is shown by the following example (see Figure 3).

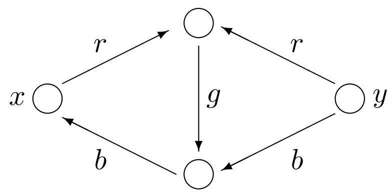

Figure 3

Remark 32. If, in Theorem 3.3, we only require D not to have C_3 and allow to have T_3, the result does not hold, as is shown by the following example (see Figure 4).

Theorem 3.4. *Let D be an m-coloured digraph resulting from the deletion of the single arc (x, y) from some m-coloured tournament. If D does not have C_3 or T_3, then $\zeta(D)$ is a KP-digraph.*

Proof. We consider two possibles cases:

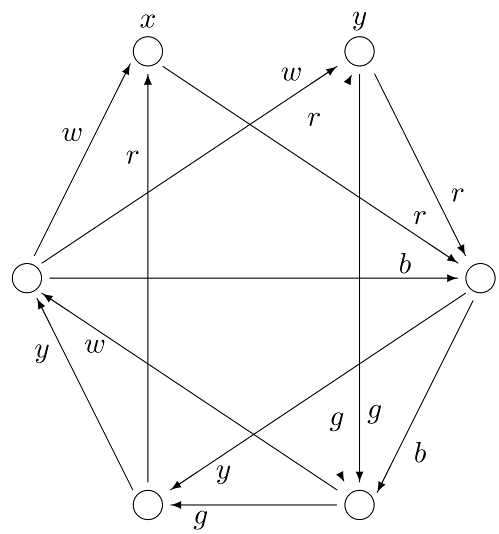

Figure 4

Case A. $\{x, y\} \in \text{ind}(\zeta(D))$.
Let $C \subseteq V(D)$ be a nonempty set of vertices. We have to prove that $\ker(\zeta(D)[C]) \neq \emptyset$.

As $D[C]$ is complete or is missing the single arc (x, y), by Theorems 21 and 3.3 we have that $\ker(\zeta(D[C])) \neq \emptyset$.

Let $B \in \ker(\zeta(D[C]))$. We will show that $B \in \ker(\zeta(D)[C])$.
$B \in \text{abs}(\zeta(D)[C])$, because $B \in \text{abs}(\zeta(D[C]))$, and, as the monochromatic paths in $D[C]$ are also monochromatic paths in $D, A(\zeta(D[C])) \subseteq A(\zeta(D)[C])$.

B has exactly one or two vertices, as the empty set is not absorbent, and in this digraph there can be no independent sets with more than two vertices.

We proceed by considering two subcases:

Subcase AA. B has one vertex.
Then $\ker(\zeta(D)[C]) \neq \emptyset$, because, as singular sets are independent, $B \in \ker(\zeta(D)[C])$. With this we conclude this subcase.

Subcase AB. B has two vertices.
Then $B = \{x, y\}$, as there cannot be other sets with two vertices independent in $\zeta(D[C])$. $B \in \text{ind}(\zeta(D))$, by assumption (A). $B \in \text{ind}(\zeta(D)[C])$, because $\zeta(D)[C][B] = \zeta(D)[B]$. (that means the subdigraph of $\zeta(D)[C]$

induced by B is the subdigraph of $\zeta(D)$ induced by B) Finally, $\ker(\zeta(D)[C]) \neq \emptyset$, because $B \in \ker(\zeta(D)[C])$. With this we conclude this subcase. Case A is now complete.

Case B. $\{x, y\} \notin \text{ind}(\zeta(D))$.
We proceed by contradiction. Assume $\zeta(D)$ is not a KP-digraph.

As every pair of vertices are adjacent in $\zeta(D), \zeta(D)$ is a complete digraph, and by Theorem 2.2, $\zeta(D)$ has an asymmetrical cycle.

Let γ be the shortest asymmetrical cycle contained in $\zeta(D)$.

For every two nonconsecutive vertices of γ, there are symmetrical arcs in $\zeta(D)$, otherwise, as $\zeta(D)$ is complete, we could get a shorter asymmetrical cycle in $\zeta(D)$.

$x, y \in V(\gamma)$, otherwise, $D[V(\gamma)]$ would be a complete digraph without C_3 or T_3, and by Theorem 2.1, there would be a vertex $v \in V(\gamma)$ such that $\{v\} \in \ker(\zeta(D[V(\gamma)]))$, and clearly $\{v\} \in \ker(\zeta(D)[V(\gamma)])$.

We get a contradiction to Theorem 3.2 as γ is an asymmetrical cycle in $\zeta(D)$, for every pair of nonconsecutive vertices of γ different from $\{x, y\}$ there are symmetrical arcs in $\zeta(D)$, and $x, y \in V(\gamma)$.

We have concluded Case (B).
The proof of Theorem 3.4 is complete. ■

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