

## DICHROMATIC NUMBER, CIRCULANT TOURNAMENTS AND ZYKOV SUMS OF DIGRAPHS

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### Abstract

The *dichromatic number*  $dc(D)$  of a digraph  $D$  is the smallest number of colours needed to colour the vertices of  $D$  so that no monochromatic directed cycle is created. In this paper the problem of computing the dichromatic number of a Zykov-sum of digraphs over a digraph  $D$  is reduced to that of computing a multicovering number of an hypergraph  $H_1(D)$  associated to  $D$  in a natural way. This result allows us to construct an infinite family of pairwise non isomorphic vertex-critical  $k$ -dichromatic circulant tournaments for every  $k \geq 3$ ,  $k \neq 7$ .

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## 1 Introduction

The *dichromatic number*  $dc(D)$  of a digraph  $D$  is the least number of colours needed to colour the vertices of  $D$  in such a way that each chromatic class is acyclic ([3, 9, 10]). It is apparent that this invariant measures in some way the complexity of the cyclic structure of digraphs. The importance of studying this invariant, introduced in [10], comes from the following fact: If  $G$  is a graph and  $G^*$  denotes the digraph obtained from  $G$  by orienting each one of the edges in both directions, then  $\chi(G) = dc(G^*)$ ; so the dichromatic number is a natural extension of the chromatic number to the class of all digraphs.

The structure of arc-critical  $k$ -dichromatic digraphs was investigated in [10] and consequently new remarkable properties of  $k$ -chromatic graphs were obtained there.

We continue here the study of vertex-critical  $k$ -dichromatic tournaments initiated in [15]. Related topics have been considered in [4, 5, 11].

Let  $H$  be an hypergraph without isolated vertices and suppose a positive integer  $\xi_u$  has been assigned to each vertex  $u$  of  $H$ ; the *covering number* of  $H$  corresponding to that assignment of weights is defined to be the minimum cardinality of a family of not necessarily different edges of  $H$  such that each vertex  $u$  belongs to at least  $\xi_u$  edges of the family.

Let  $D$  be a digraph and let  $H_1(D)$  be the hypergraph whose vertex set is  $V(D)$  and has the maximal acyclic subsets of  $V(D)$  as hyperedges. In this paper the problem of computing the dichromatic number of a Zykov-sum of digraphs over a digraph  $D$  is reduced to that of computing the covering number of  $H_1(D)$  with respect to an adequate assignment of weights (Theorem 4.2). We apply this result to construct an infinite family of pairwise non isomorphic vertex-critical  $k$ -dichromatic circulant tournaments for every  $k \geq 3$ ,  $k \neq 7$ . This improves previous results included in [15]. Other related results are also presented.

## 2 Preliminary Results and Terminology

For general concepts we refer the reader to [2].

Let  $D$  be a digraph;  $V(D)$  and  $A(D)$  will denote the sets of vertices and arcs of  $D$  respectively,  $o(D) = |V(D)|$  is the *order* of  $D$ ;  $D$  is *acyclic* provided no directed cycle is contained in  $D$ . The subdigraph of  $D$  induced by a subset  $S$  of  $V(D)$  will be denoted by  $D[S]$ ;  $S$  is said to be *acyclic* iff  $D[S]$  is acyclic. The maximal cardinality of an acyclic set of vertices of  $D$  will be denoted by  $\beta^{\rightarrow}(D)$ . A colouring of  $V(D)$  is *acyclic* if all the chromatic classes are acyclic. So the dichromatic number  $dc(D)$  of a digraph  $D$  is the minimum number of colours in an acyclic colouring of  $V(D)$ . Clearly  $dc(D^{\text{op}}) = dc(D)$  where  $D^{\text{op}}$  is obtained from  $D$  by reversing each one of its arcs.

$D$  is called  *$r$ -dichromatic* if  $dc(D) = r$  and *vertex-critical  $r$ -dichromatic* if  $dc(D) = r$  and  $dc(D - u) < r$  for every  $u \in V(D)$ .

$\mathbb{N}$  will denote the set of nonnegative integers,  $I_n = \{1, \dots, n\}$  and  $Z_n$  is the set of integers mod  $n$ . For any nonempty subset  $J$  of  $Z_n - \{0\}$ , the *circulant digraph*  $\vec{C}_n(J)$  is defined by  $V(\vec{C}_n(J)) = Z_n$  and  $A(\vec{C}_n(J)) = \{(i, j) : i, j \in Z_n \text{ and } j - i \in J\}$ . In particular,  $\vec{C}_n(\{1\})$  is the directed cycle

$\vec{C}_n$ ;  $\vec{C}_{2m+1}(J)$  is a circulant tournament whenever  $|\{j, -j\} \cap J| = 1$  for every  $j \in Z_{2m+1} - \{0\}$ . If  $i, j \in V(\vec{C}_n)$ ,  $A_{i,j}$  will denote the directed  $ij$ -path in  $\vec{C}_n$ . For  $j \in I_m$ ,  $I_{m,j}$  will denote the set  $I_m \cup \{2m + 1 - j\} - \{j\}$  considered as a subset of  $Z_{2m+1}$ .

In [12] it was proved that there is only one 4-dichromatic oriented graph of order at most 11, namely  $\vec{C}_{11}(I_{5,2})$ ; this tournament is not only vertex-critical but also arc-critical. In [13] it was proved that  $\vec{C}_{6m+1}(I_{3m,2m})$  is a vertex-critical 4-dichromatic circulant tournament for  $m \geq 2$ . In a previous paper [15] an infinite family of vertex-critical  $r$ -dichromatic regular tournaments was constructed for each  $r \geq 3$ ,  $r \neq 4$ . However these tournaments were circulants only for  $r = 3, 5, 8$ .

We will need the following

**Lemma 2.1** [13]. *For any two integers  $r, s$  such that  $1 \leq s < r$  holds  $\beta^{\vec{C}}(H_{r,s}) = r$  where  $H_{r,s}$  is the tournament defined by  $V(H_{r,s}) = \{1, 2, \dots, r + s\}$  and  $A(H_{r,s}) = \{(i, j) : (i < j \text{ and } j - i \neq r)\} \cup \{(i + r, i) : i \leq s\}$ .*

### 3 Multicoverings of Hypergraphs

If  $H = (V(H), E(H))$  is an hypergraph, the *rank*  $\rho(H)$  of  $H$  is defined to be the maximum cardinality of an edge of  $H$ ;  $H$  is an  $r$ -*graph* if each one of its edges has cardinality  $r$ .

Let  $H$  be a finite hypergraph without isolated points. A function  $\xi: V(H) \rightarrow \mathbb{N}$  will be called a *weight function* (w.f.) on  $H$ ;  $\xi$  will be said to be *degenerate* if  $\xi^{-1}(0) \neq \emptyset$ . We define  $\|\xi\| = \sum_{w \in V(H)} \xi(w)$  and denote by  $\mathbf{k}$  the w.f. on  $H$ , which has constant value  $k$ . Let  $(\alpha_j)_{j \in J}$  be a family of edges of  $H$  and  $u \in V(H)$ ; define  $J_u = \{j \in J : u \in \alpha_j\}$ . We will say that  $(\alpha_j)_{j \in J}$  is a  $\xi$ -*covering* of  $H$  whenever  $|J_u| \geq \xi(u)$  for every  $u \in V(H)$ . Finally, we define the  $\xi$ -*covering number*  $\tilde{n}(H, \xi)$  of  $H$  by  $\tilde{n}(H, \xi) = \min \{|J| : (\alpha_j)_{j \in J} \text{ is a } \xi\text{-covering of } H\}$ . So the  $\mathbf{k}$ -*covering number* of  $H$  is the usual (multi)covering number which has been studied in many articles (see [1]).

**Remark 31.** Note that if  $H'$  is the spanning subhypergraph of  $H$  whose edges are the maximal edges of  $H$ , then  $\tilde{n}(H, \xi) = \tilde{n}(H', \xi)$ .

**Proposition 32.**

- (i)  $\tilde{n}(H, \xi + \xi') \leq \tilde{n}(H, \xi) + \tilde{n}(H, \xi')$  and  $\tilde{n}(H, k\xi) \leq k\tilde{n}(H, \xi)$  for every positive integer  $k$ .

- (ii)  $\tilde{n}(H, \xi) \leq \tilde{n}(H, \xi')$  whenever  $\xi \leq \xi'$ .
- (iii)  $\tilde{n}(H, \xi) \geq \lceil \|\xi\|/\rho(H) \rceil$ .
- (iv) If  $H_0$  is a spanning subhypergraph of  $H$  then  $\tilde{n}(H, \xi) \leq \tilde{n}(H_0, \xi)$ .

**Proof.** Properties (i), (ii) and (iv) are obvious, Property (iii) follows from the inequality  $\rho(H)\tilde{n}(H, \xi) \geq \|\xi\|$ . ■

An hypergraph  $H$  is called *circulant* if it has an automorphism which is a cyclic permutation of  $V(H)$ . If  $r \leq m$ , the circulant  $r$ -graph  $\Lambda_{m,r}$  is defined by  $V(\Lambda_{m,r}) = \mathbb{Z}_m$  and  $E(\Lambda_{m,r}) = \{\alpha_j: j \in \mathbb{Z}_m\}$  where  $\alpha_j = \{j, j+1, \dots, j+r-1\}$  for  $j \in \mathbb{Z}_m$ . For every positive integer  $s$ , we define the w.f.  $\xi^{(s)}$  on  $\Lambda_{m,r}$  as follows: If  $sr = qm + t$  where  $t$  is the residue of  $sr$  mod  $m$ , then  $\xi^{(s)}(j) = q$  or  $q+1$  depending on whether  $j$  belongs or not to  $A_{t,m-1}$ . In particular,  $\xi^{(s)} = \mathbf{q}$  when  $t = 0$ . Notice that  $\|\xi^{(s)}\| = sr$ .

**Proposition 33.** *If  $H$  contains  $\Lambda_{m,r}$  as a spanning subhypergraph and  $\rho(H) = r$  then  $\tilde{n}(H, \xi^{(s)}) = s$  and  $\tilde{n}(H, \xi') > s$  whenever  $\|\xi'\| > \|\xi^{(s)}\|$ .*

**Proof.** The family  $\{\alpha_j: j = rj', j' = 0, 1, \dots, s\}$  is a  $\xi^{(s)}$ -covering of  $H$  and so  $\tilde{n}(H, \xi^{(s)}) \leq s$ . The equality and the second inequality follow from Proposition 3.2 (iii) and the fact that  $\|\xi'\| > \|\xi^{(s)}\| = sr$ . ■

**Proposition 34.** *Let  $k$  be a positive integer. If  $\rho(H) = r$  and  $H$  contains an isomorphic copy of  $\Lambda_{m,r}$  as a spanning subhypergraph, then  $\tilde{n}(H, \mathbf{k}) = \tilde{n}(\Lambda_{m,r}, \mathbf{k}) = \lceil km/r \rceil$ .*

**Proof.** We may assume that  $\Lambda_{m,r}$  is a spanning subhypergraph of  $H$ . The inequality  $\tilde{n}(H, \mathbf{k}) \geq \lceil km/r \rceil$  follows from Proposition 3.2 (iii). Since  $\xi^{(s)} \geq \mathbf{k}$  for  $s = \lceil km/r \rceil$ , the equality is obtained by applying Propositions 3.2 and 3.3. ■

Proposition 3.4 applies in particular to  $K_m^{(r)}$ , the complete  $r$ -graph of order  $m$ .

## 4 Zykov Sums and Dichromatic Number

Let  $D$  be a digraph and  $\alpha = (\alpha_i)_{i \in V(D)}$  a family of nonempty mutually disjoint digraphs. The Zykov sum  $\sigma(\alpha, D)$  of  $\alpha$  over  $D$  is defined by  $V(\sigma(\alpha, D)) = \bigcup_{i \in V(D)} V(\alpha_i)$ ;  $A(\sigma(\alpha, D)) = \bigcup_{i \in V(D)} A(\alpha_i) \cup \{uw: u \in V(\alpha_i), w \in V(\alpha_j), ij \in A(D)\}$ .

If the members of the family  $\alpha$  are not mutually disjoint we replace each of them by one isomorphic copy so that the new family  $\alpha'$  becomes one of mutually disjoint digraphs; nevertheless  $\sigma(\alpha, D)$  will still denote the resulting digraph  $\sigma(\alpha', D)$  which is defined up to isomorphism. The function  $p: \sigma(\alpha, D) \rightarrow D$  whose value is constant in each  $\alpha_u$  and equal to  $u$ , is a reflexive epimorphism which will be called the *natural projection* from  $\sigma(\alpha, D)$  onto  $D$ . If  $\alpha_i \cong W$  for every  $i \in V(D)$  we will write  $D[W]$  instead of  $\sigma(\alpha, D)$ .

In [10] it was proved that  $dc(D[W]) \geq dc(D) + dc(W) - 1$ . In [15],  $t(\alpha_1, \alpha_2, \alpha_3)$  denoted the same as  $\sigma(\alpha, D)$  for  $D = \vec{C}_3$  and  $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ .

Now, if  $D$  is a digraph, the hypergraph  $H_1(D)$  is defined by  $V(H_1(D)) = V(D)$ ,  $E(H_1(D)) = \{S \subseteq V(D) : S \text{ is a maximal acyclic set}\}$ .

**Proposition 41.**

- (i)  $H_1(\vec{C}_{2m+1}(I_m)) \supseteq \Lambda_{2m+1, m+1}, \quad \beta^\rightarrow(\vec{C}_{2m+1}(I_m)) = m + 1.$
- (ii)  $H_1(\vec{C}_{2m+1}(I_{m,m})) \supseteq \Lambda_{2m+1, m}, \quad \beta^\rightarrow(\vec{C}_{2m+1}(I_{m,m})) = m.$
- (iii)  $H_1(\vec{C}_{6m+1}(I_{3m, 2m})) \supseteq \Lambda_{6m+1, 2m}, \quad \beta^\rightarrow(\vec{C}_{6m+1}(I_{3m, 2m})) = 2m.$
- (iv)  $H_1(\vec{C}_{17}(I_{8,5})) \supseteq \Lambda_{17,5}, \quad \beta^\rightarrow(\vec{C}_{17}(I_{8,5})) = 5.$
- (v)  $H_1(\vec{C}_{17}(I_{8,7})) \supseteq \Lambda_{17,7}, \quad \beta^\rightarrow(\vec{C}_{17}(I_{8,7})) = 7 \text{ and}$
- (vi)  $H_1(\vec{C}_{17}(I_{8,6})) \supseteq \Lambda_{17,6}, \quad \beta^\rightarrow(\vec{C}_{17}(I_{8,6})) = 6.$

**Proof.** (i) is trivial, (ii) and (iii) were proved in [15] and [13] respectively. The inclusions of (iv), (v) and (vi) are obvious. Let  $T^j = \vec{C}_{17}(I_{8,j})$ ,  $j = 5, 6, 7$  and notice that  $A_{i(i+j-1)}$  is an acyclic set of cardinality  $j$ . Let  $S_j$  be an acyclic set of  $T^j$ . We will prove that  $|S_j| \leq j$ . We may assume that 0 is the source of  $T^j[S_j]$ . Let  $N_j$  be the out neighbourhood of 0 in  $T^j$ . So  $S_j - \{0\} \subseteq N_j$ . Notice that  $T^j[N_j - \{17-j\}] \cong H_{j-1, 8-j}$  (the correspondence  $i \rightarrow i$  for  $0 < i \leq j-1$  and  $i \rightarrow i+1$  for  $j \leq i \leq 7$  is an isomorphism from  $H_{j-1, 8-j}$  onto  $T^j[N_j - \{17-j\}]$ ) and  $j-1 > 8-j$ . So by Lemma 2.1,  $|S_j| \leq j$  whenever  $17-j \notin S_j$ . We assume that  $17-j \in S_j$ .

*Case  $j = 5$ .* We have  $12 \in S_5$ . If  $4 \in S_5$  then  $S_5 \cap \{1, 2, 3\} = \emptyset$  and since  $|S_5 \cap \{7, 8\}| \leq 1$  we obtain  $|S_5| \leq 5$ . If  $4 \notin S_5$  and  $8 \in S_5$  then  $S_5 \cap \{1, 2, 7\} = \emptyset$  and so  $|S_5| \leq 5$ . Finally if  $S_5 \cap \{4, 8\} = \emptyset$ , then since  $T^5[N_5] - \{0, 4, 8, 12\} \cong H_{3,2}$ , we conclude by Lemma 2.1 that  $|S_5| \leq 5$ . So the proof of (iv) is complete.

*Case  $j = 7$ .* We have  $10 \in S_7$ . If  $\{1, 3\} \cap S_7 \neq \emptyset$  then  $\{4, 5, 6\} \cap S_7 = \emptyset$  and again by Lemma 2.1,  $|S_7| \leq 5$ . In the remaining case  $|S_7| \leq 7$ . So (v) holds.

*Case  $j = 6$ .* Since  $11 \in S_6$ , either  $\{1, 2\} \cap S_6 = \emptyset$  or  $\{3, 4\} \cap S_6 = \emptyset$ . Moreover  $|S_6 \cap \{5, 7\}| \leq 1$ , therefore  $|S_6| \leq 6$  and the proof of (vi) ends. ■

Let  $D$  be a digraph and  $Q = (Q_u)_{u \in V(D)}$  a family of digraphs. Define the w.f.  $\xi_Q: V(D) \rightarrow \mathbb{N}$  by  $\xi_Q(u) = dc(Q_u)$ .

**Theorem 42.**  $dc(\sigma(D, Q)) = \tilde{n}(H_1(D), \xi_Q)$ .

**Proof.** We may assume that  $Q$  is formed with mutually disjoint digraphs and so  $Q_u \subseteq \sigma(D, Q)$ . Let  $p: \sigma(D, Q) \rightarrow D$  be the natural projection, so  $p(Q_u) = u$  for every  $u \in V(D)$ . Let  $(\alpha_j)_{j \in J}$  be an optimal  $\xi_Q$ -covering of  $H_1(D)$ ; then  $|J| = \tilde{n}(H_1(D), \xi_Q)$ . Define a colouring  $f$  of  $\sigma(D, Q)$  with  $J$  as set of colours, as follows: For each  $u \in V(D)$ , take an acyclic colouring of  $Q_u$  with colours in  $J_u$  (this is possible because  $Q_u$  is  $\xi_Q(u)$ -dichromatic and  $|J_u| \geq \xi_Q(u)$ ). Let  $C$  be a directed cycle of  $\sigma(D, Q)$ . If  $C \subseteq Q_u$  for some  $u$ ,  $C$  is not monochromatic. Otherwise,  $p(C)$  contains a directed cycle  $C_0$ . If  $C$  were monochromatic of colour  $j$ ,  $\alpha_j \supseteq V(p(C)) \supseteq V(C_0)$  which is impossible since  $\alpha_j$  is acyclic. Then  $C$  is not monochromatic and  $f$  is an acyclic colouring. Therefore  $dc(\sigma(D, Q)) \leq \tilde{n}(H_1(D), \xi_Q)$ . Let  $J$  be a set of cardinality  $dc(\sigma(D, Q))$  and  $f: \sigma(D, Q) \rightarrow J$  an optimal acyclic colouring of  $\sigma(D, Q)$ . Denote by  $R_j$  the chromatic class of colour  $j$ . Then  $\alpha_j = p(R_j)$  is an acyclic subset of  $V(D)$  since  $R_j$  is acyclic and so  $\alpha_j \in E(H_1(D))$ . Since  $J_u = \{j: u \in \alpha_j\}$ ,  $j \in J_u$  if and only if  $R_j \cap V(Q_u)$  is nonempty, then  $|J_u| \geq dc(Q_u) = \xi_Q(u)$  and  $(\alpha_j)_{j \in J}$  is a  $\xi_Q$ -covering of  $H_1(D)$ . Therefore  $\tilde{n}(H_1(D), \xi_Q) \leq dc(\sigma(D, Q))$  and the proof is complete. ■

From here on, we will write  $\tilde{n}_1(D, \xi)$  instead of  $\tilde{n}(H_1(D), \xi)$ . Note that  $\tilde{n}_1(D, \mathbf{1}) = dc(D)$ .

**Corollary 43.** *If  $dc(\alpha) = k$  then  $dc(D[\alpha]) = \tilde{n}_1(D, \mathbf{k})$ .*

Let  $\xi$  be a w.f. on  $\vec{C}_3$  such that  $\xi_0 \geq \xi_1 \geq \xi_2$  where  $\xi(j) = \xi_j$ . In [15], the following result was proved.

**Proposition 44.**  $\tilde{n}_1(\vec{C}_3, \xi) = \lceil (\xi_0 + \xi_1 + \xi_2)/2 \rceil$  or  $\xi_0$  depending on whether  $\xi_0 \leq \xi_1 + \xi_2$  or  $\xi_1 + \xi_2 \leq \xi_0$ . In particular  $\tilde{n}_1(\vec{C}_3, \mathbf{k}) = \lceil 3k/2 \rceil$ .

**Proposition 45.**

- (i)  $\tilde{n}_1(\vec{C}_{2m+1}(I_m), \mathbf{k}) = \lceil k(2m+1)/(m+1) \rceil$  for  $m \geq 2$ .
- (ii)  $\tilde{n}_1(\vec{C}_{2m+1}(I_{m,m}), \mathbf{k}) = \lceil k(2m+1)/m \rceil$  for  $m \geq 3$ .
- (iii)  $\tilde{n}_1(\vec{C}_{6m+1}(I_{3m,2m}), \mathbf{k}) = \lceil k(6m+1)/2m \rceil$  for  $m \geq 2$ .

- (iv)  $\tilde{n}_1(\vec{C}_{17}(I_{8,5}), \mathbf{k}) = \lceil 17k/5 \rceil$ ,  $\tilde{n}_1(\vec{C}_{17}(I_{8,7}), \mathbf{k}) = \lceil 17k/7 \rceil$  and  $\tilde{n}_1(\vec{C}_{17}(I_{8,6}), \mathbf{k}) = \lceil 17k/6 \rceil$ .

**Proof.** The equalities follow directly from Proposition 4.1 and Proposition 3.4. ■

**4.6. An application.** If  $\xi: Z_7 \rightarrow \mathbb{N}$  is defined by  $\xi(j) = 2$  for  $j \neq 0$  and  $\xi(0) = 1$ , then for  $T = \vec{C}_7(1, 2, 4)$ ,  $\tilde{n}_1(T, \xi) \geq \lceil 13/3 \rceil = 5$  by Proposition 3.2 (iii) since  $\beta^{\rightarrow}(\vec{C}_7(1, 2, 4)) = 3$ . By Proposition 3.3,  $\tilde{n}_1(T, \xi^{(5)}) = 5$  and since  $\xi \leq \xi^{(5)}$  it follows from Proposition 3.2 that  $\tilde{n}_1(T, \xi) = 5$ . Define  $Q_0 = T_1$  and  $Q_j = \vec{C}_3$  for  $j \in Z_7 - \{0\}$ . Because of Theorem 4.2,  $\sigma(T, (Q_u)_{u \in V(T)})$  is a 5-dichromatic tournament of order 19. The minimum order of a 5-dichromatic tournament is not known, this example shows that it is not bigger than 19. It can be proved that it is at least 17.

Let  $G^*$  be the digraph obtained from a graph  $G$  by orienting each one of the edges in both directions. Some properties and the behaviour of the function  $\tilde{n}_1(G^*, \mathbf{k})$  have been studied in several papers [6, 7, 8, 17].

## 5 Subcritical and Upcritical Weight Functions

A weight function  $\xi$  on  $H$  is said to be  $H$ -*subcritical* if for every w.f.  $\xi'$  such that  $\xi' \leq \xi$  and  $\|\xi'\| = \|\xi\| - 1$ , we have  $\tilde{n}(H, \xi') < \tilde{n}(H, \xi)$  (and therefore  $\tilde{n}(H, \xi') = \tilde{n}(H, \xi) - 1$ ). For brevity we will write  $D$ -*subcritical* instead of  $H_1(D)$ -subcritical.

Notice that the w.f.  $\xi$  considered in Proposition 4.4 is  $\vec{C}_3$ -subcritical iff  $\xi_0 \leq \xi_1 + \xi_2$  and  $\xi_0 + \xi_1 + \xi_2$  is odd [15].

**Theorem 51.** *If for every  $u \in V(D)$ ,  $Q_u$  is a vertex-critical  $\xi_Q(u)$ -dichromatic digraph and  $\xi_Q$  is  $D$ -subcritical, then  $\sigma(D, Q)$  is vertex-critical  $\tilde{n}_1(D, Q)$ -dichromatic.*

**Proof.** This follows directly from Theorem 4.2. ■

It is not difficult to prove that the w.f.  $\xi$  defined in 4.6 is  $\vec{C}_7(1, 2, 4)$ -subcritical. Therefore the tournament  $\sigma(T, (Q_u)_{u \in V(T)})$  constructed there is vertex-critical.

**Theorem 52.**

- (i)  $\mathbf{k}$  is  $\vec{C}_{2m+1}(I_m)$ -subcritical iff  $k \equiv m \pmod{m+1}$  and  $m \geq 2$ .
- (ii)  $\mathbf{k}$  is  $\vec{C}_{2m+1}(I_{m,m})$ -subcritical iff  $k \equiv 1 \pmod{m}$  and  $m \geq 3$ .

- (iii)  $\mathbf{k}$  is  $\vec{C}_{6m+1}(I_{3m,2m})$ -subcritical iff  $k \equiv 1 \pmod{2m}$  and  $m \geq 2$ .
- (iv)  $\mathbf{k}$  is
  - $\vec{C}_3$ -subcritical iff  $k$  is odd,
  - $\vec{C}_{17}(I_{8,5})$ -subcritical iff  $k \equiv 3 \pmod{5}$ ,
  - $\vec{C}_{17}(I_{8,7})$ -subcritical iff  $k \equiv 5 \pmod{7}$ ,
  - $\vec{C}_{17}(I_{8,6})$ -subcritical iff  $k \equiv 5 \pmod{6}$ .

**Proof.** It follows from Proposition 4.4 that  $\mathbf{k}$  is  $\vec{C}_3$ -subcritical iff  $k$  is odd. Let  $T$  any of the tournaments of (i), (ii) or (iii) and let  $\xi$  be a w.f. such that  $\xi \leq \mathbf{k}$ ,  $\|\xi\| = \|\mathbf{k}\| - 1$ . From Proposition 4.5 it follows immediately that  $\tilde{n}(T, \xi) = \tilde{n}(T, \mathbf{k})$  unless  $k \equiv m \pmod{2m+1}$  in case (i),  $k \equiv 1 \pmod{2m+1}$  in case (ii) or  $k \equiv 1 \pmod{2m}$  in case (iii). In these last cases  $r = \beta^\rightarrow(T)$  divides  $\|\xi\|$ . Since  $\text{Aut}(T)$  is vertex transitive, we may assume that  $\xi = \xi^{(s)}$  for  $s = \|\xi\|/r$  and the assertion follows from Proposition 3.3. The remaining cases can be proved in a similar way. ■

**5.3. Another application.** Let  $\xi: Z_7 - \{0\} \rightarrow \mathbb{N}$  be defined by  $\xi(j) = 1$  for  $j \in \{1, 2, 3, 4, 5\}$  and  $\xi(6) = 2$ . It is easy to see that  $\xi$  is  $ST_6$ -subcritical where  $ST_6 = \vec{C}_7(1, 2, 4) - \{0\}$  and  $\tilde{n}_1(ST_6, \xi) = 3$ . Proceeding as in the example of 4.6, a vertex-critical 3-dichromatic tournament  $T^{(3)}$  of order 8 is obtained. Let  $T^{(m)}$  (resp:  $W^{(m)}$ ) denote a generic vertex-critical  $m$ -dichromatic tournament of even (resp: odd) order. Recall that  $t(T^{(m)}, W^{(m)}, T_1)$  is a vertex-critical  $(m + 1)$ -dichromatic tournament of even order and that there are infinitely many pairwise non isomorphic tournaments  $W^{(3)}$  [15]. Using induction, it follows that an infinite family of pairwise non isomorphic vertex-critical  $r$ - dichromatic tournaments of even order can be constructed for every integer  $r \geq 4$ . This solves a question of [15].

After considering subcritical w.f., we define in a similar way a w.f.  $\xi$  on  $H$  to be  $H$ -upcritical if for every w.f.  $\xi'$  such that  $\xi \leq \xi'$  and  $\|\xi'\| = \|\xi\| + 1$ , we have  $\tilde{n}(H, \xi) < \tilde{n}(H, \xi')$  (and therefore  $\tilde{n}(H, \xi') = \tilde{n}(H, \xi) + 1$ ). For brevity we will write  $D$ -upcritical instead of  $H_1(D)$ -upcritical.

As an example, Proposition 3.3 asserts that the w.f.  $\xi^{(s)}$  is  $H$ -upcritical. Notice that the w.f.  $\xi$  considered in Proposition 4.4, is  $\vec{C}_3$ -upcritical iff  $\xi_0 \leq \xi_1 + \xi_2$  and  $\xi_0 + \xi_1 + \xi_2$  is even [16, Lemma 2]. Lemma 3 in [16] can be easily generalized as follows.

**Theorem 53.** *If  $\xi_Q$  is  $D$ -upcritical then every acyclic  $\tilde{n}_1(D, Q)$ -colouring of  $\sigma(D, Q)$  induces in each  $Q_u$  an optimal acyclic colouring.* ■



## 6 Vertex-Critical $r$ -Dichromatic Circulant Tournaments

In this section we will prove the existence of vertex-critical  $k$ -dichromatic circulant tournaments for every  $k \geq 3, k \neq 7$ . We will use the fact that the composition of two circulant tournaments is a circulant tournament [14, Proposition 3.3].

Let  $f_0, f'_0, f_1$  and  $f'_1$  be the functions with codomain  $\mathbb{N}^2$  defined by:

- (1)  $f_0(r, m) = r(2m + 1) - 1, \quad f'_0(r, m) = r(m + 1) - 1$   
for  $r \geq 1, m \geq 2$ .
- (2)  $f_1(r, m) = r(2m + 1) + 3, \quad f'_1(r, m) = rm + 1$   
for  $r \geq 1, m \geq 3$ .

**Lemma 61.** *If  $x$  is an integer then  $x \in \text{Image}(f_0) \cup \text{Image}(f_1)$  iff  $x \geq 4$  and  $x \notin \{5, 7, 11, 15, 23\}$ .*

**Proof.** Take  $X = \text{Image}(f_0) \cup \text{Image}(f_1)$ . Clearly  $x \in X$  implies  $x \geq 4$ . If  $x$  is an even number,  $x \geq 4$ , then  $x \in \text{Image}(f_0)$ . Let  $x = 2x_1 + 1$  with  $x_1 \geq 2$  and  $x \notin X$ . Then  $2x_1 + 2$  has no odd divisor bigger than 3 and  $2x_1 - 2$  has no odd divisor bigger than 5. So,  $x_1 + 1 = 2^t \cdot i_1$  and  $x_1 - 1 = 2^s \cdot i_2$  where  $i_1 \in \{1, 3\}$  and  $i_2 \in \{1, 3, 5\}$ . It follows that either  $t \leq 1$  or  $s \leq 1$ . In the first case  $x \in \{5, 11\}$ , in the second,  $x \in \{5, 9, 13, 7, 15, 23\}$ . However  $\{9, 13\} \subseteq \text{Image}(f_1)$  and therefore  $x \in \{5, 7, 11, 15, 23\}$ . It can be easily verified that in fact these values do not belong to  $X$ . ■

Let  $D_j$  be the (acyclic) digraph whose vertices are the integers bigger than 2 and whose arcs are the pairs of the form  $(f'_j(r, m), f_j(r, m)), j = 0, 1$  and take  $D = D_0 \cup D_1$ . It is easy to prove that  $D_0$  and  $D_1$  are arc disjoint. We assign to each arc  $\tau = (f'_j(r, m), f_j(r, m))$  the weight  $\omega(\tau) = 2m + 1$  and a digraph operator  $\hat{\tau}$  so that  $\hat{\tau}(\alpha) = \vec{C}_{2m+1}(I_m)[\alpha]$  if  $j = 0$  and  $\hat{\tau}(\alpha) = \vec{C}_{2m+1}(I_{m,m})[\alpha]$  if  $j = 1$ . If  $\pi = (u_0, \tau_1, u_1, \tau_1, u_2, \dots, u_{n-1}, \tau_n, u_n)$  is a directed path in  $D$  we define  $\hat{\pi} = \hat{\tau}_n \circ \dots \circ \hat{\tau}_2 \circ \hat{\tau}_1$  and  $\omega(\pi) = \omega(\tau_n) \dots \omega(\tau_1)$ .

Using Corollary 4.3, Proposition 4.5 and Theorems 5.1 and 5.2 we obtain the following

**Lemma 62.** *If  $\alpha$  is a vertex-critical  $u_0$ -dichromatic circulant tournament then  $\hat{\pi}(\alpha)$  is a vertex-critical  $u_n$ -dichromatic circulant tournament such that  $o(\hat{\pi}(\alpha)) = o(\alpha)\omega(\pi)$ .* ■

**Remark 63.** Using Lemma 6.1 it follows immediately that the set of vertices of  $D$  with indegree 0 is  $\{3, 4, 5, 7, 11, 15, 23\}$ .

**Lemma 64.** *For each integer  $n \geq 3$ ,  $n \neq 7$  there is a directed path in  $D$  from a vertex in  $\{3, 4, 5, 11, 13, 15, 23\}$  to  $n$ .*

**Proof.** Let  $B = \{3, 4, 5, 11, 13, 15, 23\}$  and  $W = \{w \in V(D) : \text{there is a directed } Bw\text{-path in } D\}$ . Since  $(3, 6), (4, 8), (5, 9), (5, 10), (6, 12), (8, 14), (8, 16), (9, 17), (9, 18), (10, 20), (11, 19), (11, 20), (11, 21), (11, 22), (12, 24) \in A(D_1)$  then  $I_{24} - \{1, 2, 7\} \subseteq W$ . We will prove that  $K = \mathbb{N} - \{1, 2, 7\} = W$ . The proof is by induction. Let  $n \geq 25$  such that  $s \in W$  whenever  $s \leq n - 1$ ,  $s \in K$ . Because of Remark 6.3 there exists a  $k$  such that  $(k, n) \in A(D)$ . Now  $k < n$  and  $k \notin \{1, 2, 7\}$  since the only  $\{1, 2, 7\}w$ -arcs of  $D$  are  $(2, 4), (7, 13)$  and  $(7, 14)$ . Therefore  $k \in K$  and so  $n \in K$ . ■

**Proposition 65.** *For every integer  $k \in \{3, 4, 5, 11, 13, 15, 23\}$  there exists an infinite family  $\mathcal{F}_k$  of vertex-critical  $k$ -dichromatic circulant tournaments no two of them having the same order.*

**Proof.** The families  $\mathcal{F}_j$  for  $j = 3, 4$  and  $5$  are the following:

$\mathcal{F}_3 = \{\vec{C}_{2m+1}(I_{m,m}) : m \geq 3\}$ ,  $\mathcal{F}_5 = \{\vec{C}_3[\vec{C}_{2m+1}(I_{m,m})] : m \geq 3\}$  [15];  $\mathcal{F}_4 = \{\vec{C}_{6m+1}(I_{3m,2m}) : m \geq 2\}$  [13]. Define now  $\mathcal{F}_{11} = \{\vec{C}_{17}(I_{8,5})[\alpha] : \alpha \in \mathcal{F}_3\}$ ;  $\mathcal{F}_{13} = \{\vec{C}_{17}(I_{8,7})[\alpha] : \alpha \in \mathcal{F}_5\}$ ;  $\mathcal{F}_{15} = \{\vec{C}_{17}(I_{8,6})[\alpha] : \alpha \in \mathcal{F}_5\}$ . That these last 3 families satisfy the required conditions is a direct consequence of Corollary 4.3, Proposition 4.5 and Theorems 5.1 and 5.2 and the fact that for each  $j \in \{11, 13, 15\}$ , all the members of  $\mathcal{F}_j$  have different orders. Finally define the family  $\mathcal{F}_{23} = \{\vec{C}_3[\alpha] : \alpha \in \mathcal{F}_{15}\}$  which satisfies the required conditions because of Proposition 4.4 and Theorems 4.2, 5.1 and 5.2. ■

**Theorem 66.** *For every integer  $k \geq 3$ ,  $k \neq 7$  there exists an infinite family  $\mathcal{F}_k$  of pairwise non isomorphic vertex-critical  $k$ -dichromatic circulant tournaments.*

**Proof.** In fact, we will construct for each  $k \geq 3$ ,  $k \neq 7$  an infinite family  $\mathcal{F}_k$  of vertex-critical  $k$ -dichromatic circulant tournaments such that all its members have different orders. By Lemma 6.4 there is in  $D$  a directed  $uk$ -path  $\pi$  with  $u \in \{3, 4, 5, 11, 13, 15, 23\}$ . Define  $\mathcal{F}_k = \{\hat{\pi}(\alpha) : \alpha \in \mathcal{F}_u\}$ . By Lemmas 6.2 and 6.5,  $\mathcal{F}_k$  has the required properties. ■

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