Problems Column

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SOME CONJECTURES ON PERFECT GRAPHS

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The complement of a graph G is denoted by $\overline{G}$. $\chi(G)$ denotes the chromatic number and $\omega(G)$ the clique number of G. The cycles of odd length at least five are called *odd holes* and the complements of odd holes are called *odd anti-holes.*

1.

A graph G is called *perfect* if, for each induced subgraph G' of G, $\chi(G') = \omega(G')$. Classical examples of perfect graphs consist of bipartite graphs, chordal graphs and comparability graphs. Examples of *nonperfect* graphs are the odd holes and odd anti-holes. The most important result on perfect graphs is the following one, due to L. Lovász.

The Perfect Graph Theorem ([6])**.** *The complement of a perfect graph is perfect.*

The Perfect Graph Theorem used to be the so-called weak perfect graph conjecture posed by C. Berge around 1960s. His stronger conjecture on perfect graphs, which is still open, is as follows.

The Strong Perfect Graph Conjecture. *Graphs without induced odd holes and odd anti-holes are perfect.*

Clearly, if the Strong Perfect Graph Conjecture is true, it implies the Perfect Graph Theorem. See [1, 2, 3, 4] for more information on perfect graphs.

2.

The edge-version of graph perfection has been considered by L.E. Trotter [8]. It can be formulated in terms of line graphs as follows. The well-known *line graph* $L(G)$ of a graph G has the edge set of G as its vertex set, and two distinct edges of G are adjacent in $L(G)$ if and only if they have an endvertex in common; see Figure 1. A graph H is a *line graph* if there exists a graph G such that H is (isomorphic to) $L(G)$. It is well-known that line graphs can be recognized in linear time.

A graph G is called *line perfect* if its line graph $L(G)$ is perfect. It is well-known that $L(G)$ is perfect if and only if $L(G)$ contains no induced odd holes. Trotter proved that a graph is line perfect if and only if it has no (not necessary induced) odd holes. It then follows easily that line perfect graphs are perfect.

By definition, $L(G)$ is perfect if and only if, for all induced subgraphs H of $L(G)$, $\chi(H) = \omega(H)$. Since the induced subgraphs of $L(G)$ are in one-to-one correspondence with the line graphs of subgraphs of G, $L(G)$ is perfect if and only if, for all subgraphs G' of G, $\chi(L(G')) = \omega(L(G'))$.

Call a graph G *weakly line perfect* if, for all *induced* subgraphs G' of G, $\chi(L(G')) = \omega(L(G'))$. Clearly, line perfect graphs are weakly line perfect. The graph G in Figure 1 is weakly line perfect but not line perfect ($L(G)$ contains an induced odd hole of length five). It is easy to see that weakly line perfect graphs cannot have induced odd holes and odd anti-holes. Thus, the Strong Perfect Graph Conjecture implies the following

Conjecture A. *Weakly line perfect graphs are perfect.*

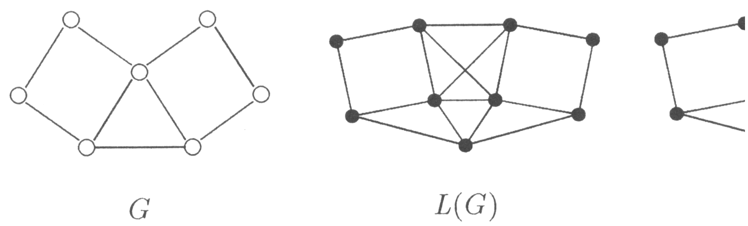

Figure 1. The line graph and the Gallai graph of a graph

3.

The *Gallai graph* $\Gamma(G)$ of a graph G has the edge set of G as its vertex set, and two distinct edges of G are adjacent in $\Gamma(G)$ if, and only if, they have an endvertex in common and the two other endvertices are nonadjacent in G; see Figure 1. Thus, $\Gamma(G)$ is a spanning subgraph of the line graph $L(G)$.

A graph H is a *Gallai graph* if there exists a graph G such that H is (isomorphic to) $\Gamma(G)$. Note that, in constrast to line graphs, not every induced subgraph of $\Gamma(G)$ is again a Gallai graph.

Problem B. *Recognize Gallai graphs in polynomial time, or prove that recognizing Gallai graphs is NP-complete.*

A nice connection between Gallai graphs and perfect graphs is the following: Call the graph G *Gallai perfect* if its Gallai graph $\Gamma(G)$ has no induced odd holes. L. Sun proved

Theorem ([7])**.** *Gallai perfect graphs are perfect.*

4.

It is shown in [5] that if $\Gamma(G)$ contains an induced odd anti-hole, it also contains an induced odd hole. Thus, the Strong Perfect Graph Conjecture implies the following

Conjecture C ([5])**.** *Gallai graphs without induced odd holes are perfect.*

If Conjecture C is true, it implies Sun's Theorem (see [5]). Also, it is easy to see that Conjecture C implies a theorem of D. König, saying that line graphs of bipartite graphs are perfect. In [5], Conjecture C is proved for Gallai graphs $\Gamma(G)$ of graphs G with $\chi(\overline{G}) \leq 4$.

5.

Sun's Theorem implies, in particular, that G is perfect if $\Gamma(G)$ is perfect. By definition, $\Gamma(G)$ is perfect if and only if, for all induced subgraphs H of $\Gamma(G)$, $\chi(H) = \omega(H)$.

Recall that not every induced subgraph of a Gallai graph is again a Gallai graph. This motivates the following definition: Call the graph G *weakly Gallai perfect* if, for all induced subgraphs G' of G, $\chi(\Gamma(G')) = \omega(\Gamma(G'))$.

The graph G in Figure 1 is weakly Gallai perfect but not Gallai perfect ($\Gamma(G)$ contains an induced odd hole of length seven). It is not clear whether all Gallai perfect graphs are weakly Gallai perfect. However, if Conjecture C is true, they are.

It is easy to see that weakly Gallai perfect graphs cannot contain odd holes and odd anti-holes (indeed, if C is an odd hole or an odd anti-hole, then $\chi(\Gamma(C)) = 3$ while $\omega(\Gamma(C)) = 2$). So, the Strong Perfect Graph Conjecture implies the following

Conjecture D. *Weakly Gallai perfect graphs are perfect.*

6.

It is shown in [5] that, for all graphs G, $\chi(\Gamma(G)) \leq \chi(\overline{G})$. Thus, if G is perfect, then $\chi(\Gamma(G')) \leq \omega(\overline{G'})$ and $\chi(\Gamma(\overline{G'})) \leq \omega(G')$ hold for all induced subgraphs G' of G. We conjecture that these properties characterize perfect graphs.

Conjecture E ([5, 2])**.** *A graph G is perfect if and only if, for all induced subgraphs G' of G, $\chi(\Gamma(G')) \leq \omega(\overline{G'})$ and $\chi(\Gamma(\overline{G'})) \leq \omega(G')$.*

This conjecture has a "semi-strong" property: It implies the Perfect Graph Theorem and it is implied by the Strong Perfect Graph Conjecture.

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