

## Unique factorization theorem

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### Abstract

A property of graphs is any class of graphs closed under isomorphism. A property of graphs is induced-hereditary and additive if it is closed under taking induced subgraphs and disjoint unions of graphs, respectively. Let  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$  be properties of graphs. A graph  $G$  is  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable ( $G$  has property  $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ ) if the vertex set  $V(G)$  of  $G$  can be partitioned into  $n$  sets  $V_1, V_2, \dots, V_n$  such that the subgraph  $G[V_i]$  of  $G$  induced by  $V_i$  belongs to  $\mathcal{P}_i$ ;  $i = 1, 2, \dots, n$ . A property  $\mathcal{R}$  is said to be reducible if there exist properties  $\mathcal{P}_1$  and  $\mathcal{P}_2$  such that  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ ; otherwise the property  $\mathcal{R}$  is irreducible. We prove that every additive and induced-hereditary property is uniquely factorizable into irreducible factors. Moreover the unique factorization implies the existence of uniquely  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graphs for any irreducible properties  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ .

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### 1. MOTIVATION AND MAIN RESULTS

A *property of graphs* is any nonempty class of graphs closed under isomorphism. A property of graphs is called *induced-hereditary* (hereditary) and *additive* if it is closed under taking induced subgraphs (subgraphs) and disjoint unions of graphs, respectively. Induced-hereditary (hereditary) properties are called also hereditary (monotone) (see [3]). Obviously, any hereditary property of graphs is induced-hereditary, too. On the other

hand, many well-known induced-hereditary classes of graphs (e.g., complete graphs, line-graphs, claw-free graphs, interval graphs, perfect graphs, etc.) are not hereditary. Let  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$  be properties of graphs. A graph  $G$  is  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable ( $G$  has property  $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ ) if the vertex set  $V(G)$  of  $G$  can be partitioned into  $n$  sets  $V_1, V_2, \dots, V_n$  such that the subgraph  $G[V_i]$  of  $G$  induced by  $V_i$  belongs to  $\mathcal{P}_i$ ,  $i = 1, 2, \dots, n$ . An induced-hereditary property  $\mathcal{R}$  is said to be *reducible* if there exist induced-hereditary properties  $\mathcal{P}_1$  and  $\mathcal{P}_2$  such that  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ , otherwise the property  $\mathcal{R}$  is *irreducible*. The notion of reducible properties have been introduced in connection with generalized graph colouring and the existence of uniquely partitionable graphs (see [6, 10, 8]).

The problem: "Is the factorization of every property into irreducible properties unique?" have been stated in the book [8] of Jensen and Toft "Graph Coloring Problems". Partial results for some subclasses of induced-hereditary properties may be found in [11, 12, 9, 13]. In May 1995 (see [11]) we proved *the unique factorization theorem* (UFT) for the additive hereditary properties with completeness at most 3, in June 1996 (see [9]) we proved UFT. The aim of this paper is to prove the unique factorization in the whole class of additive induced-hereditary properties of graphs.

**Theorem 1.** *Any reducible additive induced-hereditary property is uniquely factorizable into irreducible factors (up to the order of factors).*

Since in general for induced-hereditary properties we cannot use the concept of maximal graphs (used for hereditary properties in [13]), we define new concepts — *the operation "\*" and  $\mathcal{R}$ -decomposability number of a graph.*

**Definition.** Let  $\mathcal{R}$  be an additive induced-hereditary property. For given graphs  $G_1, G_2, \dots, G_n$ ,  $n \geq 2$ , denote by

$$G_1 * G_2 * \dots * G_n = \{G : \bigcup_{i=1}^n G_i \subseteq G \subseteq \sum_{i=1}^n G_i\},$$

where  $\bigcup_{i=1}^n G_i$  denotes the disjoint union and  $\sum_{i=1}^n G_i$  the join of the graphs  $G_1, G_2, \dots, G_n$ , respectively.

Let  $dec_{\mathcal{R}}(G) = \max\{n : \text{there exist a partition}(V_1, V_2, \dots, V_n), V_i \neq \emptyset, \text{ of } V(G) \text{ (called } \mathcal{R}\text{-decomposition of } G\text{) such that for each } k \geq 1, k.G[V_1] * k.G[V_2] * \dots * k.G[V_n] \subseteq \mathcal{R}\}$ . If  $G \notin \mathcal{R}$  we set  $dec_{\mathcal{R}}(G)$  to be zero.

A graph  $G$  is said to be  $\mathcal{R}$ -decomposable if  $dec_{\mathcal{R}} \geq 2$ ; otherwise  $G$  is  $\mathcal{R}$ -indecomposable.

These new concepts are motivated by the following observation.

Let us suppose that  $G \in \mathcal{R} = \mathcal{P} \circ \mathcal{Q}$  and let  $(V_1, V_2)$  be a  $(\mathcal{P}, \mathcal{Q})$ -partition of  $G$ . Then by additivity of  $\mathcal{P}$  and  $\mathcal{Q}$   $k.G[V_1] * k.G[V_2] \subseteq \mathcal{R}$  for every positive integer  $k$ . Thus if the property  $\mathcal{R}$  is reducible, every graph  $G \in \mathcal{R}$  with at least two vertices is  $\mathcal{R}$ -decomposable.

We shall prove that for any additive reducible induced-hereditary property also the converse assertion holds.

**Theorem 2.** *An induced-hereditary additive property  $\mathcal{R}$  is reducible if and only if all graphs in  $\mathcal{R}$  with at least two vertices are  $\mathcal{R}$ -decomposable.*

The problem of unique factorization have been from the beginning related to the investigation of the existence of uniquely partitionable graphs.

A graph  $G$  is said to be *uniquely  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable* if  $G$  has exactly one (unordered)  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition  $(V_1, V_2, \dots, V_n)$ . Let us denote by  $U(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n)$  the class of all uniquely  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graphs. In the case  $\mathcal{P}_1 = \mathcal{P}_2 = \dots = \mathcal{P}_n = \mathcal{P}$  we write  $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n = \mathcal{P}^n$  and we say that  $G$  belonging to  $U(\mathcal{P}^n)$  is uniquely  $(\mathcal{P}, n)$ -partitionable.

It turned out that the existence of uniquely partitionable graphs follows from proofs of UFT's. In this paper we prove the conjecture presented in [12].

**Theorem 3.** *Let  $\mathcal{P}$  be an additive induced-hereditary property of graphs. Then for  $n \geq 2, U(\mathcal{P}^n) \neq \emptyset$  if and only if  $\mathcal{P}$  is irreducible.*

Analogously as for hereditary properties (see [12, 5]) we prove that every reducible additive induced-hereditary property  $\mathcal{R}$  can be generated by graphs which are uniquely partitionable with respect to its irreducible factors.

**Theorem 4.** *Let  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n, n \geq 2$  be the factorization of a reducible additive induced-hereditary property  $\mathcal{R}$  into irreducible factors. Then every graph  $G \in \mathcal{R}$  is an induced subgraph of a uniquely  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable graph  $G^*$ .*

Using the result of A. Berger [2], who proved that every reducible additive induced-hereditary property  $\mathcal{P}$  has infinitely many minimal forbidden induced subgraphs, we have the following generalization of the Theorem 1 of [1].

**Corollary 5.** *Let  $\mathcal{P}$  be any induced-hereditary property of graphs defined by a finite set of connected forbidden subgraphs. Then for every positive integer  $n > 2$  there exist infinitely many uniquely  $(\mathcal{P}, n)$ -partitionable graphs.*

The notation and technical preliminary results are presented in Section 2. The proofs of the main Theorems are given in Section 3.

## 2. NOTATION AND PRELIMINARY RESULTS

All graphs considered in this paper are finite and simple (without multiple edges or loops), the class of all graphs is denoted by  $\mathcal{I}$ . We use the standard notation (see e.g. [7, 8]). In particular,  $K_n$  denotes the complete graph on  $n$  vertices,  $G \cup H$  denotes the disjoint union of graphs  $G$  and  $H$  and  $k.G$  denotes the disjoint union of  $k$  isomorphic copies of  $G$ . The symbols  $\leq$  and  $\subseteq$  stand for the relations "to be an induced subgraph" and "to be a subgraph", respectively. The join  $\sum_{i=1}^n G_i = G_1 + G_2 + \dots + G_n$  of  $n$  graphs  $G_1, G_2, \dots, G_n$  is the graph consisting of the disjoint union of  $G_i$ 's and all the edges between  $V(G_i)$  and  $V(G_j)$  for any  $1 \leq i < j \leq n$ .

A graph  $G \in \mathcal{P}$  is said to be  $\mathcal{P}$ -maximal if  $G + e \notin \mathcal{P}$  for each  $e \in E(\overline{G})$ . The structure of graphs maximal with respect to reducible hereditary properties played an important role in the proof of unique factorization of additive and hereditary properties. However for non-hereditary induced-hereditary properties we have to find another way. Let us define the related notion of  $\mathcal{P}$ -strict graphs using the operation  $*$  introduced in Section 1.

**Definition.** A graph  $G \in \mathcal{P}$  is said to be  $\mathcal{P}$ -strict if  $G * K_1 \not\subseteq \mathcal{P}$ . The class of all  $\mathcal{P}$ -strict graphs is denoted by  $\mathcal{S}(\mathcal{P})$ .

A set  $\mathcal{G} \subset \mathcal{I}$  is said to be a *generating set* of  $\mathcal{P}$  if  $G \in \mathcal{P}$  if and only if  $G$  is an induced subgraph of some graph from  $\mathcal{G}$ . The fact that  $\mathcal{G}$  is a generating set of  $\mathcal{P}$  will be written as  $[\mathcal{G}] = \mathcal{P}$ . The members of  $\mathcal{G}$  are called *generators* of  $\mathcal{P}$ .

Let us show that every graph  $G \in \mathcal{P}$  is an induced subgraph of a  $\mathcal{P}$ -strict graph and hence the class  $\mathcal{S}(\mathcal{P})$  forms a generating set of  $\mathcal{P}$ .

Obviously for any property  $\mathcal{P} \neq \mathcal{I}$  there exists a graph  $F \notin \mathcal{P}$ . For a property  $\mathcal{P}$  we can therefore define  $f(\mathcal{P})$  to be the least number of vertices of a forbidden subgraph of  $\mathcal{P}$ , i.e.  $f(\mathcal{P}) = \min\{|V(F)| : F \notin \mathcal{P}\}$ . Now it is easy to see, that for every  $G \in \mathcal{P}$  the class  $G * K_1 * \dots * K_1 \not\subseteq \mathcal{P}$  if the number of the  $K_1$ 's is  $f(\mathcal{P}) - 1$  which means that if  $G$  is not  $\mathcal{P}$ -strict,

then repeating the operation  $*$  with  $K_1$ 's after less than  $f(\mathcal{P})$  steps we will obtain a  $\mathcal{P}$ -strict graph  $G'$  such that  $G \leq G'$ .

Since  $dec_{\mathcal{R}}(G) < f(\mathcal{R})$ , this fact allows us to define the decomposability number  $dec(\mathcal{G})$  of a generating set  $\mathcal{G}$  of  $\mathcal{R}$  by

$$dec(\mathcal{G}) = \min\{dec_{\mathcal{R}}(G) : G \in \mathcal{G}\}.$$

Put  $dec(\mathcal{R}) = dec(\mathcal{S}(\mathcal{R}))$ .

The next simple Lemma will be used.

**Lemma 6.** *Let  $G$  be an  $\mathcal{R}$ -strict graph and  $G^*$  be an induced supergraph of  $G$  i.e.,  $G \leq G^*$ . Then  $G^*$  is  $\mathcal{R}$ -strict and  $dec_{\mathcal{R}}(G) \geq dec_{\mathcal{R}}(G^*)$ .*

**Proof.** The fact that  $G^* \in \mathcal{S}(\mathcal{R})$  is evident. Suppose, in contrary, that  $n = dec_{\mathcal{R}}(G) < dec_{\mathcal{R}}(G^*) = m$  and  $d = (V_1, V_2, \dots, V_m)$  be an  $\mathcal{R}$ -decomposition of  $G^*$ . Then at most  $n < m$  sets  $V_i$  of  $d$  have nonempty intersections with  $V(G)$  (otherwise  $dec_{\mathcal{R}}(G) > n$ ) and there is a vertex  $z \in V_j$  with  $V_j \cap V(G) = \emptyset$ , in contradiction with assumption:  $G$  be  $\mathcal{R}$ -strict. ■

We are going to show that there exists a generating set  $\mathcal{G}^* \subseteq \mathcal{S}(\mathcal{R})$  of  $\mathcal{R}$  which contains only graphs  $G$  with decomposability number  $dec_{\mathcal{R}}(G) = dec(\mathcal{R}) = n$  which are uniquely  $\mathcal{R}$ -decomposable (i.e., there exist exactly one  $\mathcal{R}$ -decomposition  $(V_1, V_2, \dots, V_n), V_i \neq \emptyset$ , such that for each  $k \geq 1, k.G[V_1] * k.G[V_2] * \dots * k.G[V_n] \subseteq \mathcal{R}$ ).

The final step of the proof of Theorem 2 and Theorem 1 will consist of the construction of corresponding irreducible factors. Analogously as in [13], by the construction it follows that if  $dec(\mathcal{R}) = n$ , then  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$  where the irreducible factors  $\mathcal{P}_i, i = 1, 2, \dots, n$  are uniquely determined by the structure of the generating set  $\mathcal{G}^*$ .

Our consideration requires the definitions of appropriate generating sets of  $\mathcal{R}$  derived from the set of  $\mathcal{R}$ -strict graphs. Let us present the simple Lemmas on the properties of generating sets. We omit their simple proofs analogous to those given for maximal graphs in [13] (see also [14]).

**Lemma 7.** *Let  $\mathcal{R}$  be an induced-hereditary property of graphs and let  $\mathcal{G}$  be a generating set of  $\mathcal{R}$ . If all graphs belonging to  $\mathcal{G}$  are  $\mathcal{R}$ -decomposable, then all  $\mathcal{R}$ -strict graphs are  $\mathcal{R}$ -decomposable, too.*

**Lemma 8.** *Let  $\mathcal{P}$  be an additive induced-hereditary property and  $\mathcal{G}$  be a generating set of  $\mathcal{P}$ . If  $G$  is an arbitrary graph with property  $\mathcal{P}$  then there exists a generating set  $\mathcal{G}' \subseteq \mathcal{G}$  such that each graph  $H \in \mathcal{G}'$  contains at least one copy of the graph  $G$ .*

**Lemma 9.** *Let  $\mathcal{P}$  be an induced-hereditary property of graphs. Let  $\mathcal{G}$  be a generating set of  $\mathcal{P}$  such that  $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$ . Then  $\text{dec}(\mathcal{G}) = \text{dec}(\mathcal{P})$ .*

**Lemma 10.** *Let  $\mathcal{P}$  be an additive induced-hereditary property of graphs. Let  $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$  be any generating set of  $\mathcal{P}$ . Then there exists a set  $\mathcal{G}^*$ ,  $\mathcal{G}^* \subseteq \mathcal{G}$ , which is a generating set of  $\mathcal{P}$  and contains only graphs of  $\mathcal{P}$ -decomposability number equal to  $\text{dec}(\mathcal{P})$ .*

Now, let us prove the main Lemma of this paper.

**Lemma 11.** *For every  $\mathcal{R}$ -strict graph  $G$  with  $\text{dec}_{\mathcal{R}}(G) = \text{dec}(\mathcal{R}) \geq 2$  there is a uniquely  $\mathcal{R}$ -decomposable graph  $G^* \geq G$ .*

**Proof.** Let  $G$  be a fixed  $\mathcal{R}$ -strict graph with  $\text{dec}_{\mathcal{R}}(G) = n$  and  $d_i = (V_{i1}, V_{i2}, \dots, V_{in})$ ,  $i = 0, 1, \dots, m$ ,  $m \geq 0$  be all  $\mathcal{R}$ -decompositions of  $G$ . Since  $G$  is a finite graph,  $m$  is a nonnegative integer.

We shall construct a uniquely  $\mathcal{R}$ -decomposable graph  $G^* = G^*(m)$  taking an appropriate number  $s$  of disjoint copies of  $G$  so that  $V(G^*) = V(s.G)$  and  $E(G^*) = E(s.G) \cup E^*(m)$  where new edges  $e \in E^*(m)$  are joining vertices of different copies of  $G$  only. By Lemma 6 we have  $\text{dec}_{\mathcal{R}}(G^*(m)) = \text{dec}_{\mathcal{R}}(G) = n$ .

Every  $\mathcal{R}$ -decomposition  $d = (V_1^*, V_2^*, \dots, V_n^*)$  of  $G^*$  restricted to any copy  $G$  gives some  $\mathcal{R}$ -decomposition  $d_j$  of  $G$  denoted by  $d|G = d_j$ . The aim of our construction is to add new edges  $E^*(m)$  to  $s.G$  so that the obtained graph  $G^*(m)$  will have only one  $\mathcal{R}$ -decomposition  $d$  such that  $d|G = d_0$  for each copy  $G$  of  $s.G$ .

To proceed we shall use two types of constructions:

**Construction 1.**  $G^i \Leftrightarrow G^j$ :

Let  $G^i, G^j$  be two different copies of  $G$  in  $s.G$ . Since  $G$  is  $\mathcal{R}$ -strict,  $G * K_1 \notin \mathcal{R}$ . Let us fix a graph  $F \in G * K_1$ ,  $F \notin \mathcal{R}$  and let  $N_F(z)$  be the neighbours of  $z \in V(K_1)$  in  $G$ . Let us denote by  $Z_j = V_{0j} \cap N_F(z)$ ,  $j = 1, 2, \dots, n$  the neighbours of  $z$  in  $G[V_{0j}]$  with respect to the  $\mathcal{R}$ -decomposition  $d_0$  of  $G$ . Let  $G^i, G^j$ ,  $i \neq j$  be disjoint copies of  $G$ ,  $d_0$  be the  $\mathcal{R}$ -decomposition of  $G$  and  $v$  be a vertex of  $G^i \cup G^j$ . Add new edges  $E^*(G^i \Leftrightarrow G^j)$  so that every vertex  $v \in V_{0k}$  of the corresponding  $\mathcal{R}$ -decomposition  $d_0$  is adjacent to every vertex of  $Z_j$ ,  $j \neq k$  of the other copy of  $G^i \cup G^j$ .

The resulting graph  $G^i \Leftrightarrow G^j$  has the following property: for every  $\mathcal{R}$ -decomposition  $d = (U_1, U_2, \dots, U_n)$  of  $G^i \Leftrightarrow G^j$  it holds that  $d|G^i = d_0$  if and only if  $d|G^j = d_0$ . The proof of this fact is simple, suppose that

$d|G^i = d_0$  ( $d|G^j = d_0$ ) and  $v \in V_{0k}$  of  $G^j(G^i)$  does not belong to  $U_k$ . Then  $d$  cannot be an  $\mathcal{R}$ -decomposition of  $H = G^i \Leftrightarrow G^j$  since there is a graph in  $H[U_1] * H[U_2] * \dots * H[U_n]$  which contains an induced copy of  $F$  (we can add the appropriate edges between  $v$  and  $Z_k$ ).

**Construction 2.**  $n \bullet k(r, s).G$ :

Let  $d_r$  and  $d_s$  be different  $\mathcal{R}$ -decompositions of  $G$ , denote by  $A_{ij}(r, s) = V_{r_i} \cap V_{s_j}, i, j = 1, 2, \dots, n$ . Since  $d_r \neq d_s$  at least  $n + 1$  sets  $A_{ij}(r, s)$  are nonempty. Because of  $\text{dec}_{\mathcal{R}}(G) = n$  there exists a positive integer  $k(r, s)$  such that  $k(r, s).G[A_{11}(r, s)] * k(r, s).G[A_{12}(r, s)] * \dots * k(r, s).G[A_{nn}(r, s)] \notin \mathcal{R}$ . Let fix a graph  $F(r, s) \in k(r, s).G[A_{11}(r, s)] * k(r, s).G[A_{12}(r, s)] * \dots * k(r, s).G[A_{nn}(r, s)], F(r, s) \notin \mathcal{R}$ . Denote by  $E_{ij,i'j'}(r, s)$  the set of edges of  $F(r, s)$  joining the vertices of  $k(r, s).G[A_{ij}(r, s)]$  and  $k(r, s).G[A_{i'j'}(r, s)]$ .

Let us construct the graph  $H^*(r, s) = n \bullet k(r, s).G$  taking  $n$  disjoint copies of  $H = k(r, s).G$ , denoted by  $H^j, j = 1, 2, \dots, n$ . Add new edges joining different copies of  $k(r, s).G$  so that the edges  $E_{ji,ki}(r, s), i = 1, 2, \dots, n$  be realized between the copies  $H^j$  and  $H^k, j \neq k$ , i.e. for example the edges  $E_{11,21}(r, s)$  and  $E_{12,22}(r, s)$  etc., of the graph  $F(r, s)$  are placed between  $H^1$  and  $H^2$ .

The construction 2 gives a graph  $H^* = H^*(r, s)$  without an  $\mathcal{R}$ -decomposition  $d = (W_1, W_2, \dots, W_n)$  such that  $d|G = d_s$  for each induced copy of  $G$  in  $H^*$  because otherwise the graph  $F(r, s)$  would appear in  $H^*[W_1] * H^*[W_2] * \dots * H^*[W_n]$ .

We are ready to prove the Lemma 11 by constructing  $G^*$ :

If  $m = 0$ , then  $G^* = G$  and we are done. In this case  $d_0 = (V_{01}, V_{02}, \dots, V_{0n})$  is the unique  $\mathcal{R}$ -decomposition of  $G^*$ .

If  $m \geq 1$  we proceed recurrently:

**Universal Step 0.** Let  $G^0 = G$  be a fixed copy of  $G$  and  $G(m)$  be a graph consisting of  $s$  copies of  $G$  (denoted by  $G^1, G^2, \dots, G^s$ ) (to be described recurrently below). For every  $m \geq 1$  add edges between  $G^0$  and  $G^i, i = 1, 2, \dots, s$  by Construction 1 so that  $U = V(G^0) \cup V(G^i)$  induces in resulting graph  $G^*$  the subgraph  $G^*[U] = G^0 \Leftrightarrow G^i$ . This part of the construction of  $G^*$  yields that if for a  $\mathcal{R}$ -decomposition  $d$  of  $G^*$  there exists a  $G^k$  in  $G(m)$  such that  $d|G^k = d_0$  then for every  $i = 0, 1, 2, \dots, s$   $d|G^i = d_0$  implying  $G^*$  has unique  $\mathcal{R}$ -decomposition.

**Step 1.** Let us denote by  $G(1)$  the graph  $H^*(0, 1) = n \bullet k(0, 1).G$  — see Construction 2. Let  $G^*(1)$  be obtained from  $G^0$  and  $G(1)$  according to the Step 0 ( $s = n.k(0, 1)$ ). If  $m = 1$ , then the graph  $G^*(1)$  has

unique  $\mathcal{R}$ -decomposition  $d$  since Construction 2 is forcing at least one copy of  $G$  of  $G(1)$  to have  $d|G^j = d_0$  so that by Construction 1 all copies  $G^0, G^1, \dots, G^{n.k(0,1)}$  of  $G^*(1)$  must have  $d|G^i = d_0$ .

**Step  $j$ .** For  $j \geq 2$ , let  $G(j-1)$  be the graph constructed in the Step  $j-1$ . To construct  $G(j)$  let us take  $n.k(0, j)$  disjoint copies of  $G(j-1)$  and add new edges inserting  $H^*(0, j) = n \bullet k(0, j).G$  for every choice of  $n.k(0, j)$  copies of  $G$  one by one from different copies of  $G(j-1)$ .

Let the graph  $G^*(j)$  be obtained from  $G^0$  and  $G(j)$  according to the Step 0. Let  $d$  be a  $\mathcal{R}$ -decomposition of  $G^*(j)$ . Suppose that there is a  $G^k$  such that  $d|G^k \neq d_0$ . Then  $d|G^i = d_j$  for a copy  $G^i$  of  $G$  from each copy of  $G(j-1)$ , since otherwise for every  $G^i$   $d|G^i = d_0$  by step  $j-1$ . However if every copy of  $G(j-1)$  should have a copy of  $G$  with  $d|G = d_j$ , then a copy of  $H^*(0, j)$  is forcing a contradiction.

The uniquely  $\mathcal{R}$ -decomposable graph  $G^* = G^*(m)$  is obtained in the Step  $m$ . ■

Let  $\mathcal{G}^*(\mathcal{R})$  denotes the class of all uniquely  $\mathcal{R}$ -decomposable graphs with  $\mathcal{R}$ -decomposability number  $n = \text{dec}(\mathcal{R}) \geq 2$ . By Lemma 11  $\mathcal{G}^*$  is a generating set of  $\mathcal{R}$ . Using Lemma 11 we can proceed the same way as for hereditary properties in [13].

First let us describe the structure of the generators of  $\mathcal{G}^*(\mathcal{R})$ . Let  $\mathcal{G}^* = \mathcal{G}^*(\mathcal{R}) = \{G_i; i \in I\}$  and let  $(V_i^1, V_i^2, \dots, V_i^n)$  be the unique  $\mathcal{R}$ -decomposition of  $G_i$ . The graphs  $G_i^j = G_i[V_i^j]$  are called *indecomposable-parts* of the generator  $G_i$ . The set of all indecomposable-parts of graphs belonging to  $\mathcal{G}^*$  will be denoted by  $Ip(\mathcal{R})$  so that if  $Ip(G_i) = \{G_i^j, j = 1, 2, \dots, n\}$  then  $Ip(\mathcal{R}) = \bigcup_{i \in I} Ip(G_i)$ . For  $F \in Ip(\mathcal{R})$  and  $G_k \in \mathcal{G}^*$  let us denote by  $m(F, G_k)$  the number of different (possibly isomorphic) ind-parts of  $G_k$  which  $m(F) = \max\{m(F, G_i); G_i \in \mathcal{G}^*\}$ . The positive integer  $m(F)$  is called *the multiplicity of the ind-part  $F \in Ip(\mathcal{R})$  in  $\mathcal{R}$* . Obviously for every  $F \in Ip(\mathcal{R}) : 1 \leq m(F) \leq n = \text{dec}(\mathcal{P})$ .

A technical Lemma analogous to Lemma 2.6 from [13] holds.

**Lemma 12.** *Let  $\mathcal{G}^* \subseteq \mathcal{S}(\mathcal{R})$  be the generating set of  $\mathcal{R}$  consisting of uniquely  $\mathcal{R}$ -decomposable graphs with decomposability number  $n = \text{dec}(\mathcal{R})$ . Let  $G$  be an arbitrary graph from  $\mathcal{G}^*$  and let  $(V^1, V^2, \dots, V^n)$  be its unique  $\mathcal{R}$ -decomposition. If a graph  $H \in \mathcal{G}^*$  contains  $G$  as an induced-subgraph, then the ind-parts  $G^j$  of  $G$ ,  $j \in \{1, 2, \dots, n\}$ , are induced-subgraphs of different ind-parts  $H^k$  of  $H$ ,  $k \in \{1, 2, \dots, n\}$ .*



**Proof.** If some ind-parts  $G^i, G^j, i \neq j$  are induced-subgraphs of the same ind-part  $H^k (k \in \{1, 2, \dots, n\})$  of  $H$ , then there is at least one ind-part of  $H$  which has empty intersection with  $G$ . But this contradicts the assumption that  $G$  is  $\mathcal{R}$ -strict.

Now, suppose that an ind-part, say  $G^1$ , be an induced-subgraph of at least two different ind-parts  $H^j, H^k, j \neq k$ , of  $H$ . Then  $G^1[V^1] \cap V(H^j)] * G^1[V^1 \cap V(H^k)] * G^2 * \dots * G^n \subseteq \mathcal{R}$  which is in contradiction to  $dec_{\mathcal{R}}(G) = n$ . ■

### 3. THE PROOFS OF THE MAIN RESULTS

We are prepared to prove the main results. The proof of Theorem 2 is analogous as for additive hereditary properties in [13]. We recall it to present full insight into the structure of irreducible factors.

**Proofs of Theorems 1 and 2.** Let every graph  $G \in \mathcal{R}$  with at least two vertices be  $\mathcal{R}$ -decomposable. We will find the factorization of the property  $\mathcal{R}$  into at least two irreducible factors  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ .

Let  $\mathcal{G}^* \subseteq \mathcal{S}(\mathcal{R})$  be the generating set consisting of all uniquely  $\mathcal{R}$ -decomposable graphs of decomposability number  $n = dec(\mathcal{R})$  and let  $Ip(\mathcal{R})$  be the set of all ind-parts of  $\mathcal{R}$ . We distinguish two cases:

*Case 1.* Let us suppose that there exists an ind-part  $F \in Ip(\mathcal{R})$  of multiplicity  $m(F) = k$  where  $k < dec(\mathcal{P})$ . Let  $G \in \mathcal{G}^*$  be a generator of  $\mathcal{P}$  for which  $m(F, G) = k$ . Let us consider, in accordance with Lemma 8, the generating set  $\mathcal{G}_G^* \subseteq \mathcal{G}^*$  such that  $\mathcal{G}_G^* = \{H \in \mathcal{G}^*; G \leq H\}$ . By the definition of  $\mathcal{G}_G^*$  and by Lemma 12 for every generator  $H \in \mathcal{G}_G^*$ ,  $m(F, H) = k$ . Let the induced-hereditary property  $\mathcal{Q}_1$  ( $\mathcal{Q}_2$ ) be generated by the subgraphs induced by union of vertices of  $k$  ( $n - k$ ) ind-parts of generators  $H \in \mathcal{G}_G^*$  containing (not containing) the ind-part  $F$ .

Let us show that  $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2$ . It is easy to see that  $\mathcal{R} \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2$ . Let  $H^* \in \mathcal{Q}_1 \circ \mathcal{Q}_2$ . Then  $H^* \in H_1^* * H_2^*$  where  $H_1^*$  ( $H_2^*$ ) is the subgraph induced by the union of vertices of  $k$  ( $n - k$ ) ind-parts of some generator  $H_1$  ( $H_2$ )  $\in \mathcal{G}_G^*$  which contain (do not contain) the ind-part  $F$ . Let  $G^* \in \mathcal{G}_G^*$  be such a graph that  $H_1 \cup H_2 \leq G^*$ . By Lemma 12 and by the definition of  $\mathcal{G}_G^*$   $H_1 * H_2 \subseteq \mathcal{R}$  implying  $H^* \in \mathcal{R}$ . Hence  $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2$ .

The additivity of  $\mathcal{Q}_1, \mathcal{Q}_2$ : suppose that the graphs  $H_1^*$  and  $H_2^*$  belong to  $\mathcal{Q}_1$  ( $\mathcal{Q}_2$ ). Then  $H_i^*$  is a subgraph of the join of  $k$  ( $n - k$ ) ind-part of some generator  $H_i \in \mathcal{G}_G^*$  containing (not containing) the ind-part  $F, i \in \{1, 2\}$ . If  $G^* \in \mathcal{G}_G^*$  such that  $H_1 \cup H_2 \leq G^*$ , then by Lemma 12 and by the definition

of  $\mathcal{G}_G^*$ , both  $H_1^*$  and  $H_2^*$  are induced subgraphs of  $k(n-k)$  ind-parts of  $G^*$  which contain (do not contain) the ind-part  $F$  as an induced subgraph. Then  $H_1^* \cup H_2^* \in \mathcal{Q}_1(\mathcal{Q}_2)$  and hence  $\mathcal{Q}_1, \mathcal{Q}_2$  are additive.

*Case 2.* Suppose that  $m(F) = n = \text{dec}(\mathcal{R}) \geq 2$  for each  $F \in Ip(\mathcal{R})$ . Let  $\mathcal{Q}$  be an induced-hereditary property generated by  $Ip(\mathcal{R})$ . It is easy to see that  $\mathcal{R} \subseteq \mathcal{Q}^n$ . The converse inclusion,  $\mathcal{Q}^n \subseteq \mathcal{R}$ , and the additivity of  $\mathcal{Q}$  follows analogously as in the Case 1. The proof of Theorem 2 is finished.

To complete the proof of Theorem 1 we use induction on  $n = \text{dec}(\mathcal{R})$ . If  $n = 1$ , the property  $\mathcal{R}$  is irreducible. Let us suppose that every property with decomposability number  $1 \leq k < n$  has a unique factorization into irreducible factors and let  $\mathcal{R}$  be a property with  $\text{dec}(\mathcal{R}) = n$ .

The structure of the factorization of the property  $\mathcal{R}$  depends on the multiplicities of the ind-parts of  $\mathcal{R}$  as described above. This factorization is uniquely determined because the generators of  $\mathcal{R}$  are uniquely  $\mathcal{R}$ -decomposable into ind-parts. Suppose there exists an ind-part  $F$  of  $\mathcal{R}$  with multiplicity  $m(F) = k < \text{dec}(\mathcal{R}) = n$ . Then we consider the properties  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  defined in the Case 1. By the induction hypothesis they are uniquely factorizable into irreducible factors. Since the generators of  $\mathcal{R}$  are uniquely  $(\mathcal{Q}_1, \mathcal{Q}_2)$ -partitionable, the proof is complete.

If for every ind-part  $F$  of  $\mathcal{R}$  its multiplicity  $m(F)$  in  $\mathcal{R}$  is equal to  $n$ , then  $\mathcal{R} = \mathcal{Q}^n$  by the Case 2.

**Proofs of Theorems 3 and 4.** Let  $\mathcal{R}$  be any reducible, additive induced-hereditary property. We proved above that the property  $\mathcal{R}$  can be generated by a class  $\mathcal{G}^*$  of graphs with decomposability number  $n \geq 2$  which are uniquely  $\mathcal{R}$ -decomposable into  $n$  indecomposable parts generating the corresponding irreducible factors. It means that if  $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ ,  $n \geq 2$  be the factorization of  $\mathcal{R}$  into irreducible factors, then every generator from  $\mathcal{G}^*$  is uniquely  $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable. On the other hand, let a property  $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$  be reducible, then obviously there are no uniquely  $(\mathcal{P}, n)$ -partitionable graphs since the parts belonging to  $\mathcal{P}_2$  in any  $(\mathcal{P}^n)$ -partition of  $G$  are interchangeable.

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