

## GEODETIC SETS IN GRAPHS

GARY CHARTRAND

*Department of Mathematics and Statistics*  
*Western Michigan University, Kalamazoo, MI 49008, USA*

FRANK HARARY

*Department of Computer Science*  
*New Mexico State University, Las Cruces, NM 88003, USA*

AND

PING ZHANG<sup>1</sup>

*Department of Mathematics and Statistics*  
*Western Michigan University, Kalamazoo, MI 49008, USA*

### Abstract

For two vertices  $u$  and  $v$  of a graph  $G$ , the closed interval  $I[u, v]$  consists of  $u$ ,  $v$ , and all vertices lying in some  $u - v$  geodesic in  $G$ . If  $S$  is a set of vertices of  $G$ , then  $I[S]$  is the union of all sets  $I[u, v]$  for  $u, v \in S$ . If  $I[S] = V(G)$ , then  $S$  is a geodetic set for  $G$ . The geodetic number  $g(G)$  is the minimum cardinality of a geodetic set. A set  $S$  of vertices in a graph  $G$  is uniform if the distance between every two distinct vertices of  $S$  is the same fixed number. A geodetic set is essential if for every two distinct vertices  $u, v \in S$ , there exists a third vertex  $w$  of  $G$  that lies in some  $u - v$  geodesic but in no  $x - y$  geodesic for  $x, y \in S$  and  $\{x, y\} \neq \{u, v\}$ . It is shown that for every integer  $k \geq 2$ , there exists a connected graph  $G$  with  $g(G) = k$  which contains a uniform, essential minimum geodetic set. A minimal geodetic set  $S$  has no proper subset which is a geodetic set. The maximum cardinality of a minimal geodetic set is the upper geodetic number  $g^+(G)$ . It is shown that every two integers  $a$  and  $b$  with  $2 \leq a \leq b$  are realizable as the geodetic and upper geodetic numbers, respectively, of some graph and when  $a < b$  the minimum order of such a graph is  $b + 2$ .

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## 1 Introduction

The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $u - v$  path in  $G$ . For a vertex  $v$  of  $G$ , the *eccentricity*  $e(v)$  is the distance between  $v$  and a vertex farthest from  $v$ . The minimum eccentricity among the vertices of  $G$  is the *radius*,  $rad G$ , and the maximum eccentricity is its *diameter*,  $diam G$ . A  $u - v$  path of length  $d(u, v)$  is also referred to as a  $u - v$  *geodesic*. Please see the books [2, 5] for graph notation and terminology. We define the *closed interval*  $I[u, v]$  as the set consisting of  $u$ ,  $v$ , and all vertices lying in some  $u - v$  geodesic of  $G$ , and for a nonempty subset  $S$  of  $V(G)$ ,

$$I[S] = \bigcup_{u, v \in S} I[u, v].$$

The set  $S$  is *convex* if  $I[S] = S$ . A set  $S$  of vertices of  $G$  is defined in [1, 3] to be a *geodetic set* in  $G$  if  $I[S] = V(G)$ , and a geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set in  $G$  is the *geodetic number*  $g(G)$ .

The graph  $G_1$  of Figure 1 has geodetic number 2 as  $S_1 = \{w_1, y_1\}$  is the unique minimum geodetic set of  $G_1$ . On the other hand, each 2-element subset  $S$  of the vertex set of  $G_2$  has the property that  $I[S]$  is properly contained in  $V(G_2)$ . Thus  $g(G_2) \geq 3$ . Since  $S_2 = \{u_2, v_2, x_2\}$  is a geodetic set,  $g(G_2) = 3$ .

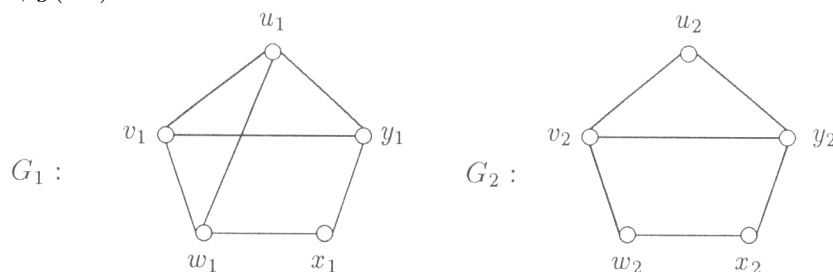


Figure 1. Illustrating the geodetic number

The closed intervals  $I[u, v]$  in a connected graph  $G$  were studied and characterized by Nebeský [7, 8] and were also investigated extensively in the book by Mulder [6], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The intervals of an oriented graph have been studied in [4].

## 2 Uniform and Essential Minimum Geodetic Sets

A graph  $F$  is called a *minimum geodetic subgraph* if there exists a graph  $G$  containing  $F$  as an induced subgraph such that  $V(F)$  is a minimum geodetic set in  $G$ . Those graphs that are minimum geodetic subgraphs were characterized in [1].

**Theorem A.** *A nontrivial graph  $F$  is a minimum geodetic subgraph if and only if every vertex of  $F$  has eccentricity 1 or no vertex of  $F$  has eccentricity 1.*

As a consequence of this theorem, there exists a graph  $G$  containing a minimum geodetic set  $S$  such that  $\langle S \rangle$  is complete or  $S$  is independent. In the former case,  $d_G(u, v) = 1$  for all distinct  $u, v \in S$ ; while in the latter case,  $d_G(u, v) \geq 2$  for all distinct  $u, v \in S$ . This is illustrated in Figure 2.

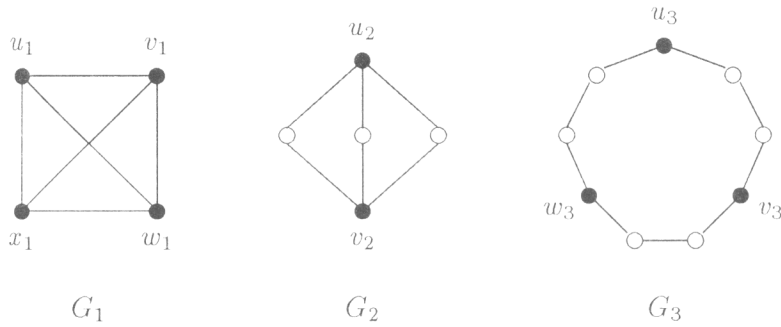
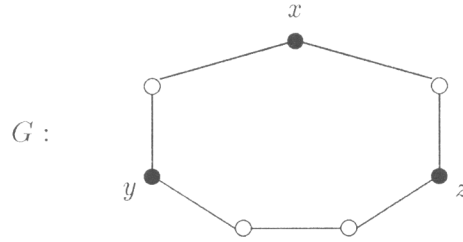


Figure 2. Graphs with uniform minimum geodetic sets

The graphs  $G_1$ ,  $G_2$ , and  $G_3$  in Figure 2 contain minimum geodetic sets  $S_1 = \{u_1, v_1, w_1, x_1\}$ ,  $S_2 = \{u_2, v_2\}$ , and  $S_3 = \{u_3, v_3, w_3\}$ , respectively, with an added property. For every two distinct vertices  $y, z \in S_i$ ,  $i = 1, 2, 3$ ,  $d_{G_i}(y, z) = i$ . This suggests the following definition. A set  $S$  of vertices in a connected graph  $G$  is *uniform* if the distance between every two vertices of  $S$  is the same fixed number. Obviously, if  $S$  is uniform, then  $\langle S \rangle$  is complete or  $S$  is independent. Hence each minimum geodetic set indicated in Figure 2 is uniform.

We define a geodetic set  $S$  to be *essential* if for every two vertices  $u, v$  in  $S$ , there exists a vertex  $w \neq u, v$  of  $G$  that lies in a  $u - v$  geodesic but in no  $x - y$  geodesic for  $x, y \in S$  and  $\{x, y\} \neq \{u, v\}$ . For example the set  $S = \{x, y, z\}$  is an essential geodetic set of the graph  $G$  of Figure 3, while  $S$  is not uniform in  $G$ .

Figure 3. A graph  $G$  with an essential geodetic set

We now show that it is possible for a graph to have a minimum geodetic set with a specified number of vertices designated as essential as well as uniform.

**Theorem 21.** *For each integer  $k \geq 2$ , there exists a connected graph  $G$  with  $g(G) = k$  which contains a uniform, essential minimum geodetic set.*

**Proof.** Let  $K_k^{(k-1)}$  denote the multigraph of order  $k$  for which every two vertices of  $K_k^{(k-1)}$  are joined by  $k-1$  edges. Let  $G_k = S(K_k^{(k-1)})$  be the subdivision graph of  $K_k^{(k-1)}$ . Clearly  $\text{diam } G_k = 3$  if  $k \geq 3$ . We show by induction that  $g(G_k) = k$  and  $V(K_k^{(k-1)})$  is a uniform, essential minimum geodetic set for  $G_k$ .

To begin the inductive proof, for  $k = 2$ , the graph  $G_2 = S(K_2^{(1)})$  is a path of order 3. Therefore,  $g(G_2) = 2$  and the two end-vertices of  $G_2$  form a uniform, essential minimum geodetic set for  $G_2$ . Now we take  $g(G_{k-1}) = k-1$ , where  $k-1 \geq 2$ , and  $V(K_{k-1}^{(k-2)})$  is a uniform, essential minimum geodetic set for  $G_{k-1}$ . We now consider  $G_k$ .

Let  $S = V(K_k^{(k-1)}) = \{v_1, v_2, \dots, v_k\}$ . For each pair  $i, j$ ,  $1 \leq i < j \leq k$ , label the  $k-1$  vertices of degree 2 that are adjacent to both  $v_i$  and  $v_j$  by  $v_{i,j}^1, v_{i,j}^2, \dots, v_{i,j}^{k-1}$ . Since  $I[S] = V(G_k)$ , it follows that  $g(G_k) \leq k$ .

Suppose, to the contrary, that  $g(G_k) = m < k$  and let  $W = \{w_1, w_2, \dots, w_m\}$  be a minimum geodetic set of  $G_k$ . We consider three cases.

*Case 1.*  $W$  is a proper subset of  $\{v_1, v_2, \dots, v_k\}$ . Then  $I[W] = V(G_m)$ , where  $G_m = S(K_m^{m-1})$  with  $V(K_m^{m-1}) = W$ . Therefore,  $I[W] \neq V(G_k)$ , contradicting the fact that  $W$  is a geodetic set of  $G_k$ .

*Case 2.*  $W = \{v_{i,j}^1, v_{i,j}^2, \dots, v_{i,j}^{k-1}\}$  where  $1 \leq i < j \leq k$ . Then  $I[W] = W \cup \{v_i, v_j\} \subset V(G_k)$ , once again contradicting the fact that  $W$  is a geodetic set of  $G_k$ .

*Case 3.* There exist integers  $i, j, p, q$ , where  $1 \leq i < j \leq k$  and  $1 \leq p < q \leq k - 1$ , such that  $v_{i,j}^p \in W$  and  $v_{i,j}^q \notin W$ . Since  $I[W] = V(G_k)$ , there exist  $x, y \in W$  such that  $v_{i,j}^q$  lies on an  $x - y$  geodesic in  $G_k$ . Since  $v_{i,j}^q \notin W$ , it follows that  $2 \leq d(x, y) \leq 3$ .

Suppose first that  $d(x, y) = 2$ . We show that

$$I[W] = I[W - \{v_{i,j}^p\}].$$

In this case,  $\{x, y\} = \{v_i, v_j\}$ , say  $x = v_i$  and  $y = v_j$ . So  $v_{i,j}^q$  lies in the geodesic  $x, v_{i,j}^q, y$  in  $G_k$ . It follows that  $v_{i,j}^p$  lies in the geodesic  $x, v_{i,j}^p, y$  in  $G_k$ , so  $v_{i,j}^p \in I[x, y]$ . Let  $v \notin W$  be a vertex that lies in some  $v_{i,j}^p - w$  geodesic in  $G_k$ , where  $w \in W$ . If  $d(v_{i,j}^p, w) = 2$ , then  $v \in \{x, y\}$ . This contradicts the fact that  $v \notin W$ , so  $d(v_{i,j}^p, w) = 3$ . Thus  $v$  lies in either the geodesic  $v_i, v, w$  or in the geodesic  $v_j, v, w$  in  $G_k$ . Therefore,  $I[W] = I[W - \{v_{i,j}^p\}]$ , contradicting the fact that  $W$  is a minimum geodetic set of  $G_k$ .

Suppose next that  $d(x, y) = 3$ . We show that a geodetic set  $W'$  of a graph  $G_{k-1}$  can be formed from  $W$ , where  $|W'| \leq k - 2$  and which will contradict the induction hypothesis.

In this case, exactly one of  $x$  and  $y$  belongs to  $\{v_i, v_j\}$ , say  $x = v_i$  and  $y \neq v_j$ . Then  $y$  is a subdivision vertex, so  $\deg y = 2$  in  $G_k$ , and  $v_{i,j}^q$  lies in the  $x - y$  geodesic  $x, v_{i,j}^q, v_j, y$  in  $G_k$ . This implies that  $v_{i,j}^p$  also lies in an  $x - y$  geodesic, namely the geodesic  $x, v_{i,j}^p, v_j, y$ , in  $G_k$ . So  $v_{i,j}^p \in I[x, y]$ . Now let  $v \notin W$  be a vertex that lies in some  $v_{i,j}^p - w$  geodesic in  $G_k$ , where  $w \in W$ . If  $d(v_{i,j}^p, w) = 2$ , then  $v = v_j$ . This implies that  $v$  lies in the  $x - y$  geodesic  $x, v_{i,j}^p, v, y$  in  $G_k$ , so  $v \in I[x, y]$  and  $d(v_{i,j}^p, w) = 3$ . Then  $w \in \{v_1, v_2, \dots, v_k\}$ , say  $w = v_h$ . Let

$$W' = W - W \cap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \leq \ell \leq k - 1\}.$$

Since  $v_{i,j}^p, y \in W \cap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \leq \ell \leq k - 1\}$ , it follows that  $|W'| \leq k - 2$ . Let  $G_{k-1} = S(K_{k-1}^{(k-2)})$ , where  $V(K_{k-1}^{(k-2)}) = \{v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$ . We show that  $I[W'] = V(G_{k-1})$ , contradicting the induction hypothesis.

Let  $v \notin W'$  be a vertex of  $G_{k-1}$ . Since  $I[W] = V(G_k)$ , it follows that  $v$  lies in some  $u - w$  geodesic  $P$  in  $G_k$ , where  $u, w \in W$ . Observe that at least one of  $u, w$  must be in  $W'$ , for otherwise,  $P$  contains no vertex in  $G_{k-1}$ . Assume first that  $u, w \in W'$ . Then  $P$  is also a geodesic in  $G_{k-1}$  giving the desired result. Therefore, exactly one of  $u$  and  $w$  belongs to  $W'$ , say  $w \in W'$ . If  $d(u, w) = 2$ , then  $v \in \{v_i, v_h\}$ , contradicting  $v \notin W'$ ,

therefore  $d(u, w) = 3$ . Then  $v$  lies in either the geodesic  $v_i, v, w$ , or in the geodesic  $v_h, v, w$  in  $G_{k-1}$ . It follows that  $I[W'] = V(G_{k-1})$ , which contradicts the induction hypothesis.

Therefore  $S = V(K_k^{(k-1)})$  is a minimum geodetic set of  $G_k$ . Then  $v_{i,j}^\ell$ , where  $1 \leq i < j \leq k$  and  $1 \leq \ell \leq k-1$ , lies in exactly one geodesic, namely the geodesic  $v_i, v_{i,j}^\ell, v_j$ , in  $G_k$ . Moreover,  $d(u, w) = 2$  for all  $u, w \in S$ . Therefore,  $S$  is a uniform, essential minimum geodetic set for  $G_k$ . ■

### 3 Minimal Geodetic Sets

A geodetic set  $S$  in a connected graph  $G$  is called a *minimal geodetic set* if no proper subset of  $S$  is a geodetic set. Of course, every minimum geodetic set is a minimal geodetic set, but the converse is not true. For example, let  $G = K_{2,3}$  of Figure 4 with partite sets  $V_1 = \{x, y\}$  and  $V_2 = \{u, v, w\}$ . Then  $\{u, v, w\}$  is a minimal geodetic set of  $K_{2,3}$  but is not a minimum geodetic set of  $K_{2,3}$  since  $\{x, y\}$  is its unique minimum geodetic set. We define the *upper geodetic number*  $g^+(G)$  as the maximum cardinality of a minimal geodetic set of  $G$ . Obviously,  $g(G) \leq g^+(G)$ . The graph  $G$  of Figure 4 has geodetic number 2 and upper geodetic number 3.

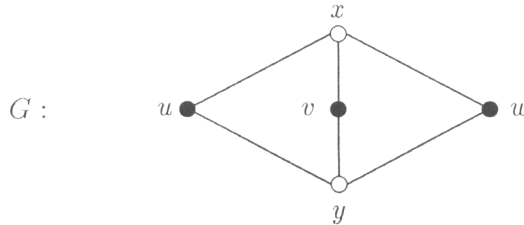


Figure 4. A graph  $G$  with a minimal geodetic set

We now show that every two integers  $a$  and  $b$  with  $2 \leq a \leq b$  are realizable as the geodetic number and upper geodetic number, respectively, of some graph. Furthermore, we determine the minimum order of such a graph. Certainly, this minimum order is at least  $b$ . Indeed, if  $a = b$ , then the only geodetic set of  $K_b$  is its vertex set; so  $g(K_b) = g^+(K_b) = b$  and the minimum order is  $b$ . Indeed, if  $G$  is a graph of order  $n$  with  $g^+(G) = n$ , then  $G = K_n$  and so  $g(G) = g^+(G)$ . Before taking this observation one step further, we present a lemma.

**Lemma 3.1.** *Let  $G$  be a nontrivial connected graph of order  $n$  with  $g^+(G) = n - 1$  and let  $S$  be a minimal geodesic set of maximum cardinality such that  $V(G) - S = \{v\}$ . Then  $G$  does not contain nonadjacent vertices  $u, w \in S$  such that  $u$  and  $w$  are mutually adjacent to both  $v$  and some vertex of  $S$ .*

**Proof.** Suppose, to the contrary, that there exist vertices  $x, y, z \in S$  such that  $xy \notin E(G)$  and  $x$  and  $y$  are mutually adjacent to both  $v$  and  $z$ . Then  $z$  lies in the geodesic  $x, z, y$ , while  $v$  lies in the geodesic  $x, v, y$ . Hence  $S - \{z\}$  is a geodesic set, contradicting the minimality of  $S$ . ■

**Theorem 3.2.** *Let  $G$  be a nontrivial connected graph of order  $n$ . If  $g^+(G) = n - 1$ , then  $g(G) = g^+(G)$ .*

**Proof.** Let  $V(G) = \{v_1, v_2, \dots, v_n\}$ , where  $S = \{v_1, v_2, \dots, v_{n-1}\}$  is a minimal geodesic set of maximum cardinality. First, we claim that every vertex in  $S$  is adjacent to  $v_n$ . Suppose, to the contrary, that some  $v \in S$  is not adjacent to  $v_n$ . Among the pairs  $x, y$  of distinct vertices of  $S$  for which  $v$  lies in some  $x - y$  geodesic, we choose a pair such that  $d(x, y)$  is minimum. If  $v \neq x, y$ , then  $v_n$  lies in some  $u - w$  geodesic of length 2, where  $u, w \in S$  and  $u, w \neq v$ . This implies that  $S - \{v\}$  is a geodesic set, a contradiction. Therefore, either  $x = v$  or  $y = v$ , say the former. We consider two cases.

*Case 1.*  $yv_n \in E(G)$ . Then there are two subcases.

*Subcase 1.1.* Among the vertices of  $S$  adjacent to  $v_n$ , there exists some vertex  $z$  not adjacent to  $y$ .

Here  $v_n$  lies in the geodesic  $y, v_n, z$  in  $G$ . By Lemma 3.1,  $xz \notin E(G)$ . Since  $P : x, y, v_n, z$  is a path in  $G$ , it follows that  $d(x, z) \leq 3$ . Assume first that  $d(x, z) = 2$ . Then there exists a vertex  $w \in S$  adjacent to both  $x$  and  $z$ . By Lemma 3.1,  $wy \notin E(G)$ . Then  $x$  lies in the geodesic  $y, x, w$  in  $G$ , implying that  $S - \{x\}$  is a geodesic set, producing a contradiction. Therefore,  $d(x, z) = 3$ . Thus  $P$  is a geodesic and  $S - \{y\}$  is a geodesic set, which is a contradiction.

*Subcase 1.2.* Every vertex of  $S$  that is adjacent to  $v_n$  is also adjacent to  $y$ . Since  $v_n$  lies in some  $u - w$  geodesic for  $u, w \in S$ , it follows that  $\deg v_n \geq 3$ . Necessarily,  $uw \notin E(G)$ , this is impossible by Lemma 3.1.

*Case 2.*  $yv_n \notin E(G)$ .

Then  $v_n$  lies in some  $u - v$  geodesic of length 2. By Lemma 3.1,  $y$  is not adjacent to both  $u$  and  $v$ , say  $yu \notin E(G)$ . Let  $d(y, u) = \ell$  and let  $y =$

$w_0, w_1, w_2, \dots, w_\ell = u$  be a  $y - u$  geodesic. Since  $yv_n \notin E(G)$ , it follows that  $w_1 \neq v_n$ . If  $w_1 \neq v$ , then  $S - \{w_1\}$  is a geodetic set, which is a contradiction. Thus  $w_1 = v$ . Then  $y, v, v_n, u$  is a geodesic and  $S - \{v\}$  is a geodetic set, contrary to hypothesis.

This completes the proof of the claim. Therefore, for every pair  $x, y$  of nonadjacent vertices in  $S$ , the vertex  $v_n$  lies in the geodetic  $x, v_n, y$ . Clearly,  $diam(G) = 2$ .

Next we show that

$$G = (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_r}) + K_1$$

where  $n_1, n_2, \dots, n_r, r$  are positive integers with  $n_1 + n_2 + \dots + n_r = n - 1$  and  $V(K_1) = \{v_n\}$ , which implies that  $g(G) = g^+(G) = n - 1$ . Suppose, to the contrary, that this is not the case. Then there exist  $x, y, z \in S$  such that  $d(x, y) = 2$  and  $xz, zy \in E(G)$ . It follows that  $z$  and  $v_n$  both lie in some  $x - y$  geodesic. So  $S - \{z\}$  is a geodetic set, which is a contradiction. ■

We can now complete the proof of the realizability of every two integers  $a$  and  $b$  with  $2 \leq a \leq b$  as the geodetic number and upper geodetic number, respectively, of some graph.

**Theorem 3.3.** *For every two positive integers  $a$  and  $b$ , where  $2 \leq a < b$ , there exists a graph  $G$  with  $g(G) = a$  and  $g^+(G) = b$ .*

**Proof.** Let  $F = \overline{K}_{b-a+1} + \overline{K}_2$ , where  $V(K_{b-a+1}) = \{v_1, v_2, \dots, v_{b-a+1}\}$  and  $V(K_2) = \{x, y\}$ . The graph  $G$  is formed from  $F$  by adding  $a - 1$  pendant edges  $yu_i$  ( $1 \leq i \leq a - 1$ ) to the vertex  $y$  of  $F$  (see Figure 5). The graph  $G$  has the unique minimum geodetic set  $S = \{x, u_1, u_2, \dots, u_{a-1}\}$  and so  $g(G) = a$ .

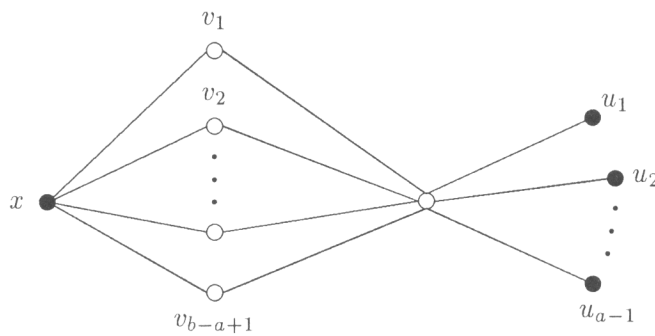


Figure 5. A graph  $G$  with  $g(G) = a$  and  $g^+(G) = b$



Now let

$$S' = \{u_1, u_2, \dots, u_{a-1}, v_1, v_2, \dots, v_{b-a+1}\}.$$

Then  $I[S'] = V(G)$ . We show that  $S'$  is a minimal geodetic set of  $G$ . Let  $v \in S'$ . We show that  $I[S' - \{v\}] \neq V(G)$ . Assume first that  $v = u_i$  for some  $i$  ( $1 \leq i \leq a - 1$ ). Then  $I[S' - \{u_i\}] = V(G) - \{u_i\}$ . So  $v = v_j$  for some  $j$  ( $1 \leq j \leq b - a + 1$ ). Then  $I[S' - \{v_j\}] = V(G) - \{v_j\}$ . It follows that  $I[S' - \{v\}] \neq V(G)$  for every  $v \in S'$ . Since  $|S'| = b$ , we have that  $g^+(G) \geq b$ .

Next we show that there is no minimal geodetic set  $W$  of  $G$  with  $|W| > b$ , which implies that  $g^+(G) = b$ . Note that the graph  $G$  has order  $n = b + 2$ . Since  $g(G) = a < b$ , it suffices to show that  $G$  does not contain an  $(n - 1)$ -element minimal geodetic set. Suppose, to the contrary, that  $W$  is a minimal geodetic set of  $G$  where  $|W| = n - 1$ . Let  $v \notin W$ . Since every geodetic set of  $G$  must contain all end-vertices of  $G$ , it follows that  $v = x$ , for otherwise, the geodetic set  $S = \{x, u_1, u_2, \dots, u_{a-1}\}$  is a proper subset of  $W$ , which contradicts the fact that  $W$  is minimal. Then  $y \in W$ . It follows that  $I[W] = I[W - \{y\}] = V(G)$ . Once again, this contradicts  $W$  being a minimal geodetic set of  $G$ . ■

The proof of Theorem 3.3 shows that if  $b - a \geq 2$  and  $k$  is an integer with  $a < k < b$ , then there need not be a graph  $G$  with  $g(G) = a$  and  $g^+(G) = b$  containing a minimal geodetic set of cardinality  $k$ , that is, a graph  $G$  need not contain an ‘intermediate’ minimal geodetic set.

The following corollary gives the smallest order of a graph satisfying the hypothesis of Theorem 3.3. The proof is a direct consequence of Theorem 3.2 and 3.3.

**Corollary 3.4.** *For every two positive integers  $a$  and  $b$ , where  $2 \leq a < b$ , the smallest order of a graph  $G$  with  $g(G) = a$  and  $g^+(G) = b$  is  $b + 2$ .*

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