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A CLASS OF TIGHT CIRCULANT TOURNAMENTS

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Abstract

A tournament is said to be *tight* whenever every 3-colouring of its vertices using the 3 colours, leaves at least one cyclic triangle all whose vertices have different colours. In this paper, we extend the class of known tight circulant tournaments.

Keywords: Circulant tournament, acyclic disconnection, vertex 3-colouring, 3-chromatic triangle, tight tournament.

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1 Introduction

Let Z_{2m+1} be the set of integers mod $2m+1$. If J is a nonempty subset of $Z_{2m+1}\backslash\{0\}$ such that $|\{j,-j\} \cap J| = 1$ for every $j \in Z_{2m+1}\backslash\{0\}$, then the circulant tournament $\overrightarrow{C}_{2m+1}(J)$ is defined by $V(\overrightarrow{C}_{2m+1}(J)) = Z_{2m+1}$, $A(\overrightarrow{C}_{2m+1}(J)) = \{(i,j) : i,j \in Z_{2m+1} \text{ and } j-i \in J\}$. Finally, for $S \subseteq I_m$, $\overrightarrow{C}_{2m+1}\langle S\rangle$ will denote the circulant tournament $\overrightarrow{C}_{2m+1}(J)$ where $J = (I_m \cup (-S))\backslash S$ and $I_m = \{1, 2, \ldots, m\} \subseteq Z_{2m+1}$.

In [5], the *acyclic disconnection* $\overrightarrow{\omega}(D)$ (resp: the $\overrightarrow{C}_3$*-free disconnection* $\overrightarrow{\omega}_3(D)$) of a digraph D, was defined to be the maximum possible number of connected components of a digraph obtained from D by deleting an acyclic set of arcs (resp: a $\overrightarrow{C}_3$-free set of arcs). It was proved there [5, Theorem 2.4] that $\overrightarrow{\omega}_3^+(D) = \overrightarrow{\omega}_3(D) + 1$ is the minimum number r such that every r-colouring of $V(D)$ using all the colours, leaves at least one heterochromatic cyclic triangle (i.e., a cyclically oriented triangle whose vertices are coloured with 3 different colours). Some related topics are considered in [6].

In [2], the heterochromatic number of a 3-graph (V, E) (hypergraph, all whose edges have cardinality 3) was defined to be the minimum number of colours r such that every vertex r-colouring using all the colours leaves at least one heterochromatic 3-edge; 3-graphs with heterochromatic number 3 were called *tight*. Tight 3-graphs have been studied in [1, 2, 3].

As remarked in [5], if T is any tournament, $\overrightarrow{\omega}_3^+(T)$ is just the heterochromatic number of the 3-graph $H_3(T) = (V(T), \tau_3(T))$ where $\tau_3(T) = \{S \subseteq V(T) : T[S] \cong \overrightarrow{C}_3\}$. We consequently define a tournament T to be *tight* whenever $\overrightarrow{\omega}_3^+(T) = 3$, namely when every 3-colouring of its vertices using the 3 colours, leaves at least one heterochromatic cyclic triangle (cyclic triangle all whose vertices have different colours).

It was proved in [5, Theorem 4.11] that for $m \geq 2$, $\overrightarrow{C}_{2m+1}\langle s \rangle$ is tight provided $s \neq 2$.

In this paper, we prove that if $1 \leq s_1 < s_2 \leq m$ then $\overrightarrow{\omega}_3^+(\overrightarrow{C}_{2m+1}\langle s_1, s_2 \rangle)$ is tight for all but a small set of pairs (s_1, s_2) (Theorem 8) and the exceptional pairs are determinated.

2 Preliminaries

We give here some definitions apart from those given in the Introduction. If D is a digraph, $V(D)$ and $A(D)$ (or simply A) will denote the sets of vertices and arcs of D respectively. If $\gamma = (0, 1, \ldots, m)$ is a directed cycle then we denote by (i, γ, j) the ij-directed path contained in γ, and by $\ell(i, \gamma, j)$ its length. A vertex r-colouring of a digraph is said to be *full* if it uses the r colours. A *heterochromatic cyclic triangle* (h.c. triangle) is a cyclic triangle whose vertices are coloured with 3 different colours. For general concepts we refer the reader to [4].

We will need the following two Lemmas:

Lemma 1. *Let f be a vertex k-colouring of the circulant tournaments $C_{2m+1}(J)$ which leaves no h.c. triangle. If α is either an automorphism or an antiautomorphism of $C_{2m+1}(J)$ then $f.\alpha$ leaves no h.c. triangle.* ■

Lemma 2. *If the circulant tournament* $C_{2m+1}(J)$ *has a full vertex* 3*-colouring* f *which leaves no h.c. triangle then it has another such* 3*-colouring* f' *such that* 0 *and* $m+1$ *belong to different chromatic classes. Moreover, if* t *belongs to a third chromatic class of* f'*, then there is another* 3*-colouring* f'' *leaving no h.c. triangle and such that* 0*,* $m+1$ *and* $m+1-t$ *belong to different chromatic classes of* f''*.*

Proof. $C_{2m+1}(J)$ contains two vertices i and $i+m+1$ belonging to different chromatic classes of f. Let α be an automorphism of $C_{2m+1}(J)$ such that $\alpha(0) = i$ and $\alpha(m+1) = i+m+1$, take $f' = f.\alpha$ and apply Lemma 1. To prove the second part let β the antiautomorphism defined by $\beta(j) = -j+m+1$, take $f'' = f'.\beta$ and apply Lemma 1. ■

Remark 1. In what follows, when we refer the reader to Lemma 2, we are thinking of the antiautomorphism β.

In [2] Neumann-Lara proved the two following results:

Theorem 1 [2]**.** *Every full vertex* 3*-colouring of the circulant tournaments,* $\overrightarrow{C}_{2n+1}(I_n)$ *and* $\overrightarrow{C}_{2n+1}\langle s\rangle$ *with* $(2n+1, s) \neq (9, 2)$ *leaves an h.c. triangle. Moreover* $\overrightarrow{w}^{+}(\overrightarrow{C}_9\langle 2\rangle) = 4$.

Theorem 2 [2]**.** *There exists a full vertex* 3*-colouring of the following circulant tournaments which leaves no h.c. triangle:* $\overrightarrow{C}_9\langle 2\rangle$, $\overrightarrow{C}_3[\overrightarrow{C}_5(1,2)]$*, and* $\overrightarrow{C}_5(1,2)[\overrightarrow{C}_3]$*. Moreover for each of these tournaments* $\overrightarrow{w}_3^{+} = 4$.

Theorem 3. *Every full vertex* 3*-colouring of the following circulant tournaments leaves an h.c. triangle:* $\overrightarrow{C}_5\langle 1,2\rangle$, $\overrightarrow{C}_7\langle 1,2\rangle$, $\overrightarrow{C}_7\langle 1,3\rangle$, $\overrightarrow{C}_7\langle 2,3\rangle$, $\overrightarrow{C}_9\langle 1,2\rangle$, $\overrightarrow{C}_9\langle 1,3\rangle$, $\overrightarrow{C}_9\langle 2,4\rangle$, $\overrightarrow{C}_9\langle 3,4\rangle$, $\overrightarrow{C}_{11}\langle 1,5\rangle$, $\overrightarrow{C}_{11}\langle 2,3\rangle$, $\overrightarrow{C}_{11}\langle 2,5\rangle$, $\overrightarrow{C}_{11}\ \langle 3,5\rangle$, $\overrightarrow{C}_{11}\langle 4,5\rangle$, $\overrightarrow{C}_{13}\langle 2,3\rangle$, $\overrightarrow{C}_{13}\langle 2,4\rangle$, $\overrightarrow{C}_{13}\langle 3,6\rangle$ *and* $\overrightarrow{C}_{13}\langle 5,6\rangle$*.*

Proof. The proof will follow from Lemma 1 and Theorem 1 by applying an automorphism to each circulant tournament ennounced in Theorem 3 which transforms it in some circulant tournament considered in Theorem 1. Along the proof of Theorem 3 and Theorem 4. We will write $D_1 \xrightarrow{i} D_2$ to mean that the function $f_i(x) = ix$ is an isomorphism from D_1 onto D_2.
$\overrightarrow{C}_5\langle 1,2\rangle \overset{-1}{\to} \overrightarrow{C}_5(I_2)$; $\overrightarrow{C}_7\langle 1,2\rangle \overset{-1}{\to} \overrightarrow{C}_7\langle 3\rangle$; $\overrightarrow{C}_7\langle 1,3\rangle \overset{-3}{\to} \overrightarrow{C}_7(I_3)$; $\overrightarrow{C}_7\langle 2,3\rangle \overset{-1}{\to} \overrightarrow{C}_7\langle 1\rangle$; $\overrightarrow{C}_9\langle 1,2\rangle \overset{-2}{\to} \overrightarrow{C}_9(I_4)$; $\overrightarrow{C}_9\langle 1,3\rangle \overset{-1}{\to} \overrightarrow{C}_9\langle 2,4\rangle \overset{-2}{\to} \overrightarrow{C}_9\langle 1,2\rangle \overset{-2}{\to} \overrightarrow{C}_9(I_4)$;

$\overrightarrow{C}_9\langle 3,4\rangle \xrightarrow{2} \overrightarrow{C}_9(I_4)$; $\overrightarrow{C}_{11}\langle 1,5\rangle \xrightarrow{8} \overrightarrow{C}_{11}\langle 1\rangle$; $\overrightarrow{C}_{11}\langle 2,3\rangle \xrightarrow{3} \overrightarrow{C}_{11}(I_5)$; $\overrightarrow{C}_{11}\langle 2,5\rangle \xrightarrow{4} \overrightarrow{C}_{11}(I_5)$; $\overrightarrow{C}_{11}\langle 3,5\rangle \xrightarrow{6} \overrightarrow{C}_{11}\langle 5\rangle$; $\overrightarrow{C}_{11}\langle 4,5\rangle \xrightarrow{2} \overrightarrow{C}_{11}\langle 5\rangle$; $\overrightarrow{C}_{13}\langle 2,3\rangle \xrightarrow{-2} \overrightarrow{C}_{13}\langle 2\rangle$; $\overrightarrow{C}_{13}\langle 2,4\rangle \xrightarrow{5} \overrightarrow{C}_{13}\langle 1\rangle$; $\overrightarrow{C}_{13}\langle 3,6\rangle \xrightarrow{4} \overrightarrow{C}_{13}\langle 5,6\rangle \xrightarrow{2} \overrightarrow{C}_{13}\langle 5\rangle$; $\overrightarrow{C}_{13}\langle 5,6\rangle \xrightarrow{2} \overrightarrow{C}_{13}\langle 5\rangle$. ■

Theorem 4. *There exists a full vertex 3-colouring of the following circulant tournaments which leaves an h.c. triangle:* $\overrightarrow{C}_9\langle 2,3\rangle$, $\overrightarrow{C}_9\langle 1,4\rangle$, $\overrightarrow{C}_{15}\langle 2,5\rangle$ *and* $\overrightarrow{C}_{15}\langle 3,4\rangle$. *Moreover* $\overrightarrow{w}_3^{+} = 4$ *for each of these tournaments.*

Proof. The proof will follow from Lemma 1 and Theorem 2 by aplying: Consider the automorphism φ: $\overrightarrow{C}_9\langle 2,3\rangle \rightarrow \overrightarrow{C}_9\langle 2\rangle$ defined as follows: $\varphi(0) = 0$, $\varphi(2) = 2$, $\varphi(3) = 6$, $\varphi(4) = 1$, $\varphi(5) = 8$, $\varphi(6) = 3$, $\varphi(7) = 7$ and $\varphi(8) = 5$; $\overrightarrow{C}_9\langle 1,4\rangle \xrightarrow{-1} \overrightarrow{C}_9\langle 2,3\rangle \xrightarrow{\varphi} \overrightarrow{C}_9\langle 2\rangle$; because of [2] $\overrightarrow{C}_{15}\langle 2,5\rangle \cong \overrightarrow{C}_3[\overrightarrow{C}_5(I_2)]$ and $\overrightarrow{C}_{13}\langle 3,4\rangle \cong \overrightarrow{C}_5(I_2)[\overrightarrow{C}_3]$. ■

3 Main Result

Theorem 5. *Every full vertex 3-colouring of the circulant tournament* $\overrightarrow{C}_{2n+1}\langle s_1,s_2\rangle$ *such that* $1 \le s_1 < s_2 \le n$ *and* $\overrightarrow{C}_{2n+1}\langle s_1,s_2\rangle \notin \left\{\overrightarrow{C}_{15}\langle 3,4\rangle, \overrightarrow{C}_{15}\langle 2,5\rangle, \overrightarrow{C}_9\langle 2,3\rangle, \overrightarrow{C}_9\langle 1,4\rangle\right\}$ *leaves an h.c. triangle.*

Proof. Consider any full vertex 3-colouring of $D = \overrightarrow{C}_{2n+1}\langle s_1,s_2\rangle$ as in the hypothesis with colors red, blue and white and denote by R, B and W (respectively) the chromatic classes. Without loss of generality, we can assume $n+1 \in R$ and $0 \in B$. Along the proof we will denote $(i \notin W, (i,j,k))$ to mean that we can assume the vertex i is not white because if the vertex i is white, then we have the h.c. triangle (i,j,k) and we are done.

The sequence $\gamma_1 = (0,1,2,\ldots,2n,0)$; will be a directed cycle when $s_1 \neq 1$ and the sequence $\gamma_2 = (0,2n,2n-1,2n-2,\ldots,0)$ a directed cycle when $s_1 = 1$.

We will make the proof by considering several cases

Case 1. Let $2 \le s_1 < s_2 \le n-1$ and there exists $i \in (0,\gamma,n+1) \cap W$ such that $\{(0,i),(i,n+1)\} \subseteq A(D)$.

Clearly, in this case $(0, i, n+1)$ is an h.c. triangle.

Case 2. Let $2 \le s_1 < s_2 \le n-1$ and the vertex $s_1 \in W$. (notice $(s_1, 0) \in A(D)$).

Subcase 2.a. Assume $s_1 + s_2 < n$.
Let $j \in (n+1, \gamma, 0)$ such that $\ell(j, \gamma, 0) = s_1$.

Since $s_1 + s_2 < n$ we have $\{(s_1, n+1),\ (n+1, j),\ (j, s_1)\} \subseteq A(D)$. $j \in W$: $(j \notin R,\ (j, s_1, 0))$, $(j \notin B,\ (j, s_1, n+1))$. Each vertex t with $t \in (0, \gamma, s_1) - \{0, s_1\}$ is blue: $(t \notin W,\ (t, n+1, 0))$, $(t \notin R,\ (t, s_1, 0))$.

Now we consider several possibilities:

If s_1 and s_2 are not consecutives $(s_2 \ne s_1 + 1)$ then $(j, 1, n+1)$ is an h.c. triangle.

If s_1 and s_2 are consecutives $(s_2 = s_1 + 1)$, we have:

Let $s_1 > 2$.
$2 \in (0, \gamma, s_1) - \{0, s_1\}$, so $2 \in B$ and $(2, n+1, j)$ is an h.c. triangle.

When $s_1 = 2$ we have $s_2 = 3$ and consider $k \in (n+1, \gamma, 0)$ such that $\ell(k, \gamma, 0) = s_2$; since $s_1 + s_2 < n$ we have $\{(k, s_1), (0, k)\} \subseteq A(D)$, and $(k \notin R, (k, s_1, 0))$.

If $(n+1, k) \in A$ then $(k \notin B, (k, s_1, n+1))$. Hence $k \in W$ and $(k, 1, n+1)$ is an h.c. triangle. When $(k, n+1) \in A$ we have $\ell(n+1, \gamma, k) = s_2$; so $2s_2 = n$, $n = 6$ and $D \cong C_{13}\langle 2, 3\rangle$.

Subcase 2.b. Assume $s_1 + s_2 \ge n+1$.
Let $k \in (0, \gamma, n+1)$ such that $\ell(k, \gamma, n) = s_2$ (notice $(n, k) \in A$), Since $s_1 + s_2 \ge n+1$ and $s_2 < n$ we have $k \in (0, \gamma, s_1) - \{0, s_1\}$; k is blue: $(k \notin R, (k, s_1, 0)), (k \notin W, (k, n+1, 0))$; n is blue: $(n \notin R, (n, k, s_1)$ when $(s_1, n) \in A$; and $(n, s_1, 0)$ when $(n, s_1) \in A)$, $(n \notin W, (n, n+1, 0))$.

Now we will prove that we can assume $(s_1, n+1) \in A$. Suppose $(n+1, s_1) \in A$; hence $\ell(s_1, \gamma, n+1) \in \{s_1, s_2\}$. When $(s_1, n) \in A$, $(n+1, s_1, n)$ is an h.c. triangle. So $(n, s_1) \in A$, $\ell(s_1, \gamma, n+1) = s_2$, $s_2 = s_1 + 1$ and $s_2 + s_1 = n+1$. Now, when $s_1 = 2$ we have $s_2 = 3$, $n+1 = 5$ and $D \cong C_9\langle 2, 3\rangle$. And when $s_1 > 2$ we consider, $n-1$; $n-1 \in W$: $(n-1 \notin R, (n-1, n, s_1))$, $(n-1 \notin B, (n-1, n+1, s_1)$ (notice $s_1 > 2$). And we have $(n-1, n+1, 0)$ an h.c. triangle. So we will assume $(s_1, n+1) \in A$. Now $k+1 \in W$: $(k+1 \notin R, (k+1, s_1, 0)), (k+1 \notin B, (k+1, s_1, n+1))$, (notice $k+1 \ne s_1$ since $(s_1, n+1) \in A$ and $(n+1, k+1) \in A$).

If $(k+1, n) \in A$, then $(k+1, n, n+1)$ is an h.c. triangle, so we will assume $(n, k+1) \in A$ (notice that $\ell(k+1, \gamma, n) = s_1, (n+1, k+2) \in A$ and $k+2 \ne s_1$).

Finally, consider $k+2 : (k+2 \notin R, (k+2, s_1, 0)), (k+2 \notin B, (k+2, s_1, n+1))$; hence $k+2$ is white and $(k+2, n, n+1)$ is an h.c. triangle.

Subcase 2.c. $s_1 + s_2 = n$.
First assume $s_1 \neq 2$.

Let $k, t \in (n+1, \gamma, 0)$ such that $\ell(k, \gamma, 0) = s_2$ and $\ell(t, \gamma, s_1) = s_2$. $k \in B : (k \notin R, (k, s_1, 0))$, $(k \notin W, (k, n+1, 0))$; $n+2 \in B : (n+2 \notin R, (n+2, k, s_1))$, $(n+2 \notin W, (n+2, k, n+1))$; $t \in B : (t \notin R, (t, k, s_1))$, $(t \notin W, (t, k, n+1)$ when $(n+1, t) \in A$ and $(t, n+1, 0)$ when $(t, n+1) \in A)$. (Notice that $(t, n+1) \in A$ implies $(0, t) \in A$ because $s_1 + s_2 + n$); also $1 \in B : (1 \notin R, (1, s_1, 0))$, $(1 \notin W, (1, n+1, 0))$.

Now we consider two possibilities:
When s_1 and s_2 are not consecutives ($s_2 \neq s_1 + 1$) we consider $2n$; $(2n \notin B, (2n, s_1, n+1))$, (Notice $(n+1, 2n) \in A$ because $s_1 \geq 2$ and $s_1 + s_2 = n$, so $s_2 \leq n-2$), $(2n \notin R, (2n, s_1, n+1))$ (Notice $(n+2, 2n) \in A$ because $\{s_1, s_2\} \neq \{2, n-2\}$). Hence $2n$ is white and then $(2n, 1, n+1)$ is an h.c. triangle, (notice again that $(2n, 1) \in A$ because $s_1 \neq 2$).

When s_1 and s_2 are consecutives ($s_2 = s_1 + 1$), observe that when $s_1 = 2$ we have $s_2 = 3$ and $D \cong \overrightarrow{C}_{11}\langle 2, 3\rangle$. So we will assume $s_1 > 2$, and consider $2n-1$; $(2n-1 \notin B, (2n-1, s_1, n+1))$ (notice $(n+1, 2n-1) \in A$ because $s_1 \neq 2$ and hence $s_2 \neq n-2$), $(2n-1 \notin R, (2n-1, s_1, n+2))$ (notice that we can assume $(n+2, 2n-1) \in A$ because if $(2n-1, n+2) \in A$ then $s_2 = n-3$, $s_1 = 3$, $s_2 = 4$, $n = 7$ and $D \cong \overrightarrow{C}_{15}\langle 3, 4\rangle$). Hence $2n-1$ is white and then $(2n-1, 1, n+1)$ is an h.c. triangle (notice that we can assume $(2n-1, 1) \in A$ because when $(1, 2n-1) \in A$ we have $s_1 = 3$, $s_2 = n-3$, $s_2 = 4$, $n = 7$ and $D \cong C_{15}\langle 3, 4\rangle$).

Now assume $s_1 = 2$, $s_2 = n-2$.
When $s_2 = s_1 + 1$ we obtain $D \cong \overrightarrow{C}_{11}\langle 2, 3\rangle$; so we will assume $s_2 \neq s_1 + 1$. $n \in B : (n \notin R, (n, 2, 0))$, $(n \notin W, (n, n+1, 0); 1 \in B : (1 \notin R, (1, 2, 0)))$; $(1 \notin W, (1, n+1, 0))$; $3 \in B : (3 \notin R, (3, 1, 2))$, $(3 \notin W, (3, n, n+1))$; $n+3 \in B : (n+3 \notin R, (n+3, 2, 0))$, $(n+3 \notin W, (n+3, n+1, 0))$; $n+2 \in B : (n+2 \notin R, (n+2, 1, 2))$, $(n+2 \notin W, (n+2, n, n+1))$; $4 \in B : (4 \notin R, (4, 2, 3))$, $(4 \notin W, (4, n+1, n+2))$; $2n \in R : (2n \notin B, (2n, 2, n+1))$ (notice that $(2n, 2) \in A$ because $s_1 = 2$ and $s_2 \neq s_1 + 1$), $(2n \notin W, (2n, 4, n+1))$ (notice that we can assume $(2n, 4) \in A$, because when $(4, 2n) \in A$ we obtain $s_2 = 5$, $n-2 = 5$, $n = 7$ and $D \cong \overrightarrow{C}_{15}\langle 2, 5\rangle$). Finally consider $n-3$; first notice that $(n-3, 2n) \in A$ because $\ell(2n, \gamma, n-3) = n-2$ and $(0, n-3) \in A$ because $s_2 \neq s_1 + 1$. We have $(n-3 \notin W, (n-3, 2n, 0))$.

We can assume $(2, n-3) \in A$ because if $(n-3, 2) \in A$ then $\ell(2, \gamma, n-3) = s_1$, $s_2 = 2s_1 + 1$ and $D \cong \overrightarrow{C}_{15}\langle 2, 5\rangle$. So $(n-3 \notin R, (n-3, n, 2))$; we conclude that $n-3$ is blue and then $(n-3, 2n, 2)$ is an h.c. triangle (notice $(2n, 2) \in A$ because $s_2 \neq s_1 + 1$).

Case 3. Let $2 \leq s_1 < s_2 \leq n-1$ and the vertex $n+1-s_1 \in W$.
This case follows directly from Case 2 by applying Lemma 2.

Case 4. Let $2 \leq s_1 < s_2 \leq n-1$ and the vertex $s_2 \in W$. (notice $(s_2, 0) \in A$).

Subcase 4.a. Assume the hypothesis on Case 4 and $s_1 + s_2 < n$. First we prove that we can assume $(s_2, n+1) \in A$. Suppose $(n+1, s_2) \in A$, then $\ell(s_2, \gamma, n+1) = s_2$ (since $s_1 + s_2 < n$), $2s_2 = n+1$ and $s_2 \neq s_1 + 1$ ($s_2 = s_1 + 1$ implies $s_1 + s_2 = n$).
$n \in R$: $(n \notin W, (0, n, n+1))$, $(n \notin B, (n, n+1, s_2))$ (notice that $(s_2, n) \in A$ because $s_1 \neq s_2 - 1$).
$s_2 - 1 \in W$: $(s_2 - 1 \notin R, (s_2 - 1, s_2, 0))$ (notice $s_1 \neq s_2 - 1$), $(s_2 - 1 \notin B, (s_2 - 1, s_2, n))$. So $(0, s_2 - 1, n+1)$ is an h.c. triangle.

We will assume $(s_2, n+1) \in A$.
Let $j \in (n+1, \gamma, 0)$ such that $\ell(j, \gamma, 0) = s_1$; since $s_1 + s_2 < n$ we have $\{(j, s_2), (n+1, j)\} \subseteq A$.
$j \in W$: $(j \notin R, (j, s_2, 0))$, $(j \notin B, (j, s_2, n+1))$.

Now consider s_1, since $s_1 + s_2 < n$ we have $\{(j, s_1), (s_1, n+1)\} \subseteq A$. and hence $(s_1 \notin R, (s_1, 0, j))$, $(s_1 \notin B, (s_1, n+1, j))$. We conclude $s_1 \in W$ and we are in Subcase 2.a.

Subcase 4.b. Assume $s_1 + s_2 \geq n+1$.
Notice that when $s_1 + s_2 = n+1$, $s_2 = n+1-s_1$, hence $n+1-s_1 \in W$ and we are in Case 3. So we will assume $s_1 + s_2 \geq n+2$. Consider $n+1-s_1$; we can assume $n+1-s_1 \notin W$ because when $n+1-s_1 \in W$ we are in Case 3; $(n+1-s_1 \notin B, (n+1-s_1, s_2, n+1))$; hence $n+1-s_1 \in R$. So when $(0, n+1-s_1) \in A$ we have $(n+1-s_1, s_2, 0)$ an h.c. triangle. Then we can assume and we will assume $(n+1-s_1, 0) \in A$, and then $2s_1 = n+1$. Consider $n+1-s_2$; $n+1-s_2 \in R$: $(n+1-s_2 \notin W, (n+1-s_2, n+1-s_1, 0))$, $(n+1-s_2 \notin B, (n+1-s_2, s_2, n+1))$ when $(s_2, n+1-s_2) \in A)$. So when $(n+1-s_2, s_2) \in A$ and $(n+1-s_2, n+1-s_1, s_2)$ when $(n+1-s_2, s_2) \in A$ we have $(n+1-s_2, s_2, 0)$ an h.c. triangle (notice $(0, n+1-s_2) \in A$ since $2s_1 = n+1$ and $s_1 + s_2 \geq n+2$ imply $n+1-s_2 \in (0, \gamma, n+1-s_1 = s_1))$. Then we can assume and we will assume $(s_2, n+1-s_2) \in A$.

Notice that s_1 and s_2 are not consecutives. When $s_2 = s_1 + 1$ we have $(n+1-s_2)+1 = n+1-s_1$; we are assuming $(n+1-s_1, 0) \in A$ hence $l(0, \gamma, n+1-s_1) = s_1$ and $(s_2, n+1-s_2) \in A$ hence $l(n+1-s_2, s_2) = s_1$ and we conclude $2s_1 + 1 = s_2$, then $s_1 + 1 = 2s_1 + 1$ and $s_1 = 0$ which is impossible.

Finally, consider $s_2 - 1$ since $s_2 \neq s_1 + 1$ we have $s_2 - 1 \neq n+1-s_1$ (notice $n+1-s_1 = s_1$);$(s_2-1 \notin W, (s_2-1, n+1, 0)), (s_2-1 \notin B, (s_2-1, s_2, n+1-s_2))$ (notice $(n+1-s_2, s_2-1) \in A$ because $l(n+1-s_2, s_2) = s_1$). Hence $s_2 - 1$ is red and then $(s_2-1, s_2, 0)$ is an h.c. triangle.

Subcase 4.c. $s_1 + s_2 = n$.

First assume $s_1 \neq 2$.

Let $k \in (n+1, \gamma, 0)$ such that $l(k, \gamma, 0) = s_1$, notice $(0, k) \in A$. $k \in B$: $(k \notin R, (k, s_2, 0)), (k \notin W, (k, n+1, 0))$ (notice $(k, n+1) \in A$ because $s_1 + s_2 = n$).

When $s_2 = s_1 + 1$ we consider $k-1$; $(k-1 \notin B, (k-1, n+1, s_2)), (k-1 \notin R, (k-1, k, s_2))$, hence $k-1$ is white and then $(k-1, n+1, 0)$ is an h.c. triangle (notice that $s_2 = s_1 + 1$ and $s_1 + s_2 = n$ imply $\{(0, k-1), (k-1, n+1)\} \subseteq A$).

So we will assume $s_2 \neq s_1 + 1$.

$n \in B$: $(n \notin W, (0, n, n+1)), (n \notin R, (n, s_2, 0))$; $s_2 - 1 \in B$: $(s_2 - 1 \notin W, (s_2 - 1, n+1, 0)$ when $(s_2 - 1, n+1) \in A$ and $(s_2 - 1, n, n+1)$ when $(n+1, s_2 - 1) \in A)$; $k - 1 \in B$: $(k-1 \notin W, (k-1, n, n+1))$ (notice that $(k-1, n) \in A$ because $s_1 + s_2 = n$), $(k-1 \notin R, (k-1, n, s_2))$.

Finally, consider $k+1$; $(k+1 \notin B, (k+1, s_2, n+1))$ (notice $(s_2, n+1) \in A$ because $s_1 + s_2 = n$ and $s_2 \neq s_1 + 1$), $(k+1 \notin W, (k+1, s_2 - 1, n+1))$

(We can assume $(s_2 - 1, n+1) \in A$ because when $(n+1, s_2-1) \in A$ we have $(n+1, s_2 - 1, s_2)$ an h.c. triangle, and we can assume $(k+1, s_2 - 1) \in A$ because when $(s_2 - 1, k+1) \in A$ we have $l(k+1, \gamma, s_2 - 1) = s_2, s_1 = 2$ and $s_2 = n - 2$). We conclude $k+1$ is red and then $(k+1, s_2, k-1)$ is an h.c. triangle. ($(k-1, k+1) \in A$ because $s_1 \neq 2$).

Now assume $s_1 = 2$ (hence $s_2 = n - 2$).

$n \in W$: $(n \notin R, (n, n-2, 0))$, $(n \notin W, (n, n+1, 0))$.

$1 \in B$: $(1 \notin R, (1, n-2, 0))$, (we can assume $(1, n-2) \in A$ because when $(n-2, 1) \in A$ we have $s_2 = s_1 + 1$, $s_2 = 3$, $s_1 = 2$ and $D \cong \overrightarrow{C}_{11}\langle 2, 3\rangle$), $(1 \notin W, (1, n+1, 0))$. $n+3 \in B$: $(n+3 \notin R, (n+3, 1, n-2))$ (We can assume $(n-2, n+3) \in A$ because when $(n+3, n-2) \in A$ we have $s_2 = 5$, $s_1 = 2$ and $D \cong \overrightarrow{C}_{15}\langle 2, 5\rangle$. And we can assume $(n+3, 1) \in A$ because when $(1, n+3) \in A$ we have $s_2 = n-1$ but we are assuming $s_2 = n-2$). $2n \in W$:

$(2n \notin R, (2n, n-2, n+3))$ (We can assume $(n+3, 2n) \in A$ because when $(2n, n+3) \in A$ we have $s_1 = n-3 = 2$, $n = 5$ and $D \cong \overrightarrow{C}_{11}\langle 2, 3\rangle$).
Finally, consider $n-4$; $(n-4 \notin R, (n-4, n, n-2))$ (We can assume $(n-4, n) \in A$ because when $(n, n-4) \in A$ we have, $s_2 = 4$, $n = 6$ and $D \cong \overrightarrow{C}_{13}\langle 2, 4\rangle$), $(n-4 \notin W, (n-4, n+1, 0))$ (We can assume $(n-4, n+1) \in A$ because otherwise we obtain $s_2 = 5$, $s_1 = 2$, $n = 7$ and $D \cong \overrightarrow{C}_{15}\langle 2, 5\rangle$. And we can assume $(0, n-4) \in A$ because in other case $s_1 = n-4 = 2$, $n = 6$ and $D \cong C_{13}\langle 2, 4\rangle$. Hence $n-4$ is blue and then $(n-4, n+1, 2n)$ is an h.c. triangle. (We can assume $(2n, n-4) \in A$ because when $(n-4, 2n) \in A$ we have $s_1 = n-3$, $n = 5$ and $D \cong \overrightarrow{C}_{11}\langle 2, 3\rangle$).

Case 5. When $2 \leq s_1 < s_2 \leq n-1$ and the vertex $n+1-s_2 \in W$. This case follows directly from Case 4 by applying Lemma 2.

Case 6. When $2 \leq s_1 < s_2 \leq n-1$ and there exists a vertex $i \in (n+1, \gamma, 0)$, $i \in W$ such that $\ell(i, \gamma, 0) \in \{s_1, s_2\}$. Since $\ell(i, \gamma, 0) = s_1$ or $\ell(i, \gamma, 0) = s_2$ we have $(0, i) \in A$. We will assume $(n+1, i) \in A$ because when $(i, n+1) \in A$ we have $(i, n+1, 0)$ an h.c. triangle.

Observe now that we can assume $n \notin R$. Because when n is red, considering the automorphism $f\colon V(D) \to V(D)$ such that $f(x) = x+n+1$ and interchanging the colors blue and red we obtain the Case 3 when $\ell(i, \gamma, 0) = s_1$ and the Case 5 when $\ell(i, \gamma, 0) = s_2$. And by Lemma 1 we obtain an h.c. triangle.
$n \in B$; it follows from the observation above and the fact $(n \notin W, (n, n+1, 0))$.

We will assume $(n, i) \in A$. Because when $(i, n) \in A$ we have $(i, n, n+1)$ an h.c. triangle.

Now we consider two possible cases:

Subcase 6.a. $s_1 + s_2 \leq n$.
Since $s_1 + s_2 \leq n$ we have $\{(s_1, n+1), (i, s_1)\} \subseteq A$.

Consider s_1; we can assume $s_1 \notin W$ because when $s_1 \in W$ we are in Case 2, $(s_1 \notin R, (s_1, 0, i))$; hence s_1 is blue and then $(s_1, n+1, i)$ is an h.c. triangle.

Subcase 6.b. Assume $s_1 + s_2 \geq n+1$.
When $\ell(i, \gamma, 0) = s_2$ or $\ell(i, \gamma, 1) = s_2$ we consider $j \in V(\gamma)$ such that $\ell(j, \gamma, i) = s_1$; since $s_1 + s_2 \geq n+1$, $(n, i) \in A$ and $(n+1, i) \in A$ we have $j \in (1, \gamma, n-1)$. If $j \in W$ then we obtain some of the cases 1 to 5 and we

are done, $(j \notin R, (j, n, i))$ (notice $(i, j) \in A$ because $\ell(j, \gamma, i) = s_1$), hence j is blue and then $(j, n+1, i)$ is an h.c. triangle. So we have $\ell(i, \gamma, 0) = s_1$ and $\ell(i, \gamma, 1) \neq s_2$ (in particular $s_2 \neq s_1 + 1$ and $(i, 1) \in A$).
Now we will prove that we can assume $(1, n) \in A$. When $(n, 1) \in A$ we have $s_1 = n - 1$ or $s_2 = n - 1$ but since $s_1 < s_2 \leq n - 1$ we conclude $s_2 = n - 1$. Since $s_2 = n - 1$ we have $(i, i+n+2) \in A$. When $\{(i+n+2, n), (i+n+2, n+1)\} \subseteq A$ we consider $i + n + 2$; since $i + n + 2 \in (0, \gamma, n - 1)$ we can assume $i + n + 2 \notin W$ (because when $i + n + 2 \in W$ we are in some of the cases 1 to 5 and we are done), $(i + n + 2 \notin B, (i + n + 2, n + 1, i))$ hence $i + n + 2 \in R$ and then $(i + n + 2, n, i)$ is an h.c. triangle. So, we have $\ell(i+n+2, \gamma, n) = s_1$ or $\ell(i+n+2, \gamma, n+1) = s_1$; in any case we have $\ell(i+n, \gamma, n) \neq s_1$ and $\ell(i+n, \gamma, n+1) \neq s_1$. Observe that $\ell(i+n, \gamma, n) \neq s_2$ because when $\ell(i+n, \gamma, n) = s_2 = n-1$ we have $i+n = n$ and then $s_1 = n-1$ which is impossible because $s_1 < s_2$. Also observe that $\ell(i+n, \gamma, n+1) \neq s_2$ because when $\ell(i+n, \gamma, n+1) = s_2 = n-1$ we obtain $i+n = 2$ and $s_1 = n-2$ but we have $s_2 \neq s_1 + 1$. We conclude that $\{(i + n, n), (i + n, n + 1)\} \subseteq A$. Now consider $i + n$; we can assume $i + n \notin W$ (see cases 1 to 5), $(i + n \notin R, (i+n, n, i))$ hence $i+n$ is blue and then $(i+n, n+1, i)$ is an h.c. triangle. So we will assume $(1, n) \in A$.

Finally, consider 1; $(1 \notin W, (0, 1, n + 1))$, $(1 \notin B, (1, n + 1, i))$ hence $1 \in R$ and then $(1, n, i)$ is an h.c. triangle.

Case 7. Let $2 \leq s_1 < s_2 \leq n-1$ and; $n+1+s_1 \in W$ or $n+1+s_2 \in W$. This case follows directly from Case 6 by applying Lemma 2.

Case 8. Let $2 \leq s_1 < s_2 \leq n - 1$ and there exists $j \in (n + 1, \gamma, 0)$ such that $j \in W$, and $\{(n + 1, j), (j, 0)\} \subseteq A$.

First we will prove that in this case we can assume $(n, j) \in A$.
Suppose $(j, n) \in A$; $(n \notin B, (n, n + 1, j))$, $(n \notin W, (n, n + 1, 0))$. Hence n is red and $\ell(n, \gamma, j) \in \{s_1, s_2\}$. And now considering the automorphism $f: V(D) \to V(D)$ such that $f(t) = t+n+1$ and interchanging the colors red and blue we obtain Case 3 or Case 5 and we are done. So we will assume $(n, j) \in A$.

Observe that we can assume $(j, 1) \in A$.
When $(1, j) \in A$ we have $(1 \notin R, (1, j, 0))$, moreover $(1 \notin W, (1, n + 1, 0))$. Hence $1 \in B$ and now considering the automorphism $f: V(D) \to V(D)$ such that $f(t) = t+n$ and interchanging the colors blue and red we obtain Case 3 or Case 5 and we are done. So we will assume $(j, 1) \in A$.
$n \in B$; $(n \notin W, (n, n + 1, 0))$, $(n \notin R, (n, j, 0))$.

$1 \in R$; $(1 \notin W, (1, n+1, 0))$ $(1 \notin B, (1, n+1, j))$.

So when $(1, n) \in A$ we have $(1, n, j)$ an h.c. triangle. Then we will assume $(1, n) \in A$. Hence $s_2 = n - 1$.

Since $s_2 = n - 1$, and $n \neq s_1$, $n \neq s_2$ we have;
$\{(j+n-1, j), (j, j+n), (j+n+1, j), (j, j+n+2)\} \subseteq A$. Since $\{j+n, j+n+1\} \subseteq V(1, \gamma, n)$ we can assume $\{j+n, j+n+1\} \cap W = \emptyset$ because if $\{j+n, j+n+1\} \cap W \neq \emptyset$ then we are in some of the Cases 1 to 5 and we are done. We conclude $j+n \notin W$ and $j+n+1 \notin W$. (i.e., $\{j+n, j+n+1\} \subseteq R \cup B$). When $j+n$ and $j+n+1$ have different colors we obtain the h.c. triangle $(j+n, j+n+1, j)$ so we can assume they have the same color and we will analyze the two possibilities:

Subcase 8.a. $\{j+n, j+n+1\} \subseteq R$.
In this case we can assume $(j+n+1, 0) \in A$ because when $(0, j+n+1) \in A$ we obtain $(0, j+n+1, j)$ an h.c. triangle. Hence $(j+n+1, 0) \in A$ and $\ell(0, \gamma, j+n+1) \in \{s_1, s_2\}$.

If $\ell(0, \gamma, j+n+1) = s_1$ then $\{(0, j+n-1), (1, j+n-1)\} \subseteq A$ and we consider $j+n-1$; $(j+n-1 \notin R, (j+n-1, j, 0))$, $(j+n-1 \notin B, (j+n-1, j, 1))$, hence $j+n-1 \in W$ and we are in some of the cases 1 to 5.

If $\ell(0, \gamma, j+n+1) = s_2$ then $j+n+1 = n-1$ (remember $s_2 = n-1$) and $j+n+2 = n$ which is impossible because $\{(j, j+n+2), (n, j)\} \subseteq A$.

Subcase 8.b. $\{j+n, j+n+1\} \subseteq B$.
In this case, we can assume $(n+1, j+n) \in A$ because when $(j+n, n+1) \in A$ we have $(j+n, n+1, j)$ an h.c. triangle.

Hence $(n+1, j+n) \in A$ and $\ell(j+n, \gamma, n+1) \in \{s_1, s_2\}$.
When $\ell(j+n, \gamma, n+1) = s_1$ we have $\{(j+n+2, n), (j+n+2, n+1)\} \subseteq A$ and we consider $j+n+2$; $(j+n+2 \notin R, (j+n+2, n, j)), (j+n+2 \notin B, (j+n+2, n+1, j))$; so $j+n+2 \in W$ and we are in some of the cases 1 to 5.

When $\ell(j+n, \gamma, n+1) = s_2$ we have $j+n = 2$ (remember $s_2 = n-1$) and $j+n-1 = 1$ which is impossible because $\{(j+n-1, j), (j, 1) \subseteq A$.

Case 9. $s_1 = 1$ and $1 \in W$ (remember we are assuming $n+1 \in R$, and $0 \in B$).

Subcase 9.a. $s_2 = n$.
In this case $(0, n+1, 1)$ is an h.c. triangle.

Subcase 9.b. $s_2 = n - 1$.

In this case we will assume $s_2 \neq 2$ because when $s_2 = 2$ we obtain $n-1=2$ and $D \cong \overrightarrow{C}_7\langle 1,2\rangle$.
$2n \in B$; ($2n \notin R$, $(2n,1,0)$) (notice $(2n,1) \in A$ because $s_2 \neq 2$), ($2n \notin W$, $(0,2n,n+1)$) (notice $(2n,n+1) \in A$ because $s_2 = n-1$).
$n \in R$; ($n \notin W$, $(n,2n,n+1)$), ($n \notin B$, $(n,1,n+1)$). Hence $(1,0,n)$ is an h.c. triangle.

Subcase 9.c. $s_2 = 2$.
In this case we will assume $n \geq 5$ because when $n=2$, $D \cong \overrightarrow{C}_5\langle 1,2\rangle$, when $n=3$, $D \cong \overrightarrow{C}_7\langle 1,2\rangle$ and when $n=4$, $D \cong \overrightarrow{C}_9\langle 1,2\rangle$. Hence we have $\ell(n+1,\gamma,2n-1) \geq 3$, $2n-1 \neq 3$, and $\ell(3,\gamma,n+1) \geq 3$.
$2n-1 \in W; (2n-1 \notin R, (1,0,2n-1)), (2n-1 \notin B, (n+1,2n-1,1))$.

Consider 3; $(3 \notin R, (3,1,0)), (3 \notin W, (3,n+1,0))$, hence 3 is blue and then $(3,n+1,2n-1)$ is an h.c. triangle.

Subcase 9.d. $s_2 \notin \{2,n-1,n\}$.
Let $j \in (n+1,\gamma,0)$ be such that $\ell(j,\gamma,0) = s_2$. We will consider two possibilities:

Let $(n+1,j) \in A$.
Since $s_2 \notin \{n-1,n\}$ we have $\{(1,n+1),(j,1)\} \subseteq A$.
$j \in W; (j \notin R, (1,0,j)), (j \notin B, (j,1,n+1))$. Now consider 2; ($2 \notin R$, $(2,1,0)$), ($2 \notin B$, $(2,n+1,j)$), hence 2 is white and $(2,n+1,0)$ is an h.c. triangle.

And let $(j,n+1) \in A$.
In this case we have $j = n+1-s_2$, $2s_2 = n$ and $(n+1-s_2,1) \in A$.
$j \in B; (j \notin R, (j,1,0)), (j \notin W, (j,n+1,0))$, consider $n+1-s_2$; $(n+1-s_2 \notin W, (i,j,n+1-s_2))$ (remember $s_2 \neq n$), $(n+1-s_2 \notin B, (n+1-s_2,1,n+1))$; hence $n+1-s_2$ is read and then $(n+1-s_2,1,0)$ is an h.c. triangle.

Case 10. $s_1 = 1$ and $n \in W$.
This case follows directly from Case 9 by applying Lemma 2.

Case 11. $s_1 = 1$ and $s_2 \in W$.
Observe that when $s_2 = n$ we obtain $(n,0,n+1)$ an h.c. triangle.

And when $s_2 = n-1$ we can assume $s_2 \neq 2$ (because $s_2 = 2 = n-1$ implies $D \cong C_7\langle 1,2\rangle$); consider n; we can assume $n \notin W$ (because when $n \in W$ we are in Case 10), ($n \notin R$, $(n,n-1,0)$), hence $n \in B$ and $(n,n-1,n+1)$ is an h.c. triangle.

So we will assume $2 \leq s_2 \leq n-2$ and consider two cases:

Subcase 11.a. $s_1 = 1$, $s_2 \in W$, $2 \le s_2 \le n-2$ and $(s_2, n+1) \in A$.
$2n \in W; (2n \notin R, (2n, s_2, 0)), (2n \notin B, (2n, s_2, n+1))$ (notice $(n+1, 2n) \in A$ because $s_2 \le n-2$.
$s_2+1 \in W; (s_2+1 \notin R, (s_2+1, s_2, 0), (s_2+1 \notin B, (s_2+1, n+1, 2n))$ when $(s_2+1, n+1) \in A$ and $(s_2+1, s_2, n+1)$ when $(n+1, s_2+1) \in A)$.
We will assume $(n+1, s_2+1) \in A$ because when $(s_2+1, n+1) \in A$ we have $(s_2+1, n+1, 0)$ an h.c. triangle. And since $s_2+1 \ne n$ we have $\ell(s_2+1, \gamma, n+1) = s_2$, and $2s_2 = n$.

Finally, consider $j \in (n+1, \gamma, 0)$ such that $\ell(j, \gamma, 0) = s_2; (j \notin W, (j, n+1, 0)), (j \notin B, (j, n+1, s_2+1))$, hence $j \in R$ and then $(j, s_2, 0)$ is an h.c. triangle.

Subcase 11.b. $s_1 = 1$, $s_2 \in W$, $2 \le s_2 \le n-2$ and $(n+1, s_2) \in A$. Since $s_2 \ne n$ we have that $\ell(s_2, \gamma, n+1) = s_2$ and $2s_2 = n+1$. Notice that we can assume $s_2 > 2$ (because when $s_2 = 2$, we have $n = 3$ and $D \cong \overrightarrow{C}_7\langle 1, 2\rangle$) and hence $\{(0, s_2-1), (s_2+1, n+1)\} \subseteq A$.
$s_2+1 \in B; (s_2+1 \notin R, (s_2+1, s_2, 0)), (s_2+1 \notin W, (s_2+1, n+1, 0))$.
$2n \in B$; $(2n \notin R, (2n, s_2, 0)), (2n \notin W, (2n, s_2+1, n+1))$.

Now consider $s_2 - 1$; $(s_2-1 \notin R, (s_2-1, 2n, s_2)), (s_2-1 \notin W, (s_2-1, n+1, 0))$, hence $s_2 - 1 \in B$ and $(s_2-1, n+1, s_2)$ is an h.c. triangle.

Case 12. $s_1 = 1$ and $n+1-s_2 \in W$.
This case follows directly from Case 11 by applying Lemma 2.

Case 13. $s_1 = 1$ and there exists $i \in (2, \gamma, n-1)$, $i \in W$ such that $\{(0, i), (i, n+1)\} \subseteq A(D)$.

When $s_2 \ne n$ we have $(0, i, n+1)$ an h.c. triangle so we will assume $s_2 = n$.

First notice that we can assume $n \ge 6$ (Because; when $n = 2$ we have $D \cong \overrightarrow{C}_5\langle 1, 2\rangle$; when $n = 3$, $D \cong \overrightarrow{C}_7\langle 1, 3\rangle$; when $n = 4$, $D \cong \overrightarrow{C}_9\langle 1, 4\rangle$; and when $n = 5$, $D \cong \overrightarrow{C}_{11}\langle 1, 5\rangle$.

First we will analyze the case $i = 2$; in this case consider $n+3$; $n+3 \in B$; $(n+3 \notin R, (n+3, 0, 2)), (n+3 \notin W, (n+3, 0, n+1))$ and now consider $n+5; (n+5 \notin R, (n+5, n+3, 2)), (n+5 \notin B, (n+5, 2, n+1))$ hence $n+5 \in W$ and $(n+5, 0, n+1)$ is an h.c. triangle (notice that $(n+5, 0) \in A$ because $n \ge 6$).

Now suppose $i \in (3, \gamma, n-1)$.
Consider 1; $1 \in R$; $(1 \notin W, (1, 0, n+1)), (1 \notin B, (1, i, n+1))$.

Let $h \in \{n+3, n+4\}$ be such that $(i,h) \in A$ (when $i = 3$ we take $h = n+4$ and when $i > 3$ we take $h = n+3$, since $s_1 = 1$ and $s_2 = n$ we have $(i,h) \in A$ and since $n \geq 6$ we have $\{(h,0),(h,1)\} \subseteq A$); and consider h; ($h \notin B$, $(h,1,i)$), ($h \notin R$, $(h,0,i)$) hence $h \in W$ and $(h,0,n+1)$ is an h.c. triangle.

Case 14. $s_1 = 1$ and $2n \in W$.

Subcase 14.a. $s_1 = 1$, $2n \in W$ and $s_2 = n$.
In this case we can assume (as in Case 13 when $s_2 = n$) $n \geq 6$.
$1 \in B$, ($1 \notin R$, $(1,0,2n)$), ($1 \notin W$, $(1,0,n+1)$). $n+2 \in B$; ($n+2 \notin R$, $(n+2,2n,1)$), ($n+2 \notin W$, $(n+2,n+1,1)$). Now consider 3; ($3 \notin W$, $(3,\ n+1,1)$), ($3 \notin B$, $(3,n+1,2n)$) hence $3 \in R$ and $(3,n+2,2n)$ is an h.c. triangle.

Subcase 14.b. $s_1 = 1$, $2n \in W$ and $s_2 = n-1$.
In this case $(0,2n,n+1)$ is an h.c. triangle.

Subcase 14.c. $s_1 = 1$, $2n \in W$ and $s_2 = 2$.
In this case we will assume $n \geq 5$. (Because when $n = 2$ when obtain $D \cong \overrightarrow{C}_5\langle 1,2\rangle$; when $n = 3$, $D \cong \overrightarrow{C}_7\langle 1,2\rangle$ and when $n = 4$, $D \cong \overrightarrow{C}_9\langle 1,2\rangle$).
$2 \in W$; ($2 \notin R, (2,0,2n)$), ($2 \notin B, (2,n+1,2n)$), now consider 3; ($3 \notin R$, $(3,2,0)$), ($3 \notin B, (3,n+1,2n)$) (notice that $(3,n+1) \in A$ because $n+1 \geq 6$). Hence $3 \in W$ and $(3,n+1,0)$ is and h.c. triangle.

Subcase 14.d. $s_1 = 1$, $2n \in W$ and $s_2 \notin \{2, n-1, n\}$.
Consider 1; we can assume that $1 \notin W$ because when $1 \in W$ we are in Case 9 and we are done, ($1 \notin R, (1,0,2n)$). Hence $1 \in B$ and $(1,n+1,2n)$ is an h.c. triangle.

Case 15. $s_1 = 1$ and $n+2 \in W$.
This Case follows directly from Case 14 by applying Lemma 2.

Case 16. $s_1 = 1$, and $i \in (n+3, \gamma, 2n-1)$ with $\ell(i,\gamma,0) = s_2$ satisfies $i \in W$.

We can assume $1 \notin W$ (see Case 9), ($1 \notin R, (1,0,i)$) hence $1 \in B$. Clearly in this case $s_2 \notin \{n, n-1\}$, so $\{(i,1),(1,n+1)\} \subseteq A$.

When $(i,n+1) \in A$ we have $(i,n+1,0)$ is an h.c. triangle and when $(n+1,i) \in A$ we obtain $(n+1,i,1)$ an h.c. triangle.

Case 17. $s_1 = 1$ and $n+1+s_2 \in W$.

This case follows directly from Case 16 and Lemma 2.

Case 18. $s_1 = 1$ and there exists $i \in (n+1, \gamma, 0) \cap W$ such that $\{(n+1, i), (i, 0)\} \subseteq A$.

Along this case we will assume without more explanation that there is no vertex $j \in (0, \gamma, n+1) \cap W$. (because when such a vertex exists we are in some of the cases 9 to 17).
Clearly, when $s_2 = n$ we have $(0, n+1, i)$ an h.c. triangle.

Subcase 18.a. $s_2 = n - 1$.
We have $\{(i+n-1, i), (i, i+n), (i+n+1, i), (i, i+n+2)\} \subseteq A$.
$i+n-1 \in B$; $(i+n-1 \notin R, (i+n-1, i, 0))$. $i+n \in B$; $(i+n \notin R, (i+n, i+n-1, i))$.
$i+n+2 \in R$; $(i+n+2 \notin B, (i+n+2, n+1, i))$. $i+n+1 \in R$; $(i+n+1 \notin B, (i+n+1, i, i+n+2))$.

When $(0, i+n+1) \in A$ we obtain $(0, i+n+1, i)$ an h.c. triangle hence we can assume $(i+n+1, 0) \in A$ and then $\ell(0, \gamma, i+n+1) \in \{s_1, S_2\}$; if $i+n+1 = 1$ we have $i+n = 0$ and $i = n+1$ which is impossible (because $n+1 \in R$ and $i \in W$); so $i+n+1 = n-1, i = 2n-1$ and $\ell(i, \gamma, 0) = 2$.

When $(i+n, n+1) \in A$ we obtain $(i+n, n+1, i)$ an h.c. triangle hence we can assume $(n+1, i+n) \in A$ and then $\ell(i+n, \gamma, n+1) \in \{s_1, s_2\}$; if $i+n = n$ we have $i = 0$ which is impossible ($i \in W$ and $0 \in B$); so $i+n = 2$, $i = n+3$ and $\ell(n+1, \gamma, i) = 2$.

Since $\ell(i, \gamma, 0) = \ell(n+1, \gamma, i) = 2$ we conclude $n = 4$ and $D \cong \overrightarrow{C}_9\langle 1, 3\rangle$.

Subcase 18.b Assume the hypothesis on Case 18, $s_2 \notin \{n, n-1\}$.
Since $s_2 \notin \{n, n-1\}$ we have $\{(i, i+n-1), (i, i+n), (i+n+1, i), (i+n+2, i)\} \subseteq A$.

First suppose $s_2 = 2$; in this case: $(0, i+n+2) \in A$ and $(i+n+2 \notin R, (0, i+n+2, i))$ hence $i+n+2 \in B$; $(i+n-1, n+1) \in A$ and $(i+n-1 \notin B, (i+n-1, n+1, i))$ hence $i+n-1 \in R$. Also we have $(i+n-1, i+n+2) \in A$ and then $(i+n-1, i+n+2, i)$ is an h.c. triangle.

Now suppose $s_2 \neq 2$; in this case $(i+n-1, i+n+1, i)$ is a triangle, so we can assume $\{i+n-1, i+n+1\} \subseteq R$ or $\{i+n-1, i+n+1\} \subseteq B$.

When $\{i+n-1, i+n+1\} \subseteq R$ we have $(i+n+1, 0) \in A$ (because when $(0, i+n+1) \in A$ we obtain $(0, i+n+1, i)$ an h.c. triangle), and since $i+n+1 > 1$, $\ell(i+n+1, \gamma, 0) = s_2$. It follows that $(0, i+n+2) \in A$, $(i+n+2 \notin R, (i+n+2, i, 0))$ and $i+n+2 \in B$.
Since $i+n+2 \in B$ we have $i+n \in B$, $(i+n \notin R, (i+n, i+n+2, i))$.
$i+n \in B$ implies $(n+1, i+n) \in A$ (in other case $(i+n, n+1, i)$ is an h.c.

triangle), and $\ell(i+n, \gamma, n+1) = s_2$ (because $i \neq 0$ and then $i+n \neq n$). So; when $s_2 \neq 3$ we have $(i+n-1, i+n+2, i)$ an h.c. triangle and when $s_2 = 3$ we obtain $n+1 = 5$ and $D \cong \overrightarrow{C}_9\langle 1, 3\rangle$.

When $\{i+n-1, i+n+1\} \subseteq B$ we have $(n+1, i+n-1) \in A$ (otherwise $(i+n-1, n+1, i)$ is an h.c. triangle) and since $i+n-1 \neq n$ we obtain $\ell(i+n-1, \gamma, n+1) = s_2$. Since $i+n \neq n$ we observe that $(i+n, n+1) \in A$ and then $i+n \in R$; $(i+n \notin B, (i+n, n+1, i))$; it follows $i+n+2 \in R$; $(i+n+2 \notin B, (i+n, i+n+2, i))$, and we can assume $(i+n+2, 0) \in A$ (when $(0, i+n+2) \in A$ the triangle $(0, i+n+2, i)$ is an h.c. triangle), and then $i+n+2 = s_2$ (clearly $i+n+2 \neq 1$). Finally, observe that when $s_2 \neq 3(i+n-1, i+n+2, i)$ is an h.c. triangle and when $s_2 = 3$ we obtain $n = 2$ (remember $i+n+2 = s_2$ and $i+n-1 = n+1-s_2$) which is impossible because $s_2 \leq n$.

Case 19. $s_2 = n$, $s_1 \neq 1$ and $s_1 \in W$.

Subcase 19.a. $s_2 = n$, $s_1 \neq 1$, $s_1 \in W$ and $2s_1 < n$.
Let $j \in (n+1, \gamma, 0)$ be such that $\ell(j, \gamma, 0) = s_1$.

We have $\{(s_1, n+1), (n+1, j), (0, j)\} \subseteq A$.
$j \in W$; $(j \notin R,\ (j, s_1, 0))$, $(j \notin B,\ (j, s_1, n+1))$. Notice $s_1 \neq n-1$ because $n \geq 2$, then we have $\{(2n, s_1),\ (n+1, 2n)\} \subseteq A$. And consider $2n$; $(2n \notin W,\ (2n, 0, n+1))$, $(2n \notin B,\ (2n, s_1, n+1))$ hence $2n \in R$ and then $(2n, 0, j)$ is an h.c. triangle.

Subcase 19.b. $s_2 = n$, $s_1 \neq 1$, $s_1 \in W$ and $2s_1 = n$.
In this case we will assume $s_1 \leq n-2$ (because when $s_1 = n-1$ we obtain $D \cong \overrightarrow{C}_5\langle 1, 2\rangle$).
$2n \in R; (2n \notin W, (2n, 0, n+1)), (2n \notin B, (2n, s_1, n+1))$.
$1 \in W$; $(1 \notin R,\ (1, s_1, 0))$, $(1 \notin B,\ (1, s_1, n+1))$. $n+2 \in B$; $(n+2 \notin W,\ (n+2, 0, n+1))$, $(n+2 \notin R,\ (n+2, 0, 1))$. Finally, consider j; $(j \notin W,\ (j, 2n, 0))$, $(j \notin R,\ (j, 1, n+2))$ hence $j \in B$ and $(j, 2n, i)$ is an h.c. triangle.

Subcase 19.c. $s_2 = n$, $s_1 \neq 1$, $s_1 \in W$ and $2s_1 > n$.
When $2s_1 = n+1$ we have $(0, n+1, s_1)$ an h.c. triangle. So we will assume $2s_1 \geq n+2$. (notice that $2s_1 \geq n+2$ implies $n+1-s_1 \in (0, \gamma, s_1)$).

Consider $n+1-s_1$; $n+1-s_1 \in W$; $(n+1-s_1 \notin R,\ (n+1-s_1, s_1, 0))$, $(n+1-s_1 \notin B,\ (n+1-s_1, s_1, n+1))$. Here we consider two possibilities:

Let $s_1 = n-1$.
We will assume $n \geq 4$ (because when $n = 2$, $D \cong C_5\langle 1, 2\rangle$ and when $n = 3$, $D \cong C_7\langle 2, 3\rangle$). Observe that in this case $n+1-s_1 = 2$.

$n \in W$; $(n \notin R, (n,0,2)), (n \notin B, (n, n+1, 2))$. Consider the vertex 4; $4 \in W$; $(4 \notin R, (4,n,0))$ (when $n = 4$ we are done because we proved $n \in W$), $(4 \notin B, (4, n+1, 2))$. Now consider $n+3$; $n+3 \in B$; $(n+3 \notin R, (n+3, 0, 2)), (n+3 \notin W, (n+3, 0, n+1))$. We conclude that $(n+3, 4, n+1)$ is an h.c. triangle.

And let $s_1 \leq n-2$.
First we prove that $(n+1-s_1+1) \in W$. When $n+1-s_1+1 = s_1$ we are done, when $n+1-s_1+1 \neq s_1$ we have $(n+1-s_1+1 \notin R, (n+1-s_1+1, s_1, 0)), (n+1-s_1+1 \notin B, (n+1-s_1+1, n+1, n+1-s_1))$.

Now $1 \in W$; $(1 \notin R, (1, s_1, 0))$, $(1 \notin B, (n+1, 1, s_1))$. Finally, $n+2 \in B$; $(n+2 \notin R, (n+2, 0, 1)), (n+2 \notin W, (n+2, 0, n+1))$. We conclude that $(n+2, n+1-s_1+1, n+1)$ is an h.c. triangle.

Case 20. $s_2 = n$, $s_1 \neq 1$ and $n+1-s_1 \in W$.
This case follows directly from Lemma 2 and Case 19.

Case 21. $s_2 = n$, $s_1 \neq 1$ and the vertex $i \in (n+1, \gamma, 0)$ such that $\ell(i, \gamma, 0) = s_1$ is white.

Subcase 21.a. $2s_1 < n$.
$(s_1 \notin R, (s_1, 0, i)), (s_1 \notin B, (s_1, n+1, i))$ hence $s_1 \in W$ and we are in Case 19.

Case 21.b. $2s_1 = n$.
In this case we will assume $s_1 \neq n-1$ (because when $s_1 = n-1$ we obtain $D \cong \overrightarrow{C}_5\langle 1, 2\rangle$).
$n+2 \in R$; $(n+2 \notin B, (n+2, i, n+1)), (n+2 \notin W, (n+2, 0, n+1))$. $2n \in B$; $(2n \notin R,\ (2n, 0, i))$, $(2n \notin W, (2n, 0, n+1))$. $s_1 \in B$; $(s_1 \in R,\ (s_1, i, 2n))$, $(s_1 \notin W,\ (s_1, n+1, 2n))$. $1 \in B$; $(1 \notin R,\ (1, j, i))$, $(1 \notin W,\ (1, n+2, 0))$.

Hence we have $(1, n+2, i)$ an h.c. triangle.

Subcase 21.c. $2s_1 \geq n+1$.
Let $s_1 = n-1$.

In this case we will assume $n \geq 4$. (Because when $n = 2$, $D \cong \overrightarrow{C}_5\langle 1, 2\rangle$ and when $n = 3$, $D \cong \overrightarrow{C}_7\langle 2, 3\rangle$).

In this case $i = n+2$ and $\{(0, n+2), (2n, n+1)\} \subseteq A(D)$, moreover since $n \geq 4$ we have $n+3 < 2n$.
$2n \in W$; $(2n \notin R, (2n, 0, n+2))$, $(2n \notin B,\ (2n, n+1, n+2))$. $n+3 \in W$; $(n+3 \notin R,\ (n+3, 0, n+2))$, $(n+3 \notin B,\ (n+3, 2n, n+1))$. So we have $(0, n+1, n+3)$ an h.c. triangle.

And let $s_1 \leq n-2$.
$n+1+s_1 \in W$; $(n+1+s_1 \notin R, (n+1+s_1, 0, i))$, $(n+1+s_1 \notin B, (n+1+s_1, n+1, i))$, $n+2 \in R$; $(n+2 \notin B, (n+2, n+1+s_1, n+1))$, $(n+2 \notin W, (n+2, 0, n+1))$, $i+1 \in W$, when $i+1 = n+1+s_1$ we have $i+1 \in W$ and when $i+1 \neq n+1+s_1$ we have; $(i+1 \notin R, (i+1, 0, i))$, $(i+1 \notin B, (i+1, n+1+s_1, n+1))$. $1 \in R$; $(1 \notin B, (1, n+2, i)), (1 \notin W, (1, n+2, 0))$. So we obtain $(1, i+1, 0)$ an h.c. triangle.

Case 22. $s_2 = n$, $s_1 \neq 1$ and $n+1+s_1 \in W$.
This case follows directly from Lemma 2 and Case 21.

Case 23. $s_2 = n$, $s_1 \neq 1$ and there exists $i \in (n+1, \gamma, 0) \cap W$ such that $\{(n+1, i), (i, 0)\} \subseteq A(D)$.

In this case $(0, n+1, i)$ is an h.c. triangle.

Case 24. $s_2 = n$, $s_1 \neq 1$ and there exists $i \in (0, \gamma, n+1) \cap W$ such that $\{(0, i), (i, n+1)\} \subseteq A(D)$.

In this case we will assume that $V(n+1, \gamma, 0) \cap W = \emptyset$ (because when there exists $x \in V(n+1, \gamma, 0) \cap W$ we are in some of the previous cases).

Subcase 24.a. $s_1 = n-1$.
In this case we will assume $n \geq 7$ (When $n = 2$, $D \cong \overrightarrow{C}_5\langle 1, 2\rangle$; when $n = 3$, $D \cong \overrightarrow{C}_7\langle 2, 3\rangle$; when $n = 4$, $D \cong \overrightarrow{C}_9\langle 3, 4\rangle$; when $n = 5$, $D \cong \overrightarrow{C}_{11}\langle 4, 5\rangle$ and when $n = 6$, $D \cong \overrightarrow{C}_{13}\langle 5, 6\rangle$).

Since $s_1 = n-1$ we have $\{(i+n-1, i), (i, i+n+2)\} \subseteq A$.
$i+n-1 \in R$; $(i+n-1 \notin B, (i+n-1, i, n+1))$ (Notice that since $s_1 = n-1$, the hypothesis on Case 24 imply $i \in (3, \gamma, n-2)$). $i+n+2 \in B$; $(i+n+2 \notin R, (i+n+2, 0, i))$.

When $i+n+3 \neq 0$ and $n+2 \neq i+n-1$, we have $i+n+3 \in B$; $(i+n+3 \notin R, (i+n+3, i, i+n+2))$. $n+2 \in R$; $(n+2 \notin B, (n+2, i+n-1, i))$; and then $(n+2, i+n+3, i)$ is an h.c. triangle.

When $i+n+3 = 0$ we have $i = n-2$ and since $n \geq 7$ we also have $n+2 \neq i+n-1$ and $n+3 \neq i+n-1$. Consider $n+3$; $(n+3 \notin B, (n+3, i+n-1, i))$ hence $n+3 \in R$ and $(n+3, 0, i)$ is an h.c. triangle.

When $n+2 = i+n-1$ we have $i = 3$ and since $n \geq 7$ we have $2n \neq i+n+2$ and $2n-1 \neq i+n-2$. Consider $2n-1$; $(2n-1 \notin R, (2n-1, i, i+n+2))$ hence $2n-1 \in B$ and $(2n-1, i, n+1)$ is an h.c. triangle.

Subcase 24.b. $s_1 \leq n-2$.

Since $s_2 = n$ and $s_1 \leq n-2$ we have $\{(i, i+n-1), (i+n, i), (i, i+n+1), (i+n+2, i)\} \subseteq A$.

Let $i+n+2 = 0$.

In this case we have $i = n-1$, $i+n+1 = 2n \in B$; $(i+n+1 \notin R, (i+n+1, i+n+2, 0))$, $n \in B$; $(n \notin R, (n, 0, n-1))$, $(n \notin W, (n, n+1, i+n+1))$, $i+n \in B$, $(i+n \notin R, (i+n, n-1, n))$, $i+n-1 \in B$; $(i+n-1 \notin R, (i+n-1, i+n, i))$; now notice that we can assume $(i+n, n+1) \in A$ (When $(n+1, i+n) \in A$, $(n+1, i+n, i)$ is and h.c. triangle), hence $\ell(n+1, \gamma, i+n) = s_1 = n-2$. Finally, consider $i+n-2$; we can assume $i+n-2 > n+1$ (when $i+n-2 = n$, $D \cong \overrightarrow{C}_7\langle 1, 3\rangle$ and when $i+n-2 = n+1$, $D \cong \overrightarrow{C}_9\langle 2, 4\rangle$); since $s_1 = n-2$ and $s_2 = n$ we have $\{(i+n-2, i), (n, i+n-2), (n+1, i+n-2)\} \subseteq A$; then $(i+n-2 \notin B, (i+n-2, i, n+1))$, so $i+n-2 \in R$ and $(i+n-2, i, n)$ is an h.c. triangle.

And let $i+n+2 \neq 0$.

First we prove that we can assume $(n+2, i) \in A$.

Suppose $(i, n+2) \in A$; then $(n+2 \notin R, (n+2, 0, i))$, so $n+2 \in B$. Now consider $i+n$; $i+n \neq n+2((i, n+2) \in A,$ and $(i+n, i) \in A), i+n \neq n+1(s_2 = n)$.

When $\{(n+2, i+n), (n+1, i+n)\} \subseteq A$ we have $(i+n \notin B, (i+n, i, n+1))$ hence $i+n \in R$ and $(i+n, i, n+2)$ is an h.c. triangle, so we have $(i+n, n+1) \in A$ or $(i+n, n+2) \in A$ and then $\ell(n+1, \gamma, i+n) = s_1$ or $\ell(n+2, \gamma, i+n) = s_1$; in any case and since $i+n+2 \neq 0$ we have $\{(n+2, i+n+2), (n+1, i+n+2)\} \subseteq A$. Finally, consider $i+n+2$, $(i+n+2 \notin R, (i+n+2, i, n+2))$ hence $i+n+2 \in B$ and $(i+n+2, i, n+1)$ is an h.c. triangle.

Now we prove that we can assume $(i, 2n) \in A$.

Suppose $(2n, i) \in A$, then $(2n \notin B, (2n, i, n+1))$, hence $2n \in R$. When $\{(i+n+1, 0), (i+n+1, 2n)\} \subseteq A$ (Notice that since $i+n+2 \neq 0$ we have $i+n+1 < 2n$), we have $(i+n+1 \notin R, (i+n+1, 0, i))$ hence $i+n+1 \in B$ and $(i+n+1, 2n, i)$ is an h.c. triangle. So we have $(0, i+n+1) \in A$ or $(2n, i+n+1) \in A$ (and since $i+n+1 \neq n+1$ we have $\ell(i+n+1, \gamma, 0) = s_1$ or $\ell(i+n+1, \gamma, 2n) = s_1$). So when $i+n-1 \neq n+1$ we have $\{(i+n-1, 0), (i+n-1, 2n)\} \subseteq A$ and consider $i+n-1$, $(i+n-1 \notin R, (i+n-1, 0, i))$ hence $i+n-1 \in B$ and $(i+n-1, 2n, i)$ is an h.c. triangle. Now we analyze the case when $i+n-1 = n+1$ and $(0, i+n+1) \in A$; in this case $s_1 = n-2$ and consider $i+n+3$; Since $s_1 = n-2$ we have $(i, i+n+3) \in A$ and we can assume $i+n+3 < 2n$ (when $i+n+3 = 0$ we have $n = 4$ and $D \cong \overrightarrow{C}_9\langle 2, 4\rangle$ and when $i+n+3 = 2n$, we have $n = 5$ and $D \cong \overrightarrow{C}_{11}\langle 3, 5\rangle$), $(i+n+3 \notin R, (i+n+3, 0, i))$ hence $i+n+3 \in B$ and $(i+n+3, 2n, i)$

is an h.c. triangle. Finally, analyze the case when $i+n-1=n+1$ and $(2n,i+n+1)\in A$ in this case $s_1=n-3$ and consider $i+n+4$ we have $(i,i+n+4)\in A$ and we can assume $i+n+4<2n$ (when $i+n+4=0$ we obtain $n=5$ and $D\cong \overrightarrow{C}_{11}\langle 2,5\rangle$ and when $i+n+4=2n$ we obtain $n=6$ and $D\cong \overrightarrow{C}_{13}\langle 3,6\rangle$); $(i+n+4\notin R,(i+n+4,0,i))$ hence $i+n+4\in B$ and $(i+n+4,2n,i)$ is an h.c. triangle.

So we can assume $\ell(2n,\gamma,i)=\ell(i,\gamma,n+2)=s_1$.
$n+2\in R$; $(n+2\notin B,(n+2,i,n+1))$. $2n\in B$; $(2n\notin R,(2n,0,i))$.

Finally, consider 1; $(1\notin W,(0,1,n+2))$, $(1\notin R,(1,i,2n))$ hence $1\in B$ and $(1,i,n+1)$ is an h.c. triangle. ■

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