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# SOME NEWS ABOUT THE INDEPENDENCE NUMBER OF A GRAPH

Jochen Harant

*Department of Mathematics, Technical University of Ilmenau*
*D-98684 Ilmenau, Germany*

**Abstract**

For a finite undirected graph $G$ on $n$ vertices some continuous optimization problems taken over the $n$-dimensional cube are presented and it is proved that their optimum values equal the independence number of $G$.

**Keywords:** graph, independence.

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## 1 Introduction and Results

Let $G$ be a finite simple and undirected graph on $V(G) = \{1, 2, ..., n\}$ with its edge set $E(G)$. A subset $I$ of $V(G)$, such that the subgraph of $G$ induced by $I$ is edgeless, is called an *independent set* of $G$, and the maximum cardinality of an independent set of $G$ is named the *independence number* $\alpha(G)$ of $G$. $N(i)$ and $d_i$ denote the set and the number of neighbours of $i \in V(G)$ in $G$, respectively, and let $\Delta(G) = \max\{d_i \mid i \in V(G)\}$ and $C^n = \{(x_1, x_2, ..., x_n) \mid 0 \leq x_i \leq 1,\ i = 1, 2, ..., n\}$. For events $A$ and $B$ and for a random variable $Z$ of an arbitrary random space, $P(A)$, $P(A|B)$, and $\mathcal{E}(Z)$ denote the probability of $A$, the conditional probability of $A$ given $B$, and the expectation of $Z$, respectively. Since the computation of $\alpha(G)$ is difficult (INDEPENDENT SET is an NP-complete problem; see [6]), much work was done to establish bounds on $\alpha(G)$ (e.g., see [1, 3, 4, 5, 8, 10, 12, 13, 14, 15, 16, 17]), to find efficient algorithms forming a large independent set of $G$ (e.g., see [2, 7, 8, 9, 10, 12]), or to replace the combinatorial optimization problem to determine $\alpha(G)$ by a continuous one (e.g., see [9, 11]). The last approach leads to bounds on $\alpha(G)$ as well as to efficient algorithms (e.g., see [8, 9]). In the present paper some new continuous optimization problems taken over $C^n$ are presented and it is proved that their optimum values equal $\alpha(G)$. Theorem 1 gives a remarkable result of T.S. Motzkin and E.G. Straus [11] and Theorem 2 is proved in [9].

**Theorem 1.**

$$\alpha(G) = \max_{(0,0,...,0)\neq(x_1,x_2,...,x_n)\in C^n} \frac{\left(\sum_{i\in V(G)} x_i\right)^2}{\sum_{i\in V(G)} x_i^2 + 2\sum_{ij\in E(G)} x_i x_j}.$$

**Theorem 2.** $\alpha(G) = \max_{(x_1,x_2,...,x_n)\in C^n} \sum_{i\in V(G)} (x_i \prod_{j\in N(i)} (1-x_j))$.

A classical lower bound on $\alpha(G)$ due to Y. Caro and V.K. Wei [3, 17] is given by the following theorem.

**Theorem 3.** $\alpha(G) \geq \sum_{i\in V(G)} \frac{1}{1+d_i}$.

The next Theorems 4, 5, 6, and 7 are the main results of the present paper.

**Theorem 4.** $\alpha(G) = \max_{(x_1,x_2,...,x_n)\in C^n} e_G(x_1, x_2, ..., x_n)$, *where*

$$e_G(x_1, x_2, ..., x_n) = \sum_{i\in V(G)} \left(\frac{x_i}{1+\sum_{j\in N(i)} x_j} + \frac{(1-x_i)\prod_{j\in N(i)}(1-x_j)}{1+\sum_{j\in N(i)}\prod_{l\in N(j)\setminus(N(i)\cup\{i\})}(1-x_l)}\right).$$

**Theorem 5.** $\alpha(G) = \max_{(x_1,x_2,...,x_n)\in C^n} f_G(x_1, x_2, ..., x_n)$, *where*

$$f_G(x_1, x_2, ..., x_n) = \sum_{i\in V(G)} \left(x_i + \frac{(1-x_i)\prod_{j\in N(i)}(1-x_j)}{1+\sum_{j\in N(i)}\prod_{l\in N(j)\setminus(N(i)\cup\{i\})}(1-x_l)}\right) - \sum_{ij\in E(G)} x_i x_j.$$

The following Theorem 6 looks more "complicate", but it is "stronger" than Theorem 4 and Theorem 5 (see Remark 1).

**Theorem 6.** $\alpha(G) = \max_{(x_1,x_2,...,x_n)\in C^n} g_G(x_1, x_2, ..., x_n)$, *where*

$$g_G(x_1, x_2, ..., x_n) = \sum_{i\in V(G)} \left((x_i + \frac{1-x_i}{1+\sum_{j\in N(i)}\prod_{l\in N(j)\setminus(N(i)\cup\{i\})}(1-x_l)}\right)\prod_{j\in N(i)}(1-x_j))$$

$$+\sum_{i\in V'} \frac{x_i(1-\prod_{j\in N(i)}(1-x_j))^2}{1-\prod_{j\in N(i)}(1-x_j)+\sum_{j\in N(i)} x_j} \text{ and } V' = \left\{ i \in V(G) \mid \sum_{j\in N(i)} x_j > 0 \right\}.$$

A "weaker" (see Remark 1), but a more "transparent" and (see Remark 2) an "algorithmically realizable" version of Theorem 5 is the following one.

**Theorem 7.** $\alpha(G) = \max_{(x_1,x_2,...,x_n)\in C^n} h_G(x_1, x_2, ..., x_n)$, *where* $h_G(x_1, x_2, ..., x_n) = \sum_{i\in V(G)} x_i - \sum_{ij\in E(G)} x_i x_j$.

# 2 Proofs

Throughout the proofs we will use the well-known fact that for a random subset $M$ of a given finite set $N$,

$$\mathcal{E}(|M|) = \sum_{y\in N} P(y \in M) = \sum_{k=0}^{|N|} kP(|M| = k).$$

Let $I$ be a maximum independent set of $G$ and let $x_i^* = 1$ if $i \in I$ and $x_i^* = 0$ if $i \notin I$. Since $(1 - x_i^*)\prod_{j\in N(i)}(1 - x_j^*) = 0$ for $i \in V(G)$ and $\sum_{ij\in E(G)} x_i^* x_j^* = 0$, we obtain

**Lemma 1.** $\alpha(G) = e_G(x_1^*, x_2^*, ..., x_n^*) = f_G(x_1^*, x_2^*, ..., x_n^*) = g_G(x_1^*, x_2^*, ..., x_n^*)$ $= h_G(x_1^*, x_2^*, ..., x_n^*)$.

With Lemma 1, it is clear that Theorem 7 follows from Theorem 5.

Now, let $(x_1, x_2, ..., x_n)$ be an arbitrary member of $C^n$. We form a set $X \subseteq V(G)$ by random and independent choice of $i \in V(G)$, where $P(i \in X) = x_i$. Let $H_1$, $H_2$, and $H_3$ be the subgraph of $G$ induced by the vertices of $X$, by the vertices $i \in X$ with $N(i) \cap X \neq \emptyset$, and by the vertices $i \notin X$ with $N(i) \cap X = \emptyset$, respectively. Furthermore, let $Y$ be a smallest subset of $V(H_2)$ covering all edges of $H_2$, i.e., the graph induced by $V(H_2) - Y$ is edgeless, and let $I_1$ and $I_3$ be a maximum independent set of $H_1$ and $H_3$, respectively. It can be seen easily that $|Y| = |V(H_2)| - \alpha(H_2)$, $|Y| \leq |E(H_2)|$ and that $(X - Y) \cup I_3$ and $I_1 \cup I_3$ are independent sets of $G$. Because of these remarks and the property of the expectation to be an average value, we have Lemma 2 as follows.

**Lemma 2.** $\alpha(G) \geq \mathcal{E}(|X-Y|)+\mathcal{E}(\alpha(H_3))$, $\alpha(G) \geq \mathcal{E}(\alpha(H_1))+\mathcal{E}(\alpha(H_3))$, $\mathcal{E}(|X-Y|) = \mathcal{E}(|X|)-\mathcal{E}(|Y|) \geq \mathcal{E}(|X|)-\mathcal{E}(|E(H_2)|)$, *and* $\mathcal{E}(|X-Y|) = \mathcal{E}(|X|)-\mathcal{E}(|V(H_2)|)+\mathcal{E}(\alpha(H_2))$.

Lower bounds on $\mathcal{E}(\alpha(H_1))$, $\mathcal{E}(\alpha(H_2))$, and $\mathcal{E}(\alpha(H_3))$ are given in Lemma 3.

**Lemma 3.** $\mathcal{E}(\alpha(H_1)) \geq \sum_{i\in V(G)} \frac{x_i}{1+\sum_{j\in N(i)} x_j}$,

$\mathcal{E}(\alpha(H_2)) \geq \sum_{i\in V'} \frac{x_i(1-\prod_{j\in N(i)}(1-x_j))^2}{1-\prod_{j\in N(i)}(1-x_j)+\sum_{j\in N(i)} x_j}$, *where* $V' = \{i \in V(G) | \sum_{j\in N(i)} x_j > 0\}$,

*and*

$\mathcal{E}(\alpha(H_3)) \geq \sum_{i\in V(G)} \frac{(1-x_i)\prod_{j\in N(i)}(1-x_j)}{1+\sum_{j\in N(i)}\prod_{l\in N(j)\setminus(N(i)\cup\{i\})}(1-x_l)}$.

***Proof.*** For $i \in V(G)$ define the random variable $Z_i^1$ with $Z_i^1 = \frac{1}{1+k}$ if $i \in X$ and $|N(i) \cap X| = k \geq 0$, and $Z_i^1 = 0$ if $i \notin X$. Using Theorem 3,

$\mathcal{E}(\alpha(H_1)) \geq \mathcal{E}(\sum_{i\in V(G)} Z_i^1) = \sum_{i\in V(G)} \mathcal{E}(Z_i^1)$

$= \sum_{i\in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k)$

$= \sum_{i\in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k)$

$= \sum_{i\in V(G)} x_i \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k))$.

For $i \in V(G)$ we have $\sum_{k=0}^{d_i} P(|N(i) \cap X| = k) = 1$. With Jensen's inequality $\sum_{l=1}^{m} \tau_l \phi(y_l) \geq \phi(\sum_{l=1}^{m} \tau_l y_l)$ for any convex function $\phi$ and any $\tau_l \geq 0$ for $l = 1, 2, ..., m$ with $\sum_{l=1}^{m} \tau_l = 1$,

$\mathcal{E}(\alpha(H_1)) \geq \sum_{i\in V(G)} x_i \frac{1}{1+\sum_{k=0}^{d_i} kP(|N(i)\cap X|=k)} = \sum_{i\in V(G)} \frac{x_i}{1+\sum_{j\in N(i)} x_j}$.

Now, let $V' = \{i \in V(G) | \sum_{j\in N(i)} x_j > 0\}$. For $i \in V(G)$ let $Z_i^2$ be the random variable with $Z_i^2 = \frac{1}{1+k}$ if $i \in X$ and $|N(i) \cap X| = k \geq 1$, and $Z_i^2 = 0$ otherwise. Then,

$$\mathcal{E}(\alpha(H_2)) \geq \mathcal{E}(\sum_{i\in V(G)} Z_i^2) = \sum_{i\in V(G)} \mathcal{E}(Z_i^2)$$
$$= \sum_{i\in V(G)} \sum_{k=1}^{d_i} \tfrac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k)$$
$$= \sum_{i\in V(G)} \sum_{k=1}^{d_i} \tfrac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k)$$
$$= \sum_{i\in V(G)} x_i \sum_{k=1}^{d_i} \tfrac{1}{1+k} P(|N(i) \cap X| = k).$$

$P(|N(i) \cap X| = 0) + \sum_{k=1}^{d_i} P(|N(i) \cap X| = k) = 1$ for $i \in V(G)$ and with $\mu_i = P(|N(i) \cap X| = 0) = \prod_{j\in N(i)} (1 - x_j)$ and $\sigma_{ik} = P(|N(i) \cap X| = k)$,

$$\mathcal{E}(\alpha(H_2)) \geq \sum_{i\in V(G)} x_i \sum_{k=1}^{d_i} \tfrac{1}{1+k} \sigma_{ik} = \sum_{i\in V(G),\mu_i<1} x_i \sum_{k=1}^{d_i} \tfrac{1}{1+k} \sigma_{ik}$$
$$= \sum_{i\in V'} x_i \sum_{k=1}^{d_i} \tfrac{1}{1+k} \sigma_{ik} = \sum_{i\in V'} x_i(1-\mu_i) \sum_{k=1}^{d_i} \tfrac{\sigma_{ik}}{(1+k)(1-\mu_i)}.$$

For $\lambda_{ik} = \frac{\sigma_{ik}}{1-\mu_i}$ we have $\lambda_{ik} \geq 0$, $\sum_{k=1}^{d_i} \lambda_{ik} = 1$ if $i \in V'$, and again using Jensen's inequality,

$$\mathcal{E}(\alpha(H_2)) \geq \sum_{i\in V'} x_i(1-\mu_i) \frac{1}{1+\sum_{k=1}^{d_i} k\lambda_{ik}}$$
$$= \sum_{i\in V'} \frac{x_i(1-\prod_{j\in N(i)}(1-x_j))^2}{1-\prod_{j\in N(i)}(1-x_j)+\sum_{k=1}^{d_i} kP(|N(i)\cap X|=k)} = \sum_{i\in V'} \frac{x_i(1-\prod_{j\in N(i)}(1-x_j))^2}{1-\prod_{j\in N(i)}(1-x_j)+\sum_{j\in N(i)} x_j}.$$

Finally, let us consider the random variable $Z_i^3$ with $Z_i^3 = \frac{1}{1+k}$ if $i \in V(H_3)$ and $|N(i) \cap V(H_3)| = k \geq 0$, and $Z_i^3 = 0$ if $i \notin V(H_3)$. Then

$$\mathcal{E}(\alpha(H_3)) \geq \mathcal{E}(\sum_{i\in V(G)} Z_i^3) = \sum_{i\in V(G)} \mathcal{E}(Z_i^3)$$
$$= \sum i \in V(G) \sum_{k=0}^{d_i} \tfrac{1}{1+k} P(i \in V(H_3) \text{ and } |N(i) \cap V(H_3)| = k)$$
$$= \sum_{i\in V(G)} \sum_{k=0}^{d_i} \tfrac{1}{1+k} P(i \in V(H_3)) P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3))$$
$$= \sum_{i\in V(G)} ((1-x_i) \prod_{j\in N(i)} (1-x_j) \sum_{k=0}^{d_i} \tfrac{1}{1+k} P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)))$$

$$\geq \sum_{i\in V(G)} ((1-x_i) \prod_{j\in N(i)} (1-x_j) \frac{1}{1+\sum_{k=0}^{d_i} kP(|N(i)\cap V(H_3)|=k \mid i\in V(H_3))})$$

$$= \sum_{i\in V(G)} ((1-x_i) \prod_{j\in N(i)} (1-x_j) \frac{1}{1+\sum_{j\in N(i)} \prod_{l\in N(j)\setminus(N(i)\cup\{i\})} (1-x_l)}),$$

and Lemma 3 is proved. ■

Theorem 4, 5, and 6 follow with $\mathcal{E}(|X|) = \sum_{i\in V(G)} x_i$, $\mathcal{E}(|E(H_2)|) = \sum_{ij\in E(G)} x_i x_j$, $\mathcal{E}(|V(H_2)|) = \sum_{i\in V(G)} x_i(1 - \prod_{j\in N(i)} (1-x_j))$, Lemma 1, 2, and 3.

## 3 Remarks

For $\phi, \psi \in \{e, f, g, h\}$ define $\phi \leq \psi$ if $\phi_G(x_1, x_2, ..., x_n) \leq \psi_G(x_1, x_2, ..., x_n)$ for every graph $G$ on $n$ vertices and for every $(x_1, x_2, ..., x_n) \in C^n$. We write $\phi <> \psi$ if neither $\phi \leq \psi$ nor $\psi \leq \phi$.

**Remark 1.** *$h \leq f \leq g$, $e \leq g$ and $e <> f$.*

***Proof.*** We will use the following Lemma 4, which can be seen easily by induction on $r$.

**Lemma 4.** *For an integer $r \geq 1$ and $a_1, a_2, ..., a_r \in [0,1]$, $\sum_{q=1}^{r} a_q + \prod_{q=1}^{r} (1-a_q) \geq 1$.*

The inequality $h \leq f$ is obvious. To see $f \leq g$, first notice that $\sum_{i\in V(G)} x_i - \sum_{ij\in E(G)} x_i x_j = \sum_{i\in V(G)} x_i(1 - \frac{1}{2}\sum_{j\in N(i)} x_j)$. If $\sum_{j\in N(i)} x_j = 0$ for an $i \in V(G)$ then $x_i = x_i(\prod_{j\in N(i)} (1-x_j)) = x_i(1 - \frac{1}{2}\sum_{j\in N(i)} x_j)$. Hence, with the abbreviation $\mu_i = \prod_{j\in N(i)} (1-x_j)$ and $\rho_i = \sum_{j\in N(i)} x_j$ for $i \in V(G)$ we have to show $\sum_{i\in V'} (x_i(\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i})) \geq \sum_{i\in V'} (x_i(1 - \frac{1}{2}\rho_i))$, where again $V' = \{i \in V(G) | \sum_{j\in N(i)} x_j > 0\}$.

Using Lemma 4, even $\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i} \geq 1 - \frac{1}{2}\rho_i$ for all $i \in V'$.

To prove $e \leq g$, we have to show

$$\sum_{i\in V(G)} \frac{x_i}{1+\sum\limits_{j\in N(i)} x_j} \le \sum_{i\in V(G)} x_i \prod_{j\in N(i)} (1-x_j) + \sum_{i\in V'} \frac{x_i(1-\prod\limits_{j\in N(i)}(1-x_j))^2}{1-\prod\limits_{j\in N(i)}(1-x_j)+\sum\limits_{j\in N(i)} x_j}.$$

Since $\frac{x_i}{1+\sum\limits_{j\in N(i)} x_j} = x_i \prod\limits_{j\in N(i)}(1-x_j)$ if $\sum\limits_{j\in N(i)} x_j = 0$, it is sufficient to establish $\frac{1}{1+\rho_i} \le \mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i}$ if $\sum\limits_{j\in N(i)} x_j > 0$, what is verified easily.

For a cycle $C_n$ on $n$ vertices $e_{C_n}(\frac{1}{3},\frac{1}{3},...,\frac{1}{3}) < f_{C_n}(\frac{1}{3},\frac{1}{3},...,\frac{1}{3})$, $e_{C_n}(\frac{2}{3},\frac{2}{3},...,\frac{2}{3}) > f_{C_n}(\frac{2}{3},\frac{2}{3},...,\frac{2}{3})$ and Remark 1 is proved. ■

With $h \le f$ and $e <> f$ it is clear that $e \le h$ does not hold. It remains open, whether $h <> e$ or $h \le e$.

Theorems 1, 2, 4, 5, 6, and 7 are of that type that the independence number $\alpha(G)$ of a graph $G$ on $n$ vertices equals the optimum value of a continuous optimization problem $O(G)$ to maximize a certain function $\phi_G$ over $C^n$. Hence, $\phi_G(x_1, x_2, ..., x_n)$ is a lower bound on $\alpha(G)$ for every $(x_1, x_2, ..., x_n) \in C^n$. Let $(x'_1, x'_2, ..., x'_n) \in C^n$ be the solution of an arbitrary approximation algorithm for $O(G)$. How to find an independent set $I$ of $G$ in polynomial time such that $|I| \ge \phi_G(x'_1, x'_2, ..., x'_n)$ ? In [8] and [9] efficient algorithms forming $I$ with $|I| \ge \phi_G(x'_1, x'_2, ..., x'_n)$ are given if $O(G)$ is the optimization problem of Theorem 1 or of Theorem 2. Remark 2 shows that this is also possible if we consider the case $\phi_G = h_G$. In case $\phi_G = e_G$, $\phi_G = f_G$ or $\phi_G = g_G$ the problem remains open, whether such an algorithm exists.

**Remark 2.** There is an $\mathcal{O}(\Delta(G)n)$-algorithm with
INPUT: $(x_1, x_2, ..., x_n) \in C^n$,
OUTPUT: an independent set $I \subseteq V(G)$ with $|I| \ge \sum\limits_{i\in V(G)} x_i - \sum\limits_{ij\in E(G)} x_i x_j$.

***Proof.*** First we give the Algorithm:

1. For $i = 1$ to $n$ do if $\sum_{j\in N(i)} x_j < 1$ then $x_i := 1$ else $x_i := 0$.
2. For $i = 1$ to $n$ do if ($x_i = 1$ and $\prod_{j\in N(i)}(1-x_j) = 0$) then $x_i := 0$.
3. $I := \{i \in V(G) \mid x_i = 1\}$.

STOP

It is obvious that the algorithm in an $\mathcal{O}(\Delta(G)n)$-algorithm. For the input vector $(x_1, x_2, ..., x_n) \in C^n$ set

$$\sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j = a.$$

After step 1, the current $(x_1, x_2, ..., x_n)$ is a 0-1-vector and

$$\sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j \geq a$$

because

$\frac{\partial}{\partial x_i}(\sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j) = 1 - \sum_{j\in N(i)} x_j$, i.e., $\sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j$ is multilinear.

In step 2, $\prod_{j\in N(i)} (1 - x_j) = 0$ if and only if there is at least one $j \in N(i)$ such that $x_j = 1$. With $x_i = 0$ instead of $x_i = 1$ the sum $\sum_{k\in V(G)} x_k$ decreases by 1 and the sum $\sum_{kj\in E(G)} x_k x_j$ decreases by at least 1, hence $\sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j$ does not decrease.

After step 2,

$x_k x_j = 0$ for all $kj \in E(G), |I| = \sum_{k\in V(G)} x_k = \sum_{k\in V(G)} x_k - \sum_{kj\in E(G)} x_k x_j \geq a$

and Remark 2 is proved. ■

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