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ON THE FACTORIZATION OF REDUCIBLE PROPERTIES OF GRAPHS INTO IRREDUCIBLE FACTORS

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Abstract

A hereditary property $\mathcal{R}$ of graphs is said to be *reducible* if there exist hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $G \in \mathcal{R}$ if and only if the set of vertices of G can be partitioned into $V(G) = V_1 \cup V_2$ so that $\langle V_1 \rangle \in \mathcal{P}_1$ and $\langle V_2 \rangle \in \mathcal{P}_2$. The problem of the factorization of reducible properties into irreducible factors is investigated.

Keywords: hereditary property of graphs, additivity, reducibility, vertex partition.

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1. Introduction

We consider finite undirected graphs without loops and multiple edges. In general, we use the notation and terminology of [2].

Let $\mathcal{I}$ is the set of all mutually non-isomorphic graphs. If $\mathcal{P}$ is a nonempty subset of $\mathcal{I}$, then $\mathcal{P}$ also denotes the property that a graph G is a member of $\mathcal{P}$. A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$ and $\mathcal{P}$ is called *additive* if for each graph G whose all components have property $\mathcal{P}$ it follows $G \in \mathcal{P}$, too (see [1]).

Many known properties of graphs are both hereditary and additive. Let us mention some of them:

$$\begin{aligned} \mathcal{O} &= \{G \in \mathcal{I} | G \text{ is totally disconnected}\}, \\ \mathcal{O}^2 &= \{G \in \mathcal{I} | G \text{ is bipartite}\}, \\ \mathcal{I}_k &= \{G \in \mathcal{I} | G \text{ does not contain} K_{k+2} \text{ as a subgraph}\}. \end{aligned}$$

We shall denote the set of all additive hereditary properties by $\boldsymbol{L}^a$.

For any property $\mathcal{P} \in \boldsymbol{L}^a$, $\mathcal{P} \neq \mathcal{I}$, there is a number $c(\mathcal{P})$ called *completeness* of $\mathcal{P}$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$. For given nonnegative i, by $\boldsymbol{L}^a_i$ we denote the set of all additive hereditary properties of completeness i.

A hereditary property $\mathcal{P}$ can be uniquely determined by the set of minimal forbidden graphs which can be defined as follows :

$$\boldsymbol{F}(\mathcal{P}) = \{F \in \mathcal{I} | F \notin \mathcal{P} \text{ but each proper subgraph of } F \text{ belongs to } \mathcal{P}\}.$$

It is easy to see that $\boldsymbol{L}^a_0 = \{\mathcal{O}\}$ and $\boldsymbol{F}(\mathcal{O}) = \{K_2\}$. The structure of the set $\boldsymbol{L}^a$ of the additive hereditary properties was investigated in [1] and [4], where it is proved that the set $\boldsymbol{L}^a$, partially ordered by set inclusion, forms a complete distributive lattice.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $\langle V_i \rangle_G$ has the property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as a set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition.

A property $\mathcal{P} \in \boldsymbol{L}^a$ is called *reducible* if there exist $\mathcal{P}_1 \in \boldsymbol{L}^a$, $\mathcal{P}_2 \in \boldsymbol{L}^a$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise $\mathcal{P}$ is called *irreducible*.

Let us start with some easy observations. Lemma 1 follows immediately from the definitions.

Lemma 1. *If $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, then $c(\mathcal{P}) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 1$.*

Lemma 2. *Let $\mathcal{P}_1, \mathcal{P}_2$ are hereditary properties of graphs. If $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$, then there exists a graph $G \in \mathcal{P}_2$ such that $G \in \boldsymbol{F}(\mathcal{P}_1)$.*

Proof. It is obvious that $\mathcal{P}_2 \backslash \mathcal{P}_1$ is nonempty, because of $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$. If $G \in \mathcal{P}_2 \backslash \mathcal{P}_1$, then either $G \in \boldsymbol{F}(\mathcal{P}_1)$ or G possesses $H \in \boldsymbol{F}(\mathcal{P}_1)$ as a subgraph. Since $\mathcal{P}_2$ is hereditary, it follows that $H \in \mathcal{P}_2$ and the proof is complete. ■

In connection with the Four Colour Theorem, different types of partitions of the vertices of planar graphs have been investigated. The problem of the determination of the "minimal reducible bounds for planar graphs" (see [2], p.266, [5]) is closely related to the characterization of the structure of the reducible properties of completeness 3 in the lattice $\boldsymbol{L}^a$.

The basic and natural question whether the factorization of any reducible property $\mathcal{R} \in \boldsymbol{L}^a$ into irreducible factors is unique seems to be extremaly difficult (see [2], p.266).

The aim of this paper is to prove that the factorization of any reducible property $\mathcal{R} \in \boldsymbol{L}^a$ of completeness $c(\mathcal{R}) \leq 3$ into irreducible factors is unique. We shall use the following result of [3].

Theorem 1. *Let $\mathcal{P}$ is an additive hereditary property with $c(\mathcal{P}) \leq 1$. Let $\mathcal{P}_1, \mathcal{P}_2$ are any additive hereditary properties. If $\mathcal{P} \circ \mathcal{P}_1 = \mathcal{P} \circ \mathcal{P}_2$, then $\mathcal{P}_1 = \mathcal{P}_2$.*

2. Main Results

Theorem 2. *A property $\mathcal{P} \in \boldsymbol{L}^a$ with $c(\mathcal{P}) = 1$ is reducible if and only if $\mathcal{P} = \mathcal{O}^2$* (i.e., $\mathcal{P}$ is the set of all bipartite graphs).

Proof. The proof follows immediately from Lemma 1, since $\boldsymbol{L}_0^a = \{\mathcal{O}\}$. ■

Theorem 3. *A factorization of each reducible property $\mathcal{P} \in \boldsymbol{L}_2^a$ into irreducible factors is unique, apart from the order of the factors.*

Proof. Let $\mathcal{P}$ be any reducible property, $\mathcal{P} \in \boldsymbol{L}_2^a$. Thus there exist $\mathcal{P}_1 \in \boldsymbol{L}^a, \mathcal{P}_2 \in \boldsymbol{L}^a$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. By Lemma 1 either $\mathcal{P}_1 = \mathcal{O}$, $\mathcal{P}_2 \in \boldsymbol{L}_1^a$ or $\mathcal{P}_1 \in \boldsymbol{L}_1^a$, $\mathcal{P}_2 = \mathcal{O}$. Suppose, without loss of generality, $\mathcal{P}_1 = \mathcal{O}$ and $\mathcal{P}_2 \in \boldsymbol{L}_1^a$. If there exists a property $\mathcal{P}_3$, $\mathcal{P}_3 \neq \mathcal{P}_2$, such that $\mathcal{O} \circ \mathcal{P}_3 = \mathcal{P}$, then $\mathcal{O} \circ \mathcal{P}_3 = \mathcal{O} \circ \mathcal{P}_2$ and by Theorem 1 we obtain $\mathcal{P}_3 = \mathcal{P}_2$, a contradiction.

Two cases can occur now:

(1) $\mathcal{P}_2$ is irreducible and then $\mathcal{O} \circ \mathcal{P}_2$ is the unique factorization of $\mathcal{P}$ into irreducible factors.

(2) $\mathcal{P}_2$ is reducible. By Theorem 2 $\mathcal{P}_2 = \mathcal{O}^2$, which implies that $\mathcal{O} \circ \mathcal{O} \circ \mathcal{O}$ is the unique factorization of the property $\mathcal{P}$ into irreducible factors. ■

Theorem 4. *A factorization of any reducible additive hereditary property $\mathcal{P}$ of completeness 3 into irreducible factors is unique, apart from the order of the factors.*

3. The Proof of the Main Result

The proof of Theorem 4 is based on the following Lemmas.

Lemma 3. *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be the additive hereditary properties all of completeness 1. If for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$ $\mathcal{P}_i \neq \mathcal{P}_j$, then $\mathcal{P}_1 \circ \mathcal{P}_2 \neq \mathcal{P}_3 \circ \mathcal{P}_4$.*

Proof. Because of transitivity of set inclusion there exists $i \in \{1,2,3,4\}$ such that for every $j \in \{1,2,3,4\}$, $j \neq i$, $\mathcal{P}_j \not\subset \mathcal{P}_i$. Without loss of generality, we can suppose that $i = 3$. The facts that $\mathcal{P}_1 \neq \mathcal{P}_3$ and $\mathcal{P}_2 \neq \mathcal{P}_3$ imply that both

$$\mathcal{P}_1 \not\subseteq \mathcal{P}_3 \text{ and } \mathcal{P}_2 \not\subseteq \mathcal{P}_3. \tag{1}$$

Let us suppose, on the contrary, that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{P}_3 \circ \mathcal{P}_4$. From (1) and Lemma 2 it follows that there exist graphs $G_1^{'}, G_2^{'}$ such that

$$G_1^{'} \in \mathcal{P}_1 \text{ and } G_1^{'} \notin \mathcal{P}_3, \tag{2}$$

$$G_2^{'} \in \mathcal{P}_2 \text{ and } G_2^{'} \notin \mathcal{P}_3. \tag{3}$$

Let we state $n = \max\{|V(G_1^{'})|, |V(G_2^{'})|\}$ and let the graphs G_1 and G_2 consist of n disjoint copies of $G_1^{'}$ and $G_2^{'}$, respectively. Let

$$G = G_1 + G_2. \tag{4}$$

Let us denote $V_1 = V(G_1)$, $V_2 = V(G_2)$. It is obvious that (V_1, V_2) is a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G. If the graph $G \in \mathcal{P}_3 \circ \mathcal{P}_4$, then there exists a vertex $(\mathcal{P}_3, \mathcal{P}_4)$-partition of G, say (U_1, U_2). Since for every $i \in \{1,2\}$ $G_i^{'} \notin \mathcal{P}_3$, then $G_1^{'}$ cannot be a subgraph of $\langle V_1 \cap U_1 \rangle_G$ and $G_2^{'}$ cannot be a subgraph of $\langle V_2 \cap U_1 \rangle_G$. Moreover, as $E(G_1^{'}) \neq \emptyset$ then $G_1^{'}$ cannot be a subgraph of $\langle V_1 \cap U_2 \rangle_G$, otherwise necessarilly $V_2 \cap U_2 = \emptyset$ what implies that $G_2^{'} \subseteq \langle V_2 \cap U_1 \rangle_G$, a contradiction. By the similar reason $G_2^{'}$ cannot be a subgraph of $\langle V_2 \cap U_2 \rangle_G$. These imply, according to the number of components of $G_1^{'}$ and $G_2^{'}$ in the graphs G_1 and G_2, respectively, that

$$|V_i \cap U_j| \geq n, \quad \text{for every } \ i, j \in \{1,2\}. \tag{5}$$

Two cases can appear.

Case 1. Both graphs $G_1^{'}$ and $G_2^{'}$ are bipartite.
Thus, according to the choice of n, we have

$$G_1^{'} \subseteq K_{n,n} \subseteq \langle U_1 \rangle_G,$$

$$G_2^{'} \subseteq K_{n,n} \subseteq \langle U_2 \rangle_G,$$

this contradicts our assumption $\langle U_1 \rangle_G \in \mathcal{P}_3$ and $\langle U_2 \rangle_G \in \mathcal{P}_4$.

Case 2. At least one of the graphs $G_1^{'}$ and $G_2^{'}$ is not bipartite.
Suppose, without loss of generality, that $G_1^{'}$ is not bipartite. Then necessarily either $E(\langle V_1 \cap U_1 \rangle_G) \neq \emptyset$ or $E(\langle V_1 \cap U_2 \rangle_G) \neq \emptyset$. In accordance with (5)

these facts imply that either $K_3 \subseteq \langle U_1 \rangle_G$ or $K_3 \subseteq \langle U_2 \rangle_G$, which contradicts our assumption $\langle U_1 \rangle_G \in \mathcal{P}_3$ and $\langle U_2 \rangle_G \in \mathcal{P}_4$.

Since there exists no vertex $(\mathcal{P}_3, \mathcal{P}_4)$-partition of G, thus $G \notin \mathcal{P}_3 \circ \mathcal{P}_4$. Because of $G \in \mathcal{P}_1.\mathcal{P}_2$, the proof is complete. ■

The proof of the following simple but helpful Lemma is trivial and thus we omit it.

Lemma 4. *Let G_1, G_2 are two arbitrary nonbipartite graphs. Let $G = G_1 + G_2$ be the join of G_1 and G_2. Then for every $\mathcal{O}$-independent subset S of a vertex set $V(G)$, the graph $K_4 \subseteq G$-S.*

Lemma 5. *Let $\mathcal{P}$ be any reducible additive hereditary property of completeness* 3. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the irreducible additive hereditary properties both of completeness* 1. *If $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$ then there exists no additive hereditary property $\mathcal{P}'$, $c(\mathcal{P}') = 2$, such that $\mathcal{P} = \mathcal{O} \circ \mathcal{P}'$.*

Proof. Let us suppose that $\mathcal{P}_1, \mathcal{P}_2$ are the irreducible additive hereditary properties of completeness 1. Three cases can occur. We shall prove the Lemma for every case separately.

Case 1. $\mathcal{P}_1 \subset \mathcal{O}^2$ and $\mathcal{P}_2 \subset \mathcal{O}^2$.

In this case there exist the graphs $K_{p,p}$ and $K_{q,q}$ such that $K_{p,p} \notin \mathcal{P}_1$ and $K_{q,q} \notin \mathcal{P}_2$. Let us assume, on the contrary, that there exists an additive hereditary property $\mathcal{P}'$ of completeness 2 such that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$. Then each graph $G \in \mathcal{O} \circ \mathcal{P}'$ must belong to $\mathcal{P}_1 \circ \mathcal{P}_2$, too. Let us take the graph G as follows:

$$G = D_n + \bigcup_{i=1}^{n} K_3,$$

where D_n is a totally disconnected graph of order $n = \max\{p, q\}$. It is easy to see that $G \in \mathcal{O} \circ \mathcal{P}'$ for every $\mathcal{P}' \in \boldsymbol{L}_2^a$. We must only realize that $K_3 \in \mathcal{P}'$ whenever $c(\mathcal{P}') = 2$. Then (V_1, V_2), if $V_1 = V(D_n)$ and $V_2 = V(\cup_{i=1}^{n} K_3)$, is a vertex $(\mathcal{O}, \mathcal{P}')$-partition of G. Now we shall prove that $G \notin \mathcal{P}_1 \circ \mathcal{P}_2$. Let us assume, on the contrary, that G has $(\mathcal{P}_1, \mathcal{P}_2)$-partition of a vertex set $V(G)$, denote it (U_1, U_2). It is obvious that the sets of vertices of each copy of K_3 must be partitioned into two nonempty subsets. Then $|U_1 \cap V_2| \geq n$ and $|U_2 \cap V_2| \geq n$, too. Moreover, as K_3 is not bipartite, then either $E(\langle U_1 \cap V_2 \rangle_G)$ or $E(\langle U_2 \cap V_2 \rangle_G)$ is nonempty. This implies that then necessarily either $U_1 \cap V_1 = \emptyset$ or $U_2 \cap V_2 = \emptyset$, respectively (otherwise $K_3 \subseteq \langle U_1 \cap V_2 \rangle_G$ or $K_3 \subseteq \langle U_2 \cap V_2 \rangle_G$). Then either $K_{n,n} \subseteq \langle U_1 \rangle_G$ or $K_{n,n} \subseteq \langle U_2 \rangle_G$, respectively, but as it was stated before, neither $K_{n,n} \notin \mathcal{P}_1$, nor $K_{n,n} \notin \mathcal{P}_2$. This is a contradiction to the assumption that $\langle U_1\rangle_G \in \mathcal{P}_1$ and $\langle U_2\rangle_G \in \mathcal{P}_2$.

So we found the graph G which belongs to each additive hereditary property $\mathcal{O} \circ \mathcal{P}'$, $\mathcal{P}' \in \boldsymbol{L}_2^a$, but $G \notin \mathcal{P}_1 \circ \mathcal{P}_2$. This refutes our assumption about the existence of such a property $\mathcal{P}' \in \boldsymbol{L}_2^a$ that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$.

Case 2. $\mathcal{P}_1 \not\subseteq \mathcal{O}^2$ and $\mathcal{P}_2 \not\subseteq \mathcal{O}^2$.
If both $\mathcal{P}_1 \not\subseteq \mathcal{O}^2$ and $\mathcal{P}_2 \not\subseteq \mathcal{O}^2$ then there exist non bipartite graphs $G_1 \in \mathcal{P}_1$, $G_2 \in \mathcal{P}_2$. Let us construct the graph G

$$G = G_1 + G_2.$$

Then by Lemma 4, for every partition (V_1, V_2) of a vertex set $V(G)$ such that $\langle V_1\rangle_G \in \mathcal{O}$, the graph $K_4 \subseteq \langle V_2\rangle_G$. This implies that there exists no additive hereditary property $\mathcal{P}'$, $c(\mathcal{P}') = 2$ and $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$ holds.

Case 3. Either ($\mathcal{P}_1 \subset \mathcal{O}^2$ and $\mathcal{P}_2 \not\subseteq \mathcal{O}^2$) or ($\mathcal{P}_1 \not\subseteq \mathcal{O}^2$ and $\mathcal{P}_2 \subset \mathcal{O}^2$).
Let us assume that $\mathcal{P}_1 \subset \mathcal{O}^2$ and $\mathcal{P}_2 \not\subseteq \mathcal{O}^2$. In case $\mathcal{P}_2 \subset \mathcal{O}^2$ and $\mathcal{P}_1 \not\subseteq \mathcal{O}^2$ the proof goes in analogical way.

If $\mathcal{P}_1 \subset \mathcal{O}^2$, then there exists a natural number m such that $K_{m,m} \notin \mathcal{P}_1$. Let us define

$$n = \min\{m \in \boldsymbol{N} | K_{m,m} \notin \mathcal{P}_1\} - 1.$$

Let G_2 be the graph with property $\mathcal{P}_2$ such that G_2 is not bipartite. Now we define the graph G^* as follows:

$$G^* = K_{n,n} + G_2.$$

If we denote $W_1 = V(K_{n,n})$ and $W_2 = V(G_2)$, then it is easy to see that (W_1, W_2) is a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G^*. Let us suppose, on the contrary, that there exists a property $\mathcal{P}' \in \boldsymbol{L}_2^a$ such that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$ holds. As $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$ and $G^* \in \mathcal{P}_1 \circ \mathcal{P}_2$, then $G^* \in \mathcal{O} \circ \mathcal{P}'$. This implies that there exists $\mathcal{O}$-independent subset S of a vertex set $V(G^*)$ so that G^*-S $\in \mathcal{P}'$. Because of $K_4 \notin \mathcal{P}'$ the vertex set S has to be a subset of $V(K_{n,n})$ such that $K_{n,n}$-S $\in \mathcal{O}$. Thus $|S| = n$ and $|V(K_{n,n}\text{-S})| = n$. This implies

$$D_n + G_2 \in \mathcal{P}'.$$

We showed, supposing the existence of $\mathcal{P}' \in \boldsymbol{L}_2^a$: $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$, that the graph $D_n + G_2$ has a property $\mathcal{P}'$.

Let us define the graphs G_3, G_4, G_5 and G as follows :

$$G_3 = D_n + G_2,\ G_4 = D_{n+1},\ G_5 = G_3 \cup G_3,\ G = G_4 + G_5.$$

As $\mathcal{O}$ and $\mathcal{P}'$ are both additive hereditary properties, then $G_4 \in \mathcal{O}$ and $G_5 \in \mathcal{P}'$. Further, it is obvious that the graph $G \in \mathcal{O} \circ \mathcal{P}'$ and then it has a vertex $(\mathcal{O}, \mathcal{P}')$-partition, say (V_1, V_2). Let $V_1 = V(G_4)$ and $V_2 = V(G_5)$. We shall prove that $G \notin \mathcal{P}_1 \circ \mathcal{P}_2$. Let us suppose, on the contrary, that $G \in \mathcal{P}_1 \circ \mathcal{P}_2$. Then there exists some vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G. Let (U_1, U_2) be the vertex partition mentioned above. As graph G_5 is not bipartite it follows immediately that either $\langle V_2 \cap U_1 \rangle_G \notin \mathcal{O}$ or $\langle V_2 \cap U_2 \rangle_G \notin \mathcal{O}$. We will distinguish the following two subcases.

Subcase 3.1. $\langle V_2 \cap U_2 \rangle_G \notin \mathcal{O}$.
Then the condition $V_1 \subseteq U_1$ has to be fulfilled. Because of $K_3 \notin \mathcal{P}_1$, it is obvious that $V_2 \cap U_1$ has to be an independent set of vertices. As also $K_3 \notin \mathcal{P}_2$, we are forced to move just all the vertices of the subgraph D_n of each copy of the graph G_3 in the graph G_5 into the set $V_2 \cap U_1$ (otherwise $K_3 \subseteq \langle V_2 \cap U_2 \rangle_G$). This implies that

$$K_{n+1,n+1} \subseteq \langle U_1 \rangle_G$$

which is a contradiction because of $K_{n+1,n+1} \notin \mathcal{P}_1$.

Subcase 3.2. $\langle V_2 \cap U_1 \rangle_G \notin \mathcal{O}$.
Then the condition $V_1 \subseteq U_2$ must be fulfilled. By an easy observation we can see that all the vertices of the subgraphs D_n of each copy of G_3 in G_5 must belong to the set $V_2 \cap U_2$ (otherwise $K_3 \subseteq \langle V_2 \cap U_1 \rangle_G$). So the thing is completed because the graph

$$G_2 \subseteq \langle U_1 \rangle_G$$

which contradicts our supposition $\langle U_1 \rangle_G \in \mathcal{P}_1$.

We showed that the graph $G \in \mathcal{O} \circ \mathcal{P}'$ and $G \notin \mathcal{P}_1 \circ \mathcal{P}_2$. This fact contradicts our assumption that there exists $\mathcal{P}' \in \boldsymbol{L}_2^a$ in such a way that the equation $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$ holds.

As there are no more possibilities for existence of the properties $\mathcal{P}_1$ and $\mathcal{P}_2$, the proof is complete. ■

Lemma 6. *Let $\mathcal{P}'$ be an irreducible additive hereditary property, $c(\mathcal{P}') = 2$. Then there exist no additive hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$, both of completeness* 1, *such that $\mathcal{O} \circ \mathcal{P}' = \mathcal{P}_1 \circ \mathcal{P}_2$.*

Proof. Let $\mathcal{P}' \in \boldsymbol{L}_2^a$ be an arbitrary irreducible property. Let us suppose, on the contrary, that there exist additive hereditary properties $\mathcal{P}_1 \in \boldsymbol{L}_1^a$, $\mathcal{P}_2 \in \boldsymbol{L}_1^a$ such that $\mathcal{O} \circ \mathcal{P}' = \mathcal{P}_1 \circ \mathcal{P}_2$. There are two possibilities for the existence of the above mentioned properties $\mathcal{P}_1$ and $\mathcal{P}_2$.

Case 1. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ are irreducible.
By Lemma 5 there exists no additive hereditary property $\mathcal{P}^* \in \boldsymbol{L}_2^a$ such that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}^*$. This fact contradicts our assumption $\mathcal{O} \circ \mathcal{P}' = \mathcal{P}_1 \circ \mathcal{P}_2$.

Case 2. Either $\mathcal{P}_1$ or $\mathcal{P}_2$ is reducible.
We can suppose, without loss of generality, that $\mathcal{P}_1$ is reducible. By Theorem 2 we have that $\mathcal{P}_1 = \mathcal{O} \circ \mathcal{O}$. Thus

$$\mathcal{O} \circ \mathcal{O} \circ \mathcal{P}_2 = \mathcal{O} \circ \mathcal{P}'$$

and by cancellation of $\mathcal{O}$ (see [3]) we obtain

$$\mathcal{P}' = \mathcal{O} \circ \mathcal{P}_2$$

which is a contradiction to our assumption that $\mathcal{P}'$ is irreducible. ∎

***The Proof of Theorem* 4.** Let $\mathcal{P}$ be any reducible property, $\mathcal{P} \in \boldsymbol{L}_3^a$, so that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Then from Lemma 1 either $c(\mathcal{P}_1) = c(\mathcal{P}_2) = 1$ or $\mathcal{P}_1 = \mathcal{O}$ and $c(\mathcal{P}_2) = 2$. We shall distinguish three cases.

Case 1. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ are irreducible.
Then the uniqueness of a factorization $\mathcal{P}_1 \circ \mathcal{P}_2$ of $\mathcal{P}$ into irreducible factors follows by Lemmas 3, 5 and 6 using the cancellation law according to Theorem 1.

Case 2. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ are reducible.
Then by Theorem 2, $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{O}^2$ and $\mathcal{P} = \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} \circ \mathcal{O}$. This factorization is unique by Lemmas 3, 5, 6 and by Theorem 1.

Case 3. Either $\mathcal{P}_1$ or $\mathcal{P}_2$ is reducible. Without loss of generality, let $\mathcal{P}_1$ be a reducible property and $\mathcal{P}_2$ be an irreducible property.

(a) If $c(\mathcal{P}_1) = 1$, then $\mathcal{P}_1 = \mathcal{O}^2$ and $\mathcal{P} = \mathcal{O} \circ \mathcal{O} \circ \mathcal{P}_2$ is a unique factorization of $\mathcal{P}$ into irreducible factors.

(b) If $c(\mathcal{P}_1) = 2$, then there exists a property $\mathcal{P}'$ such that $\mathcal{P}_1 = \mathcal{O} \circ \mathcal{P}'$ and $\mathcal{P}_2 = \mathcal{O}$. Thus, according to whether $\mathcal{P}'$ is a reducible or an irreducible property, either $\mathcal{P} = \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} \circ \mathcal{O}$ or $\mathcal{P} = \mathcal{O} \circ \mathcal{O} \circ \mathcal{P}'$ is a unique factorization of $\mathcal{P}$ into irreducible factors. ∎

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