

## GENERALIZED LIST COLOURINGS OF GRAPHS

MIECZYSLAW BOROWIECKI

EWA DRGAS-BURCHARDT

*Institute of Mathematics*  
*Technical University*  
*Podgórna 50, 65-246 Zielona Góra, Poland*

and

PETER MIHÓK

*Department of Geometry and Algebra*  
*P.J. Šafárik University*  
*041 54 Košice, Slovakia*

### Abstract

We prove: (1) that  $\text{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$  can be arbitrarily large, where  $\text{ch}_{\mathcal{P}}(G)$  and  $\chi_{\mathcal{P}}(G)$  are  $\mathcal{P}$ -choice and  $\mathcal{P}$ -chromatic numbers, respectively, (2) the  $(\mathcal{P}, L)$ -colouring version of Brooks' and Gallai's theorems.

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### 1. INTRODUCTION AND NOTATION

We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph  $G$  is denoted by  $V(G)$  and the edge set by  $E(G)$ . The notation  $H \subseteq G$  means that  $H$  is a subgraph of  $G$ . The vertex induced (we will say briefly: induced) subgraph  $H$  of  $G$  is denoted by  $H \leq G$ . We say that  $G$  contains  $H$  whenever  $G$  contains a subgraph isomorphic to  $H$ . In general, we follow the notation and terminology of [5].

Let  $\mathcal{I}$  denote the set of all mutually nonisomorphic graphs. If  $\mathcal{P}$  is a nonempty subset of  $\mathcal{I}$ , then  $\mathcal{P}$  will also denote the property that a graph is

a member of the set  $\mathcal{P}$ . We shall use the terms *set of graphs* and *property of graphs* interchangeably.

A property  $\mathcal{P}$  of graphs is said to be *induced hereditary* (shortly: *hereditary*) if whenever  $G \in \mathcal{P}$  and  $H$  is a vertex induced subgraph of  $G$ , then also  $H \in \mathcal{P}$ . For convenience, the empty set  $\emptyset$  will be regarded as the set inducing the subgraph with any property  $\mathcal{P}$ .

A property  $\mathcal{P}$  is *additive*, if for each graph  $G$  all of whose components have the property  $\mathcal{P}$  it follows that  $G \in \mathcal{P}$ . Let us denote by  $\mathbf{M}$  and  $\mathbf{M}^a$  the set of all hereditary and additive hereditary properties, respectively. Any hereditary property  $\mathcal{P}$  of graphs is uniquely determined by the set  $\mathcal{C}(\mathcal{P})$  of all *forbidden subgraphs* defined by

$$\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H)\}.$$

It is easy to prove that a property  $\mathcal{P} \in \mathbf{M}$  is additive if and only if all forbidden subgraphs  $H \in \mathcal{C}(\mathcal{P})$  are connected.

A hereditary property  $\mathcal{P} \in \mathbf{M}$  is said to be nontrivial if  $\mathcal{P} \neq \mathcal{I}$ . For a nontrivial property  $\mathcal{P} \in \mathbf{M}$  there exists a number  $c(\mathcal{P})$  called the *completeness* of  $\mathcal{P}$  defined as  $\sup\{k : K_{k+1} \in \mathcal{P}\}$ , and  $c(\mathcal{P}) = \infty$  if for every  $k$ ,  $K_{k+2} \in \mathcal{P}$ . Let  $\delta(\mathcal{P}) = \min\{\delta(H) : H \in \mathcal{C}(\mathcal{P})\}$ .

Let us denote by  $\mathcal{O} = \{G : G \in \mathcal{I}, E(G) = \emptyset\}$ . For this property we have  $\mathcal{C}(\mathcal{O}) = \{K_2\}$  and  $\delta(\mathcal{O}) = 1$ .

A  $\mathcal{P}$ -*partition* (*colouring*) of a graph  $G$  is a partition  $(V_1, \dots, V_n)$  of  $V(G)$  such that the subgraph  $\langle V_i \rangle$  induced by the set  $V_i$  has property  $\mathcal{P}$  for each  $i = 1, \dots, n$ . If  $(V_1, \dots, V_n)$  is a  $\mathcal{P}$ -partition of a graph  $G$ , then the corresponding vertex colouring  $f$  is defined by  $f(v) = i$  whenever  $v \in V_i$ , for  $i = 1, \dots, n$ . The smallest integer  $n$  for which  $G$  has  $\mathcal{P}$ -partition is called the  $\mathcal{P}$ -*chromatic* (or  $\mathcal{P}$ -*vertex-partition*) *number* of  $G$  and is denoted by  $\chi_{\mathcal{P}}(G)$ . The  $\mathcal{O}$ -chromatic number is the ordinary chromatic number. See [1] for a survey and more details.

Let  $G$  be a graph and let  $L(v)$  be a list of colours (as above, positive integers) prescribed for the vertex  $v$ , and  $\mathcal{P} \in \mathbf{M}$ . A  $(\mathcal{P}, L)$ -*colouring* is a graph  $\mathcal{P}$ -colouring  $f(v)$  with the additional requirement that for all  $v \in V(G)$ ,  $f(v) \in L(v)$ . If  $G$  admits a  $(\mathcal{P}, L)$ -colouring, then  $G$  is said to be  $(\mathcal{P}, L)$ -*colourable*. The graph  $G$  is  $(k, \mathcal{P})$ -*choosable* if it is  $(\mathcal{P}, L)$ -colourable for every list  $L$  of  $G$  satisfying  $|L(v)| = k$  for every  $v \in V(G)$ . The  $\mathcal{P}$ -*choice number*  $\text{ch}_{\mathcal{P}}(G)$  is the smallest natural number  $k$  such that  $G$  is  $(k, \mathcal{P})$ -choosable.

Vizing [6] and Erdős, Rubin and Taylor [3] independently introduce the idea of considering  $(\mathcal{O}, L)$ -colouring and  $(k, \mathcal{O})$ -choosability.

The aim of this paper is to prove some extensions of two basic theorems in the colouring theory of graphs, namely the Brooks [2] and Gallai [4] theorems. If  $L(v)$  is the same for all vertices of  $G$ , this results generalize also the corresponding results of [1]. Moreover, we prove that  $\text{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$  can be arbitrarily large.

2. BEHAVIOUR OF THE  $\mathcal{P}$ -CHOICE NUMBER

To prove the main theorem of this section, use well-known Pigeonhole Principle in the following form.

**Pigeonhole Principle.** *Suppose that  $q_1, \dots, q_t$  are positive integers. If  $X = X_1 \cup \dots \cup X_t$  is a partition of the set  $X$  and  $|X| \geq (\sum_{i=1}^t q_i) - t + 1$ , then  $|X_i| \geq q_i$  for some  $i \in \{1, \dots, t\}$ .*

**Theorem 1.** *Let  $\mathcal{P} \in \mathbf{M}^a$  and  $1 \leq c(\mathcal{P}) < \infty$ . Then for any nonnegative integer  $s$  there exists a graph  $G_s$  such that  $\text{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > s$ .*

**Proof.** For a given  $c(\mathcal{P})$  let the function  $g(l)$  be defined by

$$g(l) = \left\lceil \frac{l(l-1) - 2 + c(\mathcal{P})(c(\mathcal{P}) + 3)}{2c(\mathcal{P})(c(\mathcal{P}) + 3)} \right\rceil - 1 - l$$

The function  $g(l)$  tends to infinity with  $l \rightarrow \infty$ . Hence, for given  $s$  and  $c(\mathcal{P})$  there is an integer  $l_0 \geq 3$  such that  $g(l_0) \geq s$ . For this integer  $l_0$ , let

$$(*) \quad b = \left\lceil \frac{l_0(l_0 - 1) - 2 + c(\mathcal{P})(c(\mathcal{P}) + 3)}{2c(\mathcal{P})(c(\mathcal{P}) + 3)} \right\rceil - 1$$

Note that  $b \geq s + l_0 \geq 3$ .

Let the graph  $G_s$  be given by the join  $H_1 + \dots + H_{l_0}$  of totally disconnected graphs  $H_i, i = 1, \dots, l_0$ , all of order  $t = \binom{2b-1}{b}$ , i.e.,  $G_s$  is a complete  $l_0$ -partite graph with  $t$  elements in each part.

Since  $\mathcal{P} \in \mathbf{M}^a$  and  $K_1 \in \mathcal{P}$ , then  $H_i \in \mathcal{P}$  for  $i = 1, \dots, l_0$ . Hence,  $\chi_{\mathcal{P}}(G_s) \leq l_0$ . Note that it is sufficient to prove that  $G_s$  is not  $(b, \mathcal{P})$ -choosable, i.e.,  $\text{ch}_{\mathcal{P}}(G_s) > b$ . By this and (\*) we have  $\text{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > b - l_0 \geq s$ .

To prove this, let  $C = \{1, \dots, 2b - 1\}$  be a set of colours and let  $[C]^b = \{A : A \subseteq C, |A| = b\}$ . Suppose that this set is indexed as follows  $\{A_j : j = 1, \dots, t\}$ . Besides, let  $V(H_i) = \{v_{ij} : j = 1, \dots, t\}, i = 1, \dots, l_0$ . Define a list  $L$  for the graph  $G_s$  by  $L(v_{ij}) = A_j$  for all  $i$  and  $j$ . Let  $f$  be a  $(\mathcal{P}, L)$ -colouring of  $G_s$ . Note that

(a) For any colouring of the graph  $H_i$  from its lists we need at least  $b$  different colours for any  $i = 1, \dots, l_0$ .

(b) For  $i_1, i_2 \in \{1, \dots, l_0\}, i_1 \neq i_2$ , there is  $j_1, j_2 \in \{1, \dots, t\}$  and a colour  $c \in C$  such that  $f(v_{i_1 j_1}) = f(v_{i_2 j_2}) = c$ . This follows by (a) and  $|C| = 2b - 1$ . The colour  $c$  is said to be  $\{i_1, i_2\}$ -colour. Let  $X = \{\{i_1, i_2\} : \{i_1, i_2\}\text{-colour}\}$  and  $X_c = \{\{i_1, i_2\} : c \text{ is } \{i_1, i_2\}\text{-colour}\}, c \in C$ .

Let be given a sequence of integers  $q_1 = \dots = q_{2b-1} = \binom{c(\mathcal{P}) + 2}{2}$ .

Note that  $X = X_1 \cup \dots \cup X_{2b-1}$ . By above and (\*) we have

$$|X| = \binom{l_0}{2} \geq \sum_{i=1}^{2b-1} q_i - (2b - 1) + 1.$$

Hence, by Pigeonhole Principle it follows that there is  $c_0 \in C$ , such that  $|X_{c_0}| \geq q_{c_0} = \binom{c(\mathcal{P}) + 2}{2}$ . It implies that there are at least  $c(\mathcal{P}) + 2$  pairwise different integers  $i_1, \dots, i_{c(\mathcal{P})+2} \in \{1, \dots, l_0\}$  and for any  $i_r$  there is an integer  $j, 1 \leq j \leq t$ , such that  $f(v_{i_r j}) = c_0$ . By above and the definition of  $G_s$ , the subgraph of  $G_s$  induced by vertices with the assigned colour  $c_0$  contains a complete graph of order  $c(\mathcal{P}) + 2$ , i.e., it is not  $(b, \mathcal{P})$ -choosable. Hence,  $\text{ch}_{\mathcal{P}}(G_s) > b$ . ■

### 3. $(\mathcal{P}, L)$ -CRITICAL GRAPHS

For a nontrivial property  $\mathcal{P} \in \mathbf{M}$  a graph  $G$  is said to be  $(\mathcal{P}, L)$ -critical if  $G$  has no  $(\mathcal{P}, L)$ -colouring but  $G - v$  is  $(\mathcal{P}, L)$ -colourable for all  $v \in V(G)$ .

**Lemma 1.** *If  $\mathcal{P} \in \mathbf{M}$  and  $G$  is  $(\mathcal{P}, L)$ -critical, then  $d_G(v) \geq \delta(\mathcal{P}) |L(v)|$  for any vertex  $v$  of  $G$ .*

**Proof.** Suppose that  $d_G(u) < \delta(\mathcal{P}) |L(u)|$  for a vertex  $u \in V(G)$ . Then there is  $i \in L(u)$  which is used in colouring of the vertices of  $N_G(u)$  less than  $\delta(\mathcal{P})$  times. Therefore, the vertex  $u$  can be coloured by  $i, \langle V_i \rangle_G \in \mathcal{P}$  and  $G$  is not  $(\mathcal{P}, L)$ -critical, a contradiction. ■

Let  $\mathcal{P} \in \mathbf{M}$ ,  $G$  be  $(\mathcal{P}, L)$ -critical and  $x \in V(G)$ . Define a new list assignment

$$L^x(v) = \begin{cases} \{l\}, & v = x, \\ L(v), & \text{otherwise,} \end{cases}$$

where  $l \notin L(v)$  for  $v \in V(G)$ .

Since, by the definition,  $G - w$  is  $(\mathcal{P}, L)$ -colourable, then  $G$  is  $(\mathcal{P}, L^w)$ -colourable. Thus, for a vertex  $w$  of a  $(\mathcal{P}, L)$ -critical graph  $G$  we shall say that  $f$  is a  $(\mathcal{P}, L^w)$ -colouring of  $G$ , whenever  $f(w) = l$  and  $f(v) \in L(v)$  for  $v \neq w$ .

Note that the list  $L^x$  is always created from the list  $L$  by assignment of the colour  $l$  to the vertex  $x$  and preserving the remaining assignments.

Let us denote by  $S(G) = \{v : v \in V(G), d_G(v) = \delta(\mathcal{P}) \mid L(v)\}$ .

From Lemma 1 we have immediately the following lemma.

**Lemma 2.** *Let  $G$  be a  $(\mathcal{P}, L)$ -critical,  $w \in S(G)$ . Then for any  $(\mathcal{P}, L^w)$ -colouring  $f$  of  $G$ ,  $|N_G(w) \cap V_i| = \delta(\mathcal{P})$  for any  $i \in L(w)$ , where  $V_i = \{v : v \in V(G), f(v) = i\}$ . ■*

**Lemma 3.** *Let  $G$  be a  $(\mathcal{P}, L)$ -critical,  $u, v \in S(G)$ ,  $uv \in E(G)$ , and let  $f$  be a  $(\mathcal{P}, L^v)$ -colouring of  $G$ . Then there is a  $(\mathcal{P}, L^u)$ -colouring  $f'$  of  $G$  such that  $f'(v) = f(u)$  and  $f'(w) = f(w)$  for all  $w \in V(G) - \{u, v\}$ .*

**Proof.** Since  $u, v \in S(G)$ , by Lemma 2 we have  $f(u) \in L(v)$ . From this, the definition of the  $(\mathcal{P}, L^x)$ -colouring and again Lemma 2, the required  $f'$  colouring follows. ■

**Lemma 4.** *Let  $G$  be a  $(\mathcal{P}, L)$ -critical and  $Q : v_0v_1\dots v_m$  be a walk in  $G$  such that  $V(Q) = \{v_0, v_1, \dots, v_m\} \subseteq S(G)$ . Let  $f$  be a  $(\mathcal{P}, L^{v_0})$ -colouring of  $G$ . Then there is a  $(\mathcal{P}, L^{v_m})$ -colouring  $f'$  of  $G$  such that  $f'(v_i) = f(v_{i+1})$  for  $i = 1, \dots, m-1$ ,  $f'(v_m) = f(v_0) = l$  and  $f'(w) = f(w)$  for all  $w \in V(G) - V(Q)$ .*

**Proof.** By applying Lemma 3 to the consecutive adjacent vertices of  $Q$  we obtain the required  $f'$  colouring. ■

The procedure described by Lemma 4 will be called  $(\mathcal{P}, L^x)$ -recolouring of  $Q$ .

**Lemma 5.** *Let  $G$  be a  $(\mathcal{P}, L)$ -critical and  $C : v_0v_1\dots v_mv_0$  be a cycle in  $G$  with  $V(C) = \{v_0, v_1, \dots, v_m\} \subseteq S(G)$  and let  $f$  be a  $(\mathcal{P}, L^{v_0})$ -colouring of  $G$ . Then there is a  $(\mathcal{P}, L^{v_0})$ -colouring  $f'$  of  $G$  such that  $f'(v_i) = f(v_{i+1})$  for  $i = 0, 1, \dots, m-1$ ,  $f'(v_m) = f(v_1)$ ,  $f'(w) = f(w)$  for all  $w \in V(G) - V(C)$ .*

**Proof.** By  $(\mathcal{P}, L^x)$  recolouring of  $Q : v_0v_1 \dots v_mv_0$  it follows. ■

**Lemma 6.** *Let  $C : v_0v_1 \dots v_mv_0$  be an even cycle in a  $(\mathcal{P}, L)$ -critical graph  $G$  with  $V(C) \subseteq S(G)$ . If there exists a vertex  $v_j$  which is not incident to any diagonal of  $C$ , then in any  $(\mathcal{P}, L^{v_j})$ -colouring  $f$  of  $G$  all vertices of  $C$  but  $v_j$  have the same colour.*

**Proof.** By the assumption,  $N_C(v_j) = \{v_{j-1}, v_{j+1}\}$ . Let  $f$  be an arbitrary  $(\mathcal{P}, L^{v_j})$ -colouring of  $G$ . By Lemma 2 the vertex  $v_j$  has  $\delta(\mathcal{P})$  neighbours in each colour from its list. Lemma 4 implies that this property is preserved after  $(\mathcal{P}, L^x)$ -recolouring of a walk  $W : v_{j+1} \dots v_mv_0 \dots v_{j-1}$ . Since  $C$  is an even cycle we have that all vertices of  $C$  but  $v_j$  have the same colour. ■

**Lemma 7.** (Dirac, see [4], p.170). *If each even cycle in the block  $B$  of a graph  $G$  has at least two diagonals in  $G$ , then the block  $B$  is a complete subgraph of  $G$ .* ■

**Theorem 2.** *Let  $\mathcal{P} \in \mathbf{M}$ ,  $G$  be a  $(\mathcal{P}, L)$ -critical graph. Then any block of  $\langle S(G) \rangle_G$  is one of the following types:*

- (i)  $B$  is a complete graph,
- (ii)  $B$  is a  $\delta(\mathcal{P})$ -regular graph belonging to  $\mathcal{C}(\mathcal{P})$ ,
- (iii)  $B \in \mathcal{P}$  and  $\Delta(B) \leq \delta(\mathcal{P})$ ,
- (iv)  $B$  is an odd cycle.

**Proof.** We have considered three cases.

*Case 1.* The block  $B$  of  $\langle S(G) \rangle_G$  contains no even cycles.

Then either  $B = K_2$  or  $B$  contains an odd cycle  $C_{2p+1}$ . In the case when  $B \neq C_{2p+1}$ , either  $C_{2p+1}$  has a diagonal in  $\langle S(G) \rangle_G$  or there exists a vertex  $u$  of  $B$  not belonging to  $C_{2p+1}$ . In both cases there is an even cycle in  $B$ , a contradiction.

*Case 2.* There is an even cycle  $C_{2p} : v_0v_1 \dots v_{2p-1}v_0$  in  $B$  which contains a vertex  $v_j$  not incident to any diagonal of  $C_{2p}$ .

Let  $f$  be a  $(\mathcal{P}, L^{v_j})$ -colouring of  $G$ . By Lemma 6 all vertices of  $C_{2p}$  but  $v_j$  have the same colour. Let us suppose that  $f(v_i) = a, i \neq j$ . We are going to prove that all vertices of  $B$  but  $v_j$  are coloured in  $f$  by  $a$ . Suppose that there exists a vertex  $z \neq v_j$  such that  $f(z) \neq a$ . Then since  $B$  is 2-connected and  $z \notin V(C_{2p})$  there exists a cycle  $C' : v_jv_{j+1} \dots z \dots v_j$ . By applying Lemma 5 to the cycle  $C'$  we can obtain  $(\mathcal{P}, L^{v_j})$ -colouring of  $G$  such that the vertices  $v_{j+1}, \dots, v_{2p-1}, v_0, \dots, v_j$  of  $C_{2p}$  are not coloured the same, a contradiction. Thus, by Lemma 2, we have  $\Delta(B) \leq \delta(\mathcal{P})$ . If  $B$  is  $\delta(\mathcal{P})$ -regular, then

$B \in \mathcal{C}(\mathcal{P})$ ; otherwise  $v_j$  could be recoloured by  $a$ , which contradicts that  $G$  is  $(\mathcal{P}, L)$ -critical. If there is a vertex  $u \in V(B)$  with  $d_B(u) < \delta(\mathcal{P})$ , then according to Lemma 4 by  $(\mathcal{P}, L^x)$ -recolouring of a walk  $Q : v_j \dots u$ , we obtain  $(\mathcal{P}, L^u)$ -colouring of  $G$  with all vertices of  $B$  but  $u$  coloured by  $a$ . Since  $d_B(u) < \delta(\mathcal{P})$ , we have  $B \in \mathcal{P}$ .

*Case 3.* Each vertex of any even cycle  $C$  in  $B$  is incident with at least one diagonal of  $C$ .

In this case, by Lemma 7,  $B$  is a complete subgraph of  $G$ . ■

4.  $(k, \mathcal{P})$ -CHOICE CRITICAL GRAPHS. GENERALIZATIONS OF GALLAI'S AND BROOKS' THEOREMS

For a nontrivial property  $\mathcal{P} \in \mathbf{M}$ , a graph  $G$  is said to be (*vertex*)  $(k, \mathcal{P})$ -*choice critical* if  $\text{ch}_{\mathcal{P}}(G) = k \geq 2$  but  $\text{ch}_{\mathcal{P}}(G - v) < k$  for all vertices  $v$  of  $G$ . According to the previous definitions, it follows immediately that if  $G$  is  $(k + 1, \mathcal{P})$ -choice critical, then  $G$  is  $(\mathcal{P}, L)$ -critical with some list  $|L(v)| = k$  for all  $v \in V(G)$ .

Hence, by Theorem 2 we have the following generalization of Gallai's Theorem.

**Theorem 3.** *Let  $\mathcal{P} \in \mathbf{M}$  and  $G$  be a  $(k, \mathcal{P})$ -choice critical graph. Then any block of  $\langle S(G) \rangle_G$  is one of the following types:*

- (i)  $B$  is a complete graph,
- (ii)  $B$  is a  $\delta(\mathcal{P})$ -regular graph belonging to  $\mathcal{C}(\mathcal{P})$ ,
- (iii)  $B \in \mathcal{P}$  and  $\Delta(B) \leq \delta(\mathcal{P})$ ,
- (iv)  $B$  is an odd cycle.

Note that in Theorem 3,  $S(G) = \{v : v \in V(G), d_G(v) = \delta(\mathcal{P})(k - 1)\}$ .

**Lemma 8.** *Let  $\mathcal{P} \in \mathbf{M}^a$  and let  $G$  be a connected graph with  $|L(v)| \geq \frac{d_G(v)}{\delta(\mathcal{P})}$  for all  $v \in V(G)$ , and let there exist a vertex  $v_0 \in V(G)$  such that  $|L(v_0)| \geq \frac{d_G(v_0)+1}{\delta(\mathcal{P})}$ . Then  $G$  is  $(\mathcal{P}, L)$ -colourable.*

**Proof.** The proof is by induction on the order of  $G$ . Let  $|V(G)| = 1$ . Then  $G = K_1$ ,  $|L(v)| \geq 1$  for  $v \in V(G)$  and Lemma is true.

Assume the Lemma holds for all graphs of order  $\leq n$ . Let  $|V(G)| = n + 1$  and let  $L$  be a list satisfying the assumptions of the Lemma. Consider the graph  $G - v_0$  and its list  $\tilde{L}(v) = L(v)$  for all  $v \in V(G) - \{v_0\}$ . Since  $G$  is connected, then in each component  $G'$  of  $G - v_0$  there is a vertex  $u$  such that  $uv_0 \in E(G)$ . Thus,

$$|\tilde{L}(u)| \geq \frac{d_G(u)}{\delta(\mathcal{P})} = \frac{d_{G'}(u)+1}{\delta(\mathcal{P})}.$$

For the remaining vertices of each component

$$|\tilde{L}(v)| \geq \frac{d_G(v)}{\delta(\mathcal{P})} \geq \frac{d_{G'}(v)}{\delta(\mathcal{P})} \text{ holds.}$$

Then, by the induction hypothesis, each component  $G'$  of  $G - v_0$  is  $(\mathcal{P}, \tilde{L})$ -colourable. Since  $\mathcal{P} \in \mathbf{M}^a$ , we have  $(\mathcal{P}, \tilde{L})$ -colourability  $f'$  of  $G - v_0$ . We will prove that  $f'$  can be extended to  $(\mathcal{P}, L)$ -colourability  $f$  of  $G$ . Suppose, contrary to our claim. It implies that the vertex  $v_0$  has at least  $\delta(\mathcal{P})$  neighbours in each colour class  $V_i$  for  $i \in L(v_0)$ . Thus,

$$d_G(v_0) \geq \delta(\mathcal{P}) |L(v_0)| \geq \delta(\mathcal{P}) \frac{d_G(v_0)+1}{\delta(\mathcal{P})} = d_G(v_0) + 1, \text{ a contradiction.} \quad \blacksquare$$

**Theorem 4.** *Let  $\mathcal{P} \in \mathbf{M}^a$  and  $G$  be a connected graph other than*

- (i) *a complete graph of order  $n\delta(\mathcal{P}) + 1, n \geq 0$ ,*
- (ii) *a  $\delta(\mathcal{P})$ -regular graph belonging to  $\mathcal{C}(\mathcal{P})$ ,*
- (iii) *an odd cycle if  $\mathcal{P} = \mathcal{O}$ .*

*Then*

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil.$$

**Proof.** By Lemma 8, the Theorem is true for all not regular graphs and for any graph  $G$  with  $\Delta(G) \not\equiv 0 \pmod{\delta(\mathcal{P})}$ . So, let  $G$  be a regular graph with  $\Delta(G) = (k-1)\delta(\mathcal{P})$ . Since  $G \neq K_1$  we have  $k \geq 2$ . Now, suppose the assertion of the Theorem is false for a list  $L$ , with  $|L(v)| = k-1$  for all  $v \in V(G)$ . By Lemma 8 it follows that for each vertex  $v \in V(G)$  and each component  $G'$  of  $G - v$  there exists  $(\mathcal{P}, L)$ -colouring. Since,  $\mathcal{P} \in \mathbf{M}^a$  we have  $\text{ch}_{\mathcal{P}}(G - v) \leq k-1$  for all  $v \in V(G)$ . Thus,  $\text{ch}_{\mathcal{P}}(G) = k$ , i.e.,  $G$  is  $(k, \mathcal{P})$ -choice critical. Since  $S(G) = V(G)$ , then by Theorem 3, for  $k=2$  it follows that  $G$  is a  $\delta(\mathcal{P})$ -regular graph with  $\text{ch}_{\mathcal{P}}(G) > 1$ . Hence,  $G \in \mathcal{C}(\mathcal{P})$ , a contradiction.

If  $k \geq 3$ , also by Theorem 3, we have that  $G$  is a complete graph of order  $(k-1)\delta(\mathcal{P}) + 1$  or  $G$  is an odd cycle (only in the case when  $\mathcal{P} = \mathcal{O}$ ), a contradiction. ■

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