DISTINGUISHING GRAPHS BY THE NUMBER OF HOMOMORPHISMS

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Abstract

A homomorphism from one graph to another is a map that sends vertices to vertices and edges to edges. We denote the number of homomorphisms from G to H by $|G \to H|$. If $\mathcal{F}$ is a collection of graphs, we say that $\mathcal{F}$ *distinguishes* graphs G and H if there is some member X of $\mathcal{F}$ such that $|G \to X| \neq |H \to X|$. $\mathcal{F}$ is a *distinguishing family* if it distinguishes all pairs of graphs.

We show that various collections of graphs are a distinguishing family.

Keywords: graph homomorphism, chromatic number

1991 Mathematics Subject Classification: 05C15

Suppose $\mathcal{F}$ is the collection of all complete graphs. The number of maps from G to a complete graph K_n is the number of colorings with n colors. If two graphs are not distinguished by $\mathcal{F}$, then they have the same chromatic polynomial. For instance, trees with the same number of vertices are not distinguished by $\mathcal{F}$.

A family of graphs is a *distinguishing family* if it distinguishes all pairs of graphs. We first show that there is a distinguishing family.

Theorem 1. *The collection of all graphs is a distinguishing family.*

Proof. If X is any graph with vertices $v_1, v_2, ..., v_n$ and $a_1, a_2, ..., a_n$ is any sequence of positive integers, let $X(a_1, a_2, ..., a_n)$ be the graph obtained from X by replacing v_i by a_i distinct points, and joining these a_i points to each of the a_j points corresponding to v_j iff v_i is adjacent to v_j. $X(a_1, a_2, ..., a_n)$ is a generalized composition graph. Let p be the map that sends to v_i the a_i points of $X(a_1, a_2, ..., a_n)$ corresponding to v_i.

Suppose that graphs G and H satisfy $|G \to X| = |H \to X|$ for all X. Suppose there are s maps $g_1, g_2, ..., g_s$, from G to X, and that there are $n_{i,j}$ points of G that map to v_i by the map g_j. The number of maps $\widetilde{g}_j : G \to X(a_1, a_2, ..., a_n)$ such that $p\widetilde{g}_j = g_j$ is exactly $\prod_i a_i^{n_{i,j}}$, and so

$$|G \to X(a_1, a_2, ..., a_n)| = \sum_{j=1}^{s} \prod_{i=1}^{n} a_i^{n_{i,j}}$$

Now take X to be $G(a_1, a_2, ..., a_n)$. Let $r_{i,j}$ (respectively $m_{i,j}$) be the number of vertices of G (respectively H) that map to v_i by the j-th map from G (respectively H) to G. We have

$$\sum_j \prod_i a_i^{r_{i,j}} = \sum_j \prod_i a_i^{m_{i,j}} \tag{1}$$

Since these are polynomials in the a_i, and they agree for infinitely many values, they must be identical. The identity map from G to G determines the monomial $a_1, a_2, ..., a_n$ in the left hand side of (1), and so there must be a map f from H to G such that f maps the vertices of G onto the vertices of H. Such an f is 1–1, so H is a subgraph of G. Similarly, G is a subgraph of H, and so they are isomorphic. ■

Lovász [Lov71] proves that if $|G \to X| = |H \to X|$ for all graphs X with $|V(X)| \leq \max(|V(G)|, |V(H)|)$ then G and H are isomorphic. This result implies Theorem 1, but not the following corollaries.

Corollary 2. *For any fixed integer N, all graphs with at least N vertices from a distinguishing family.*

Corollary 3. *All graphs with an even number of vertices form a distinguishing family.*

The next result is a consequence of the fact that the chromatic numbers of G and $G(a_1, a_2, ..., a_n)$ are equal.

Corollary 4. *If G and H have chromatic numbers $\boldsymbol{g}$ and $\boldsymbol{h}$, where $\boldsymbol{g} \leq \boldsymbol{h}$, then G and H can be distinguished by a graph of chromatic number at most $\boldsymbol{h}$.*

Corollary 5. *For any fixed integer N, the set of all connected graphs with at least N vertices is a distinguishing family for the collection of all connected graphs.*

It would be interesting to find a minimal family that distinguishes all graphs.

References

[Lov71] L. Lovász, *On the cancellation law among finite relational structures*, Periodica Math. Hung. **1** (1971) 145–156.

Received 25 May 1994