PROBLEMS REMAINING NP–COMPLETE FOR SPARSE OR DENSE GRAPHS

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Abstract

For each fixed pair $\alpha, c > 0$ let INDEPENDENT SET $(m \leq cn^\alpha)$ and INDEPENDENT SET $(m \geq \binom{n}{2} - cn^\alpha)$ be the problem INDEPENDENT SET restricted to graphs on n vertices with $m \leq cn^\alpha$ or $m \geq \binom{n}{2} - cn^\alpha$ edges, respectively. Analogously, HAMILTONIAN CIRCUIT $(m \leq n + cn^\alpha)$ and HAMILTONIAN PATH $(m \leq n + cn^\alpha)$ are the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with $m \leq n + cn^\alpha$ edges. For each $\epsilon > 0$ let HAMILTONIAN CIRCUIT $(m \geq (1-\epsilon)\binom{n}{2})$ and HAMILTONIAN PATH $(m \geq (1-\epsilon)\binom{n}{2})$ be the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with $m \geq (1-\epsilon)\binom{n}{2}$ edges.

We prove that these six restricted problems remain NP–complete. Finally, we consider sufficient conditions for a graph to have a Hamiltonian circuit. These conditions are based on degree sums and neighborhood unions of independent vertices, respectively. Lowering the required bounds the problem HAMILTONIAN CIRCUIT jumps from 'easy' to 'NP–complete'.

Keywords: Computational Complexity, NP–Completeness, Hamiltonian Circuit, Hamiltonian Path, Independent Set

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1. Motivation and Notation

One of the most well–known problems in the theory of NP–completeness is the k–Satisfiability problem:

k–SATISFIABILITY

INSTANCE: A set V of Boolean variables and a formula F of r different clauses in conjunctive normal form where each clause contains k literals in disjunctive normal form.
QUESTION: Is there a satisfying truth assignment for F?

Recently it has been shown that k–SATISFIABILITY remains NP–complete when restricted to sparse as well as to dense formulas (cf. [7],[9]).

Let (k,s)–SATISFIABILITY be the k–SATISFIABILITY problem restricted to formulas F where each variable occurs at most s times. In [7] Kratochvíl, Savický and Tuza proved the following result for sparse formulas.

Theorem 1.1. *For each integer $k \geq 3$ there exists an integer $f(k)$ such that*

1. *every instance of (k,s)–SATISFIABILITY is satisfiable for $s \leq f(k)$;*
2. *(k,s)–SATISFIABILITY is NP–complete for $s \geq f(k)+1$.*

Note that (k,s)–SATISFIABILITY is solvable in polynomial time for each $s \leq k$ (cf. [7]).

Let k–SATISFIABILITY $(r > r_0)$ be the k–SATISFIABILITY problem restricted to formulas F with $r > r_0$ clauses. In [9] we proved the following result for dense formulas.

Theorem 1.2. *For each $k \geq 3$ and each $l \geq 4$ with $n \geq lk^2$ k–SATISFIABILITY $(r > \binom{n}{k}(2^k - 1 - 4/k))$ is NP–complete.*

Note that each formula F of k–SATISFIABILITY $(r > \binom{n}{k}(2^k-1))$ is unsatisfiable since there always exists a set of k variables such that all 2^k possible clauses over the corresponding literals belong to F.

In section 2 we prove that the problems INDEPENDENT SET, HAMILTONIAN CIRCUIT and HAMILTONIAN PATH remain NP–complete when restricted to sparse or dense graphs. In section 3 we state several sufficient conditions in terms of degree–sums and neighborhood unions of vertices for a graph to have a Hamiltonian circuit, which can be checked in polynomial time. We show that HAMILTONIAN CIRCUIT becomes NP–complete when the corresponding bounds required for the degree–sums and neighborhood unions are lowered.

In this paper we only consider undirected and simple graphs (i.e., graphs without loops and multiple edges). Let G be a graph. By $V(G)$ we denote the *vertex–set* of G, and by $E(G)$ the *edge–set* of G. The cardinalities of $V(G)$ and $E(G)$ will be denoted by n and m, respectively. For a vertex $v \in V(G)$ the *neighborhood* $N(v)$ of v is the set of all vertices adjacent to v. The *degree* of a vertex v is denoted by $d_G(v) = |N(v)|$, or shortly, $d(v)$. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. For $k \leq \alpha(G)$ we define

$$\sigma_k(G) = \min\{\sum_{v \in S} d(v) \mid \text{S is an independent set of } k \text{ vertices}\}$$

and

$$NC_k(G) = \min\{|\bigcup_{v \in S} N(v)| \mid \text{S is an independent set of } k \text{ vertices}\}.$$

In these definitions we follow the convention that the minimum over an empty set is $+\infty$.

For further terminology and notations not defined here we refer to [6] (concerning complexity) and to [2](concerning graph theory), respectively.

2. Three Problems in Graph Theory

We now show that the problems INDEPENDENT SET, HAMILTONIAN CIRCUIT and HAMILTONIAN PATH remain NP–complete when restricted to sparse or dense graphs. Our technique of proof will be standard as described in [6]. In each proof we choose a known NP–complete problem Π_2 and transform it to one of the considered problems Π_1. Clearly, in all cases our problem Π_1 belongs to NP under restriction.

For each fixed pair $\alpha, c > 0$ let INDEPENDENT SET ($m \leq cn^\alpha$) and INDEPENDENT SET ($m \geq \binom{n}{2} - cn^\alpha$) be the problem INDEPENDENT SET restricted to graphs with $m \leq cn^\alpha$ or $m \geq \binom{n}{2} - cn^\alpha$ edges, respectively. Analogously, for each fixed pair $\alpha, c > 0$, HAMILTONIAN CIRCUIT ($m \leq n + cn^\alpha$) and HAMILTONIAN PATH ($m \leq n + cn^\alpha$) are the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with $m \leq n + cn^\alpha$ edges. For each $\epsilon > 0$ let HAMILTONIAN CIRCUIT ($m \geq (1-\epsilon)\binom{n}{2}$) and HAMILTONIAN PATH ($m \geq (1-\epsilon)\binom{n}{2}$) be the problems HAMILTONIAN CIRCUIT and HAMILTONIAN PATH restricted to graphs with $m \geq (1-\epsilon)\binom{n}{2}$ edges.

Theorem 2.1. *INDEPENDENT SET* $(m \le cn^\alpha)$ *is NP–complete.*

Proof. We transform INDEPENDENT SET to INDEPENDENT SET $(m \le cn^\alpha)$. Let $G_1 = (V_1, E_1)$ be a graph on n_1 vertices and m_1 edges making up an arbitrary instance of INDEPENDENT SET. We now construct a graph $G_2 = (V_2, E_2)$ on n_2 vertices and $m_2 = m_1$ edges by adding $n_2 - n_1$ isolated vertices such that $n_2 \ge max\{n_1, \lceil (\frac{1}{c}\binom{n_1}{2})^{1/\alpha} \rceil\}$. Then $\alpha(G) = \alpha(G_1) + (n_2 - n_1)$ and G_2 has $m_2 = m_1 \le \binom{n_1}{2} \le cn_2^\alpha$ edges. For each fixed pair $\alpha, c > 0$ we have $m_2 = O(n_2^2)$ which is bounded above by a polynomial function of n_1, since $m_1 = O(n_1^2)$.

For each positive $K_1 \le n_1$ let $K_2 = K_1 + (n_2 - n_1)$. Then $0 < K_2 \le n_2$ and G_2 has an independent set of cardinality K_2 or more if and only if G_1 has an independent set of cardinality K_1 or more. ■

Note that INDEPENDENT SET $(m \le k)$ can be solved in time $O(n^k)$.

Theorem 2.2. *INDEPENDENT SET* $(m \ge \binom{n}{2} - cn^\alpha)$ *is NP–complete.*

Proof. We transform INDEPENDENT SET to INDEPENDENT SET $(m \ge \binom{n}{2} - cn^\alpha)$ and proceed as in the proof of theorem 2.1. This time we add a complete graph K_p on $p = n_2 - n_1$ vertices $v_1, v_2, \ldots, v_p$ and join them to all vertices of G_1. Thus G_2 has $m_2 = \binom{n_2}{2} - \binom{n_1}{2} + m_1 \ge \binom{n_2}{2} - \binom{n_1}{2} \ge \binom{n_2}{2} - cn_2^\alpha$ edges. Furthermore, $\alpha(G_1) = \alpha(G_2)$ by this construction. Now let $K \le n_1$ be positive. Then G_2 has an independent set of cardinality K or more if and only if G_1 has an independent set of cardinality K or more. ■

Note that INDEPENDENT SET $(m \ge \binom{n}{2} - k)$ can be solved in time $O(n^k)$.

Remark. Considering the problem CLIQUE in the complement of G, theorems 2.1 and 2.2 show that for each fixed pair $\alpha, c > 0$ CLIQUE remains NP–complete for graphs on n vertices and $m \le cn^\alpha$ or $m \ge \binom{n}{2} - cn^\alpha$ edges, respectively.

Theorem 2.3. *HAMILTONIAN CIRCUIT* $(m \le n + cn^\alpha)$ *is NP–complete.*

Proof. We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT $(m \le n + cn^\alpha)$. Let $G_1 = (V_1, E_1)$ be a graph on n_1 vertices and m_1 edges making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph $G_2 = (V_2, E_2)$ on n_2 vertices and m_2 edges by adding $p \geq max\{2, \lceil n_1^{2/\alpha} c^{-1/\alpha} \rceil - n_1\}$ vertices $v_1, v_2, \ldots, v_p$ inducing the path $v_1 v_2 \ldots v_p$ and joining v_1 and v_p to all vertices of G_1. Thus $n_2 = n_1 + p$ and $m_2 = m_1 + p - 1 + 2n_1 = m_1 + n_2 + n_1 - 1 \leq n_2 + (n_1 - 1) + \binom{n_1}{2} < n_2 + n_1^2$. Now $n_2 \geq n_1^{2/\alpha} c^{-1/\alpha} \Leftrightarrow n_1^2 \leq c n_2^{\alpha}$ and thus $m_2 \leq n_2 + c n_2^{\alpha}$. Furthermore, for each fixed pair $\alpha, c > 0$ we have $m_2 = O(n_2^2)$ which is bounded above by a polynomial function of n_1.

If G_2 has a Hamiltonian circuit C, then C contains the path $u_1 v_1 v_2 \ldots v_p u_2$ for two vertices $u_1, u_2 \in V_1$. Hence C contains also a path from u_1 to u_2 in G_1 containing all vertices of G_1. Thus G_1 has a Hamiltonian path. Conversely, if G_1 has a Hamiltonian path, say $u_1 u_2 \ldots u_{n_1}$, then $u_1 u_2 \ldots u_{n_1} v_1 v_2 \ldots v_p u_1$ is a Hamiltonian circuit in G_2. Thus G_1 has a Hamiltonian path if and only if G_2 has a Hamiltonian circuit. ∎

Note that HAMILTONIAN CIRCUIT $(m \leq n + k)$ can be solved in time $O(n^k)$.

Theorem 2.4. *HAMILTONIAN CIRCUIT* $(m \geq (1 - \epsilon)\binom{n}{2})$ *is NP–complete.*

Proof. We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT $(m \geq (1 - \epsilon)\binom{n}{2})$. Let $G_1 = (V_1, E_1)$ be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph $G_2 = (V_2, E_2)$ by adding a complete graph K_p on $p \geq 3$ vertices $v_1, v_2, \ldots, v_p$ and joining v_1 and v_2 to all vertices of G_1, where $p = n_2 - n_1$ and $n_2 = \lceil \frac{2n_1}{\epsilon} \rceil$. Thus $n_2 = n_1 + p$ and G_2 has

$$m_2 = \binom{n_2 - n_1}{2} + 2n_1 + m_1 = \binom{n_2}{2} - n_1(n_2 - \frac{5}{2}) + \frac{n_1^2}{2} + m_1$$

$$> \binom{n_2}{2} - n_1(n_2 - 1) \geq \binom{n_2}{2} - \frac{\epsilon}{2} n_2(n_2 - 1) = (1 - \epsilon)\binom{n_2}{2}$$

edges. For each fixed $\epsilon > 0$ the graph G_2 has size $O(n_2^2)$ which is bounded above by a polynomial function of n_1.

If G_2 has a Hamiltonian circuit then G_1 has a Hamiltonian path since $G[G_2 - \{v_1, v_2\}]$ consists of two components $G[\{v_3, v_4, \ldots, v_p\}]$ and G_1. If G_1 has a Hamiltonian path, say $u_1 u_2 \ldots u_{n_1}$, then $u_1 u_2 \ldots u_{n_1} v_1 v_3 v_4 \ldots v_p v_2 u_1$ is a Hamiltonian circuit in G_2. Thus G_1 has a Hamiltonian path if and only if G_2 has a Hamiltonian circuit. ∎

Theorem 2.5 *HAMILTONIAN PATH* $(m \leq n + cn^{\alpha})$ *is NP–complete.*

Proof. We transform HAMILTONIAN PATH to HAMILTONIAN PATH $(m \leq n + cn^{\alpha})$. Let $G_1 = (V_1, E_1)$ be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph $G_2 = (V_2, E_2)$ by adding $p \geq max\{1, \lceil n_1^{2/\alpha} c^{-1/\alpha} \rceil - n_1\}$ vertices $v_1, v_2, \ldots, v_p$ inducing the path $v_1 v_2 \ldots v_p$ and joining v_1 to all vertices of G_1. Thus $n_2 = n_1 + p$ and G_2 has $m_2 = m_1 + p - 1 + n_1 = m_1 + n_2 - 1 \leq n_2 - 1 + \binom{n_1}{2} < n_2 + n_1^2$ edges. As in the proof of theorem 2.3 we obtain $m_2 \leq n_2 + cn_2^{\alpha}$ and $m_2 = O(n_1^{4/\alpha})$ for each fixed pair $\alpha, c > 0$.

If G_2 has a Hamiltonian path P then P contains the path $v_p \ldots v_1 u$ for a vertex $u \in V_1$. Hence P contains also a path in G_1 starting at u and containing all vertices of V_1. Thus G_1 has a Hamiltonian path. Conversely, if G_1 has a Hamiltonian path, say $u_1 u_2 \ldots u_{n_1}$, then $u_1 u_2 \ldots u_{n_1} v_1 v_2 \ldots v_p$ is a Hamiltonian path in G_2. Thus G_1 has a Hamiltonian path if and only if G_2 has a Hamiltonian path. ■

Note that HAMILTONIAN PATH $(m \leq n + k)$ can be solved in time $O(n^k)$.

Theorem 2.6. *HAMILTONIAN PATH* $(m \geq (1 - \epsilon)\binom{n}{2})$ *is NP–complete.*

Proof. We transform HAMILTONIAN PATH to HAMILTONIAN PATH $(m \geq (1 - \epsilon)\binom{n}{2})$ and proceed as in the proof of theorem 2.4. This time, only one vertex (v_1) of the complete graph K_p is joined to all vertices of G_1. Thus $n_2 = n_1 + p$ and G_2 has $\binom{n_2 - n_1}{2} + n_1 + m_1$ edges.

If G_2 has a Hamiltonian path then G_1 has a Hamiltonian path, too, since $G[G_2 - v_1]$ consists of two components $G[\{v_2, v_3, \ldots, v_p\}]$ and G_1. If G_1 has a Hamiltonian path, say $u_1 u_2 \ldots u_{n_1}$, then $u_1 u_2 \ldots u_{n_1} v_1 v_2 \ldots v_p$ is a Hamiltonian path in G_2. Thus G_1 has a Hamiltonian path if and only if G_2 has a Hamiltonian path.

Everything else mentioned in the proof of theorem 2.4. remains valid. Thus the proof is complete. ■

3. Sufficient Conditions for Hamiltonian Circuits

There are quite a lot of sufficient conditions for a graph to have a Hamiltonian circuit. The following result is due to Bondy [1] and generalizes the well-known theorems of Dirac ($k = 0$, [4]) and Ore ($k = 1$, [8]).

Theorem 3.1. *Let G be a k-connected graph of order $n \geq 3$. If $\sigma_{k+1} \geq \frac{1}{2}((k+1)(n-1)+1)$ then G has a Hamiltonian circuit.*

A major improvement for the case $k = 2$ has been established in [5] by Flandrin, Jung and Li.

Theorem 3.2. *Let G be a 2-connected graph of order n such that*

$$d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$$

for any independent set $\{u, v, w\}$ of vertices. Then G has a Hamiltonian circuit.

Note that the conditions required in theorem 3.1. and 3.2. can be checked in polynomial time of $O(n^{k+1})$ and $O(n^3)$, respectively.

The best known sufficient condition using neighborhood unions of two independent vertices is due to Broersma, van den Heuvel and Veldman [3].

Theorem 3.3. *Let G be a graph of order n and $NC_2 \geq \frac{1}{2}n$. Then either G has a Hamiltonian circuit, or G is the Petersen graph, or G belongs to one of three families of exceptional graphs.*

Remark. It can be decided in polynomial time whether a given graph G is isomorphic to the Petersen graph (a graph on ten vertices) or belongs to one of the three families of exceptional graphs.

As in the previous section we now restrict HAMILTONIAN CIRCUIT to graphs with $\sigma_{k+1} \geq (\frac{1}{2}-\epsilon)(k+1)n$, $d(u)+d(v)+d(w) \geq (1-\epsilon)n+|N(u) \cap N(v) \cap N(w)|$ for any three independent vertices or $NC_2 \geq (\frac{1}{2} - \epsilon)n$, respectively. The corresponding problems will be HAMILTONIAN CIRCUIT $(\sigma_{k+1} \geq (\frac{1}{2}-\epsilon)(k+1)n)$, HAMILTONIAN CIRCUIT $(d(u)+d(v)+d(w) \geq n+|N(u) \cap N(v) \cap N(w)|)$ or HAMILTONIAN CIRCUIT $(NC_2 \geq (\frac{1}{2}-\epsilon)n)$, respectively.

Theorem 3.4. *HAMILTONIAN CIRCUIT $(\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n)$ is NP–complete.*

Proof. We transform HAMILTONIAN PATH to HAMILTONIAN CIRCUIT $(\sigma_{k+1} \geq (\frac{1}{2} - \epsilon)(k+1)n)$. Let $G_1 = (V_1, E_1)$ be a graph making up an arbitrary instance of HAMILTONIAN PATH. We now construct a graph $G_2 = (V_2, E_2)$ by adding a complete graph K_p on p vertices $v_1, v_2, \ldots, v_p$ and $p-1$ vertices $v_{p+1}, v_{p+2}, \ldots, v_{2p-1}$, where $p = \lceil (\frac{1}{2} - \epsilon)n_2 \rceil$ and $n_2 = \lceil \frac{n_1}{2\epsilon} \rceil$. Then $n_2 = n_1 + 2p - 1$. Let $E_2 = E_1 \cup \{v_iv_j | 1 \leq i < j \leq p\}$ $\cup \{v_iv_j | 1 \leq i \leq p, p+1 \leq j \leq 2p-1\} \cup \{v_iv | v \in V_1, 1 \leq i \leq p\}$.

Then $\sigma_{k+1} \geq (k+1)p \geq (\frac{1}{2}-\epsilon)(k+1)n_2$ for $0 \leq k+1 \leq p$, since $\sigma_1 \geq (\frac{1}{2}-\epsilon)n_2$ and $\alpha(G_2) = \alpha(G_1) + (p-1) \geq 1 + (p-1) = p$ by the construction. Thus there always exists an independent set I of size $k+1$, e.g., $I = \{v_{p+1}, v_{p+2}, \ldots, v_{p+k}\} \cup \{v\}$ for a vertex $v \in V_1$. For each fixed $\epsilon > 0$ we have $m_2 = O(n_2^2)$ which is bounded above by a polynomial function of n_1.

If G_2 has a Hamiltonian circuit then G_1 has a Hamiltonian path since $G[G_2 - \{v_1, v_2, \ldots, v_p\}]$ consists of p components $\{v_{p+1}\}, \{v_{p+2}\}, \ldots, \{v_{2p-1}\}$ and G_1. If G_1 has a Hamiltonian path, say $u_1u_2\ldots u_{n_1}$, then $u_1u_2\ldots u_{n_1}v_1 v_{p+1}v_2v_{p+2}\ldots v_{2p-1}v_pu_1$ is a Hamiltonian circuit in G_2. Thus G_1 has a Hamiltonian path if and only if G_2 has a Hamiltonian circuit. ■

Remark. For $n_1 = 2$ let G_1 consist of two isolated vertices $\{u, v\}$. Then the graph G_2 shows that the required conditions in theorems 3.1, 3.2 and 3.3 are sharp, since $G[G_2 - \{v_1, v_2, \ldots, v_p\}]$ consists of $p+1$ components $\{v_{p+1}\}, \{v_{p+2}\}, \ldots, \{v_{2p-1}\}, \{u\}$ and $\{v\}$. Thus G_2 has no Hamiltonian circuit.

Theorem 3.5. *HAMILTONIAN CIRCUIT* $(d(u)+d(v)+d(w) \geq (1-\epsilon)n + |N(u) \cap N(v) \cap N(w)|)$ *is NP–complete.*

Proof. We follow the proof of theorem 3.4 and let $\epsilon_1 = \frac{\epsilon}{2}, p = \lceil(\frac{1}{2}-\epsilon_1)n_2\rceil$. Set $S = \{v_1, v_2, \ldots, v_p\}$, $T = V_2 - S$. If $I = \{u, v, w\}$ is an independent set of three vertices, then $I \subseteq T$ by the construction of G_2. For a set $X \subseteq V(G)$ and a vertex $v \in V(G)$ let $N_X(v) := N(v) \cap X$. By $d_X(v)$ we denote the degree of v in X. Then

$$\begin{aligned} d(u)+d(v)+d(w) &= d_S(u)+d_S(v)+d_S(w)+d_T(u)+d_T(v)+d_T(w) \\ &\geq 3p + |N_T(u) \cap N_T(v) \cap N_T(w)| \\ &\geq 2(\frac{1}{2}-\epsilon_1)n_2 + |N_S(u) \cap N_S(v) \cap N_S(w)| \\ &\quad + |N_T(u) \cap N_T(v) \cap N_T(w)| \\ &= (1-\epsilon)n_2 + |N(u) \cap N(v) \cap N(w)|. \end{aligned}$$

■

Theorem 3.6. *HAMILTONIAN CIRCUIT* $(NC_2 \geq (\frac{1}{2}-\epsilon)n)$ *is NP–complete.*

Proof. We follow the proof of theorem 3.4. With $\alpha(G_2) \geq p+1 \geq 2$ and $\sigma_1(G_2) \geq (\frac{1}{2}-\epsilon)n_2$ we have $NC_2(G_2) \geq (\frac{1}{2}-\epsilon)n_2$. ■

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REFERENCES

[1] J. A. Bondy, *Longest paths and cycles in graphs of high degree*, Research Report CORR 80–16, Dept. of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada (1980).

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).

[3] H. J. Broersma, J. van den Heuvel and H. J. Veldman, *A generalization of Ore's Theorem involving neighborhood unions*, Discrete Math. **122** (1993) 37–49.

[4] G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952) 69–81.

[5] E. Flandrin, H. A. Jung and H. Li, *Hamiltonism, degree sum and neighborhood intersections*, Discrete Math. **90** (1991) 41–52.

[6] M. R. Garey and D. S. Johnson, Computers and Intractability, A Guide to the Theory of NP–Completeness (Freeman, San Francisco, 1979).

[7] J. Kratochvíl, P. Savický and Z. Tuza, *One more occurrence of variables makes satisfiability jump from trivial to NP–complete*, SIAM J. Comput. **22** (1993) 203–210.

[8] O. Ore, *Note on Hamiltonian circuits*, Amer. Math. Monthly 67 (1960) 55.

[9] I. Schiermeyer, *The k–Satisfiability problem remains NP–complete for dense families*, Discrete Math. **125** (1994) 343–346.

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