AN INEQUALITY CONCERNING EDGES OF MINOR WEIGHT IN CONVEX 3-POLYTOPES

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Dedicated to Professor E. Jucovič on the occasion of his 70th birthday.

Abstract

Let $e_{ij}$ be the number of edges in a convex 3-polytope joining the vertices of degree $i$ with the vertices of degree $j$. We prove that for every convex 3-polytope there is $20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{3}{2}e_{3,7} + 5e_{3,8} + 2\frac{3}{2}e_{3,9} + 2e_{3,10} + 16\frac{3}{2}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{3}{2}e_{4,7} + 5\frac{3}{2}e_{5,5} + 2e_{5,6} \geq 120$; moreover, each coefficient is the best possible. This result brings a final answer to the conjecture raised by B. Grünbaum in 1973.

Keywords: planar graph, convex 3-polytope, normal map.

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1. Introduction and Statement of Results

This note deals with connected planar maps. We use standard terminology and notation of graph theory, see e.g. Ore [16]. We recall, however, more specialized notions. A plane map is called normal if it contains neither vertices nor faces incident with less than 3 edges. Notice, however, that both loops and multiple edges can appear in a normal plane map. By the
Steinitz’s theorem (see e.g. Grünbaum [11], Jucovič [13]) convex 3-polytopes are distinguished among all planar maps by the property that their graphs are 3-connected. The degree of a face \( \omega \) is the number of edges incident to \( \omega \) where each cut-edge is counted twice. Similarly, each loop contributes 2 to the degree of the incident vertex. Vertices and faces of degree \( i \) are called \( i \)-vertices and \( i \)-faces, respectively. Let \( e_{i,j}(M) = e_{i,j} \) be the number of edges in a planar map \( M \) which join \( i \)-vertices and \( j \)-vertices. Recall that a convex 3-polytope is called simplicial if all its faces are 3-gons.

An excellent theorem of Kotzig [14] (see also [1,3,6,7,13,15]) states that every convex 3-polytope contains an edge of the weight (i.e., the sum of degrees of its endvertices) at most 13; in other words \( \sum_{i+j \leq 13} e_{i,j} > 0 \). This Kotzig’s result was further developed in various directions, see e.g. Borodin [1,2,3], Grünbaum [6,7,8], Grünbaum and Shephard [9], Ivančo [10], Ivančo and Jendről [11], Jucovič [12,13], Zaks [17].

Grünbaum [8] has brought an idea that a relation of the type \( \sum_{i+j \leq 13} \alpha_{i,j} e_{i,j} \geq 1 \) should hold for each convex 3-polytope (\( \alpha_{i,j} \) denotes the coefficient at \( e_{i,j} \)) and has conjectured that the following holds for every simplicial convex 3-polytope

\[
20e_{3,3} + 15e_{3,4} + 12e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 3\frac{1}{3}e_{3,9} + 2e_{3,10} \\
+ 12e_{4,4} + 7e_{4,5} + 5e_{4,6} + 4e_{4,7} + 2\frac{2}{3}e_{4,8} + 2\frac{2}{3}e_{4,9} \\
+ 4e_{5,5} + 2e_{5,6} + 1\frac{1}{3}e_{5,7} \\
+ 12e_{6,6} \geq 120.
\]

Jucovič [12] proved that for each simplicial convex 3-polytope there is

\[
20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\
+ 20e_{4,4} + 11e_{4,5} + 5e_{4,6} + 6e_{4,7} + 5e_{4,8} + 3e_{4,9} \\
+ 8e_{5,5} + 2e_{5,6} + 2e_{5,7} + 2e_{5,8} \geq 120.
\]

Later on Jucovič, in [13], proved that this inequality holds for all convex 3-polytopes.

For a wider class of planar maps which also includes convex 3-polytopes Borodin [3] has obtained.

**Theorem 1.** For each normal planar map there holds

\[
40e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\
+ 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{1}{3}e_{4,7} \\
+ 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120,
\]

moreover, each coefficient of this inequality is the best possible.
In the same paper Borodin [3] proves that for simplicial convex 3-polytopes (1) is the best possible if we put \( \alpha_{3,3} = 20 \) instead of \( \alpha_{3,3} = 40 \). For other results of this type see Borodin [1,2,3], Borodin and Sanders [5], Jucovič [13].

The main purpose of the present note is to give a final answer to the above mentioned conjecture by Grünbaum [8]. We prove the following

**Theorem 2.** For each convex 3-polytopes there holds

\[
20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} \\
+ 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} \\
+ 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120;
\]

moreover, each coefficient of this inequality is the best possible.

2. Proof of Theorem 2

We prove our Theorem 2 in a dual form. It is well known that the dual of a 3-connected planar map is also 3-connected, see e.g. Ore [16, Chapter 3] and, due to Steinitz’s theorem, it is also true for convex 3-polytopes. It is easy to check that the dual of a normal map is again normal.

For the purposes of this proof an edge \( h \) is called an \((i,j)\)-edge when it is incident with an \( i \)-gon and a \( j \)-gon. Let \( g_{i,j}(M) = g_{ij} \) denote the number of \((i,j)\)-edges in a map \( M \). If \( M^d \) is the dual to a normal map \( M \), then clearly \( e_{ij}(M^d) = g_{ij}(M) \). Let \( V(M), E(M) \) and \( F(M) \) denote the set of vertices, edges and faces of the map \( M \), respectively.

The proof is by contradiction. Replace \( e_{ij} \) with \( g_{ij} \) in the left part of (2) and denote it by \( \sum \). We want to prove that for every 3-connected planar \( M \) there is \( \sum(M) \geq 120 \). Suppose \( M \) be a counterexample having a minimum number of faces.

To obtain a contradiction we are going to look for a suitable configuration in \( M \) which will be changed locally to obtain a new 3-connected plane map \( M^* \) with \( \sum(M^*) \leq \sum(M) < 120 \) and with a fewer number of faces than in \( M \). During this transformation of the map \( M \) into \( M^* \) some edges and vertices of \( M \) are deleted, some edges change their types (an edge is of the type \((i,j)\) if it is an \((i,j)\)-edge) and some new edges and vertices can appear in \( M^* \).

Associate with an \((i,j)\)-edge \( h \) of the map \( M \) the charge \( \alpha(h, M) = \alpha_{ij} \), where \( \alpha_{ij} \) is as in (2) or \( \alpha_{i,j} = 0 \) for \( i = 3, j \geq 11 \) or \( i = 4, j \geq 8 \) or
\(i = 5, j \geq 7\) or \(i \geq 6, j \geq 6\). Hence \(\sum(M) = \sum_{h \in E(M)} \alpha(h, M)\). Let 
\[\Delta(h) = \alpha(h, M) - \alpha(h, M^*)\]
Since every 3-connected plane map is also normal Theorem 1 yields \(g_{3,3}(M) \neq 0\), i.e., \(M\) contains a \((3, 3)\)-edge \(h_0 = uv\). Denote by \(s\) and \(t\) the vertices incident to triangles incident with \(h_0\) and different from \(u\) and \(v\), see Figure 1. Let \(h_1 = us, h_2 = sv, h_3 = vt\) and \(h_4 = ut\) be edges of \(M\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

To finish our proof several cases have to be considered

\textit{Case 1.} \(\deg u \geq 4\) and \(\deg v \geq 4\).

1.1. Let \(\deg s = 3\) or \(\deg t = 3\). The required map \(M^*\) is obtained by deleting the edge \(h_0\) from \(M\), i.e., \(M^* = M - h_0\). Because \(M\) is 3-connected and at least one of the vertices \(s\) and \(t\) is a 3-vertex also \(M^*\) is 3-connected. We can easily see that \(|F(M^*)| = |F(M)| - 1\) and \(\Delta(\sum) = \sum(M) - \sum(M^*) = \alpha(h_0, M) + \sum_{i=1}^{4} (\alpha(h_i, M) - \alpha(h_i, M^*)) = \alpha_{3,3} + \sum_{i=1}^{4} \Delta(h_i) \geq 20 + 4 \cdot (-5) = 0\). The last inequality is due to the fact that if a \((3,k)\)-edge \(h\) is transformed into a \((4,k)\)-edge, its charge always decreases or is the same except of the case \(k = 3\). We also refer to the fact that \(\Delta(h_i) \geq -5\) for any edge \(h_i \in E(M)\).

1.2. \(\deg s \geq 4\) and \(\deg t \geq 4\). In this case we transform \(M\) into \(M^*\) as shown in Figure 2. We delete the edge \(h_0\) from \(M\) and split the vertex \(t\) of \(M\) into two new vertices \(t_1\) and \(t_2\) such that we obtain, in \(M^*\), \(\deg t_1 = 3\) and \(\deg t_2 = \deg t - 1\). (The reason for this transformation of \(M\) into \(M^*\) is to preserve 3-connectivity also in \(M^*\).) Let \(h', h_1, h_2, h_3\) and \(h_4\) be edges and \(\omega_1\) and \(\omega_2\) be faces of \(M^*\) as depicted in Figure 2. Without loss of generality we can assume that \(4 \leq \deg \omega_1 \leq \deg \omega_2\). Put \(\Delta' = \sum_{x \in E(M) - \{h_0, h_3, h_4\}} \Delta(x)\). Then we have \(|F(M^*)| = |F(M)| - 1\) and \(\Delta(\sum) = \alpha(h_0, M) + \Delta(h_3) + \alpha(h_1, M^*) + \sum_{i=1}^{3} \Delta(h_i) \geq 20 + 4 \cdot (-5) = 0\).

\[\frac{\Delta(h)\cdot4}{4}\]
\[ \Delta(h_4) + \Delta^* - \alpha(h', M^*) \geq 0. \]

To check it use \( \alpha(h_0, M) = \alpha_{3,3} = 20 \) and for the values \( \Delta(h_3), \Delta(h_4), \alpha(h', M^*) \) and \( \Delta^* \) see Table 1 below. To count \( \Delta^* \) we also refer to the fact that \( g_{3,3}(M) \leq 5 \) (because \( M \) is a counterexample) and consider the ”worst” case.

### Case 2. \( \deg u = 3 \) and \( \deg v \geq 4 \).

Let \( w \) be a face incident to the edges \( h_1 \) and \( h_4 \), see Figure 1.

2.1. If \( \deg w = 3 \) then \( M^* \) is obtained by removing the vertex \( u \) from \( M \), i.e. \( M^* = M - \{u\} \). We have \( \Delta(\sum) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) = 60 > 0 \) and \( |F(M^*)| = |F(M)| - 2 \).

2.2. Let \( \deg w = k \geq 4 \). If we delete the vertex \( u \) from \( M \) and then insert a new edge \( h^* = st \) we obtain a required map \( M^*, M^* = M - \{u\} + \{h^*\} \). In this case \( |F(M^*)| = |F(M)| - 1 \) and we can check that \( \Delta(\sum) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) - \alpha(h^*, M^*) + \Delta \geq 0 \). To see it, take \( \alpha(h_0, M) = \alpha_{3,3} = 20 \) and the values \( \alpha(h_1, M), \alpha(h_4, M), \alpha(h^*, M^*) \) and \( \Delta \) from the Table 2 below; here \( \Delta = \sum \Delta(g) \), where the sum is taken over all edges \( g \) incident to the face \( \omega, g \neq h_1, h_4 \). Note that during this transformation the edge \( g \) changes its type \( (n, k) \) into the type \( (n, k - 1) \) and in the counting we consider the worst case, that is \( \Delta \geq (k - 2)(\alpha_{3,k} - \alpha_{3,k-1}) \).

### Case 3. \( \deg u = \deg v = 3 \).

This assumption leads immediately to the graph of the tetrahedron or to a 2-connected planar map. In both cases we get a contradiction.

The proof that a 3-connectivity of \( M \) implies a 3-connectivity of \( M^* \) is easy and is left to the reader.
The coefficient $\alpha_{3,3} = 20$ cannot be improved as we can see from the tetrahedron. The above mentioned examples by Borodin [3] also show the impossibility to improve the other coefficient $\alpha_{i,j}$ in Theorem 2.

**Table 1**

<table>
<thead>
<tr>
<th>$\deg \omega_1$</th>
<th>$\deg \omega_2$</th>
<th>$\Delta(h_3)$</th>
<th>$\Delta(h_4)$</th>
<th>$\alpha(h', M^*)$</th>
<th>$\Delta^* \geq$</th>
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<td>4</td>
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<td>$3\frac{1}{3}$</td>
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**Table 2**

<table>
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<th>$\deg \omega$</th>
<th>$d(h_1, M)$</th>
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<th>$d(h^<em>, M^</em>)$</th>
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<td>$4 \cdot (-6)$</td>
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<td>10</td>
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</tr>
<tr>
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<td>5</td>
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<td>$6 \cdot (-1\frac{2}{3})$</td>
</tr>
<tr>
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<td>$2\frac{1}{2}$</td>
<td>5</td>
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</tr>
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References


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