ON \( k \)-FACTOR-CRITICAL GRAPHS

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Abstract

A graph is said to be \( k \)-factor-critical if the removal of any set of \( k \) vertices results in a graph with a perfect matching. We study some properties of \( k \)-factor-critical graphs and show that many results on \( q \)-extendable graphs can be improved using this concept.

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1. Introduction

The graphs \( G = (V, E) \) we consider here are undirected, simple and finite of order \( |V| = n \). A graph is even if its order is even and odd if its order is odd. The neighborhood of a vertex \( x \) is the set \( N(x) = \{y : y \in V \text{ and } xy \in E\} \) and the degree of \( x \) is the integer \( d(x) = |N(x)| \). The integer \( \delta = \min\{d(x) : x \in V\} \) is called the minimum degree of \( G \). For any set \( A \subseteq V \), \( \langle A \rangle \) denotes the subgraph of \( G \) induced by \( A \), \( G - A \) stands for \( \langle V \setminus A \rangle \), and \( c(G - A) \) denotes the number of connected components of \( G - A \). A set \( A \) such that \( c(G - A) > 1 \) is called a cutset of \( G \). The connectivity of \( G \) is the integer \( \kappa(G) = \min\{|A| : A \text{ is a cutset of } G\} \). A claw of \( G \) is an induced subgraph isomorphic to the star \( K_{1,3} \), and the claw center is the center of the star. A matching \( F \) of \( G \) is a set of independent edges. The maximum matching number of \( G \) is the integer \( \nu(G) = \max\{|F| : F \text{ is a matching of } G\} \). A perfect matching, or a 1-factor, is a matching covering all the vertices of \( G \). For convenience, we will say that a graph of order 0 has a perfect matching. A graph without perfect matching is called prime. Clearly, every odd graph is prime. For details concerning the Matching Theory, the reader is referred to [7] by Lovász and Plummer.
The concepts of factor-critical and bicritical graphs were introduced by Gallai [5] and by Lovász [6], respectively. A graph $G$ is factor-critical if $G - x$ has a perfect matching for every vertex $x$ of $G$. A graph $G$ is bicritical if $G - x - y$ has a perfect matching for every pair of distinct vertices of $G$. Motivated by the similitude and the interest of these two concepts, which lead to powerful results, the author extended them to $k$-factor-critical graphs in [3].

**Definition.** For a given integer $k$ with $0 \leq k \leq n$, a graph $G$ of order $n$ is $k$-factor-critical, in brief $k$-fc, if $G - X$ has a perfect matching for every set $X$ of $k$ vertices of $G$.

Equivalently, $G$ is $k$-factor-critical if $\langle Y \rangle$ has a perfect matching for every set $Y$ of $n - k$ vertices of $G$.

**Remarks.** 1. If a graph of order $n$ is $k$-fc, then $n$ and $k$ have the same parity, i.e., $n + k$ is even.

2. $0$-fc, $1$-fc and $2$-fc graphs are respectively graphs with a perfect matching, factor-critical graphs and bicritical graphs. The only $(n-2)$-fc graph of order $n$ is the clique $K_n$.

3. With the convention that the graph of order $0$ has a perfect matching, every graph of order $n$ is $n$-fc. The concept of $k$-factor-criticality for $k = n$ is thus not very interesting. We still admit the possibility for $k$ to be equal to $n$ to get a definition more consistent with that of factor-critical or bicritical graphs. In particular, the factor-critical components appearing in the Edmonds-Gallai structure theorem can be effectively reduced to one vertex, which corresponds to the case $k = n = 1$. But most properties of $k$-fc graphs will be proved for $n > k$.

**Examples of $k$-fc graphs.** For a given integer $k$, $0 \leq k \leq n - 3$, a graph $G$ of order $n$ is said to be $k$-hamiltonian [2] if the removal of any set of at most $k$ vertices of $G$ results in a hamiltonian graph. Since any even cycle has a perfect matching, every $k$-hamiltonian graph of order $n$ with $n + k$ even is $k$-fc. Similarly, if we remove one vertex from an odd cycle we obtain a path with a perfect matching. Hence every $k$-hamiltonian graph of order $n$ with $n + k$ odd is $(k+1)$-fc. The most famous examples of $k$-hamiltonian graphs are powers of graphs whose hamiltonian properties have been extensively studied. The $q^{th}$ power $G^q$ of a connected graph $G$ has as its vertices those of $G$, and two distinct vertices $u$ and $v$ are adjacent in $G^q$ if their distance in $G$ is at most $q$. For instance, it is known that if $G$ is connected of order $n$,
then $G^{k+2}$ is $k$-hamiltonian for any $k$, $1 \leq k \leq n - 3$ [1]. On the other hand, a graph $G_k$ that is obtained by taking an arbitrary nonhamiltonian graph having a perfect matching and by joining each of its vertices with every vertex of a clique $K_k$ is an example of a $k$-fc graph with $n + k$ even that is not $k$-hamiltonian.

In Section 2 of this paper, we study some simple properties of $k$-fc graphs. In Section 3, we extend to $k$-fc graphs the characterization of 0-fc, 1-fc and 2-fc graphs in terms of the number of odd or prime components of induced subgraphs of $G$. In Section 4, we discuss the relationship between the concepts of $k$-factor-criticality and $q$-extendability.

2. Basic Properties of $k$-FC Graphs

We begin with some easy observations.

**Theorem 2.1.** For $k \geq 2$, any $k$-fc graph of order $n > k$ is $(k - 2)$-fc.

**Proof.** Let $G$ be a $k$-fc graph of order $n > k$ (i.e., by parity, $n \geq k + 2$) and $Y$ a set of $n - (k - 2) \geq 4$ vertices of $G$. The set $Y$ is not independent since $\langle Y \setminus \{z,t\} \rangle$ has a perfect matching for any pair $\{z,t\}$ of vertices of $Y$. Let $xy$ be an edge of $\langle Y \rangle$. Since $G$ is $k$-fc, $\langle Y \setminus \{x,y\} \rangle$ has a perfect matching $F$, and $F \cup xy$ is a perfect matching of $\langle Y \rangle$.

As a first consequence we get that every $k$-fc graph of order $n > k$ is 0-fc or 1-fc, depending on its parity. In particular

**Corollary 2.2.** For $k \geq 1$, no $k$-fc graph $G$ of order $n > k$ is bipartite.

**Proof.** By theorem 2.1, $G$ is 1-fc or 2-fc, and it is known, and easy to check, that 1-fc or 2-fc graphs of order greater than 2 are not bipartite.

**Theorem 2.3.** If a graph $G$ is $k$-fc, then $G - Y$ is $(k - p)$-fc for every integer $p$ with $0 \leq p \leq k$ and every set $Y$ of $p$ vertices of $G$.

**Proof.** If $Z$ is a set of $k - p$ vertices of $G - Y$, then $X = Y \cup Z$ is a set of $k$ vertices of $G$ and thus $G - X = (G - Y) - Z$ has a perfect matching.

The next property extends Gallai’s result saying that a connected graph $G$ is 1-fc if and only if $\nu(G - x) = \nu(G)$ for every vertex $x$ of $G$ [5], to $k$-fc graphs with $k > 1$.

**Theorem 2.4.** Let $G$ be a graph of order $n > 2$ and maximum matching number $\nu$. For $2 \leq k < n$, the following three properties are equivalent.
(i) The graph $G$ is $k$-fc.

(ii) The integer $n + k$ is even and $\nu(G - X) = \frac{n - k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$ for every set $X$ of $k$ vertices of $G$.

(iii) The graph $G$ contains at least one edge, $n + k$ is even, and $\nu(G - X)$ has the same value for any set $X$ of $k$ vertices of $G$.

**Proof.** (i) $\implies$ (ii). For any $k$-fc graph $G$ of order $n > k$, $n + k$ is even and for any subset $X$ of $V$ with $|X| = k$, $\nu(G - X) = \frac{n - k}{2}$. If $n$ and $k$ are even, then $G$ is 0-fc and $\nu = \frac{n}{2}$. If $n$ and $k$ are odd, $G$ is 1-fc and $\nu = \frac{n - 1}{2}$. In both cases, $\frac{n - k}{2} = \nu - \lfloor \frac{k}{2} \rfloor$.

(ii) $\implies$ (iii). Obvious.

(iii) $\implies$ (i). Let $G$ be a graph satisfying (iii), $xy$ an edge of $G$, $X$ a set of $k$ vertices containing $x$ and $y$, and $M$ a maximum matching of $G - X$. If at least two vertices $u$ and $v$ of $G - X$ are not covered by $M$, then for $X' = (X - \{x, y\}) \cup \{u, v\}$, which is another set of $k$ vertices of $G$, $M \cup xy$ is a matching of $G - X'$ greater than $M$, in contradiction to the hypothesis. Therefore, by parity, $M$ is a perfect matching of $G - X$ and thus $G - Y$ has also a perfect matching for any other set $Y$ of $k$ vertices of $G$.

For $k = 1$, the implication (iii) $\implies$ (i) fails, as shown for instance by the graph formed by one odd clique and two isolated vertices. This explains the necessity of the hypothesis “$G$ is connected” in Gallai’s statement.

We finish this section with two connectivity results.

**Theorem 2.5.** Every $k$-fc graph of order $n > k$ is $k$-connected and this result is sharp.

**Proof.** If $k = 0$, the result is obvious, and obviously sharp since a graph with a perfect matching is not necessarily connected. Suppose now that the $k$-fc graph $G$ with $k \geq 1$ admits a cutset $S$ of $k - 1$ vertices. Let $C_1$ and $C_2$ be two components of $G - S$, and $a_i$ a vertex of $C_i$ for $i \in \{1, 2\}$. Since both $G - (S \cup \{a_1\})$ and $G - (S \cup \{a_2\})$ have a perfect matching, every component $C_i$ must be even and odd, a contradiction. Therefore each cutset of $G$ has at least $k$ vertices. The graph $G$ obtained by joining all the vertices of a clique $K_k$ to all the vertices of two disjoint even cliques $K_{2q}$ is $k$-fc and its connectivity is exactly $k$, which proves the sharpness of the result.

**Theorem 2.6.** For $k \geq 1$, every $k$-fc graph $G$ of order $n > k$ is $(k + 1)$-edge-connected.
**Proof.** By Theorem 2.5, $G$ is at least $k$-edge-connected. Suppose $G$ is not $(k + 1)$-edge-connected and let $F = \{a_ib_i\}$ be a set of $k$ edges such that $G - F$ consists of two connected components $(X)$ and $(Y)$, with $a_i \in X$ and $b_i \in Y$ for $1 \leq i \leq k$. Let $A = \{a_i; 1 \leq i \leq k\}$ and $B = \{b_i; 1 \leq i \leq k\}$. If $|A| = k$, i.e., if the $k$ vertices $a_i$ are distinct, then $A' = (A \setminus \{a_k\}) \cup \{b_k\}$ is a cutset of $G$, and, since $|A| = k$, the component $((X \setminus A) \cup \{a_k\})$ of $G - A'$ has a perfect matching. Hence $|X| - k + 1$ is even, $X \neq A$, $A$ is another cutset of $k$ vertices of $G$, and the component $(X \setminus A)$ of $G - A$ has also a perfect matching. This leads to a contradiction. Hence $|A| < k$ and $X = A$, for otherwise $A$ is a cutset of $G$ smaller than $k$. Since $G$ is $k$-connected, every vertex has degree at least $k$. Every vertex of $A$ has at most $|A| - 1$ neighbors in $A$, and thus at least $k - |A| + 1$ neighbors in $B$. Therefore, $k = |F| \geq |A|(k - |A| + 1)$, from which $(|A| - 1)(k - |A|) \leq 0$, which implies $|A| = 1$. Similarly, $Y = B$ and $|B| = 1$, which gives a final contradiction since $n \geq 3$. Hence $G$ is $(k + 1)$-edge-connected.

**Theorem 2.7.** Every $k$-fc graph of order $n > k$ has at least $\frac{(k+1)n}{2}$ edges and this bound is sharp.

**Proof.** In a $k$-fc graph, the minimum degree is at least $k + 1$ since no vertex can be isolated after the removal of $k$ vertices (for $k \geq 1$, this is also a consequence of Theorem 2.6), and thus the number of edges is at least $\frac{(k+1)n}{2}$. It is known that every $(k - 1)$-hamiltonian graph of order $n$ has minimum degree at least $k + 1$, and thus at least $\frac{(k+1)n}{2}$ edges [2]. In [15] and [8], the authors give examples of $(k - 1)$-hamiltonian graphs having exactly this number of edges. These graphs are, if $k$ is odd, powers of cycles, and if $k$ is even, powers of cycles plus all or part of the diameters where one pair of consecutive diameters is untwisted in some particular cases. As seen in the previous section, when $n + k$ is even these graphs are $k$-fc. Hence the result on the number of edges, and thus that on the edge-connectivity, of $k$-fc graphs is sharp.

### 3. Tutte and Gallai Type Properties

Let $B$ be a set of vertices of the graph $G$. We denote by $c_o(G - B)$ the number of odd components of $G - B$ and by $c_p(G - B)$ the number of its prime components. Clearly $c_o(G - B) \leq c_p(G - B)$ since a component of odd order has no perfect matching. Tutte and Gallai respectively characterized $0$-fc and $1$-fc graphs in terms of $c_o(G - B)$ and $c_p(G - B)$ where $B$ is any
subset of \( V \). In order to compare their results and to extend them to \( k \)-fc graphs, we first unify the notation.

**Definition.** For a graph \( G = (V, E) \), the properties \( Q_k \), \( Q'_k \) and \( Q''_k \) are defined as follows:

- \( Q_k : \) \( c_o(G - B) \leq |B| - k \) for any \( B \subseteq V \) with \( |B| \geq k \),
- \( Q'_k : \) \( c_p(G - B) \leq |B| - k \) for any \( B \subseteq V \) with \( |B| \geq k \),
- \( Q''_k : \) \( c_p(G - B) \leq |B| - k + 1 \) for any \( B \subseteq V \) with \( |B| \geq k \).

Note that Property \( Q'_k \) is stronger than \( Q_k \) and than \( Q''_k \).

The first result in the domain was Tutte’s Theorem.

**Theorem 3.1** (Tutte [14]). The following two properties are equivalent.

(i) The graph \( G \) is 0-fc.

(ii) The graph \( G \) satisfies Property \( Q_0 \).

There exist many proofs of Tutte’s Theorem. In one of them, Gallai implicitly gave another characterization of 0-fc graphs, and a characterization of 1-fc graphs.

**Theorem 3.2** (Gallai [5]). 1. The following two properties are equivalent.

(i) The graph \( G \) is 0-fc.

(ii) The graph \( G \) satisfies Property \( Q'_0 \).

2. If \( G \) is not 0-fc, there exists a subset \( B \) of \( V \) with \( c_p(G - B) > |B| \) such that every prime component of \( G - B \) is 1-fc.

**Theorem 3.3** (Gallai [5]). The following two properties are equivalent.

(i) The graph \( G \) is 1-fc.

(ii) The graph \( G \) is connected, not 0-fc, and satisfies Property \( Q''_1 \).

Lovász gave a similar characterization of 2-fc graphs.

**Theorem 3.4** (Lovász [6]). The following two properties are equivalent.

(i) The graph \( G \) is 2-fc.

(ii) The graph \( G \) satisfies Property \( Q_2 \).

Starting from Tutte’s and Gallai’s results, we extend Theorems 3.1, 3.2 and 3.4 (related to \( Q_k \) and \( Q'_k \)) to \( k \)-fc graphs in Theorem 3.5 and we extend Theorem 3.3 (related to \( Q''_k \)) to \( k \)-fc graphs in Corollary 3.6.
Theorem 3.5. 1. For a graph $G$ of order $n \geq k$, the following three properties are equivalent.

(i) The graph $G$ is $k$-fc.

(ii) The graph $G$ satisfies Property $Q_k$.

(iii) The graph $G$ satisfies Property $Q'_k$.

2. If $G$ is not $k$-fc, then, for every subset $S$ of $V$ of maximum order among all the subsets $B$ such that $|B| \geq k$ and $c_p(G - B) > |B| - k$, every prime component of $G - S$ is 1-fc. If moreover $n + k$ is even, then $S$ is a cutset and $c_o(G - S) > |S| - k + 1$.

Proof. 1. For $k = 0$, the equivalence follows from Theorems 3.1 and 3.2. For $k = n$, the three properties are always satisfied. We suppose henceforth $1 \leq k < n$.

(i) $\implies$ (iii). Let $G$ be a $k$-fc graph, $B$ any set of at least $k$ vertices of $G$, and $X$ an arbitrary set of $k$ vertices of $B$. Put $B' = B \setminus X$ and $V' = V \setminus X$. Hence $V' \setminus B' = V \setminus B$. By (i), $\langle V' \rangle$ has a perfect matching. By Theorem 3.2, $c_p(V' \setminus B') \leq |B'|$ and thus $c_p(G - B) \leq |B| - k$. Therefore $G$ satisfies $Q'_k$.

(iii) $\implies$ (ii). Obvious since $Q'_k$ implies $Q_k$.

(ii) $\implies$ (i). Suppose $G$ satisfies $Q_k$. Let $X$ be any set of $k$ vertices of $G$, $Y$ any subset of $V \setminus X$ and $B = X \cup Y$. By (ii), $c_o(G - B) \leq |B| - k$, or, equivalently, $c_o((G - X) - Y) \leq |Y|$ for any $Y \subseteq V \setminus X$. By Theorem 3.1, $G - X$ admits a perfect matching and thus $G$ is $k$-fc.

2. Let $G$ be a graph of order $n$ that is not $k$-fc. By the first part of the theorem, there is a subset $B$ of $V$ such that $|B| \geq k$ and $c_p(G - B) > |B| - k$.

Among all such sets $B$, choose a set $S$ of maximum order and suppose that some prime component $\Gamma$ of $G - S$ is not 1-fc. By Theorem 3.3, $\Gamma$ contains a subset $C$ with $|C| \geq 1$ and $c_p(\Gamma - C) \geq |C| + 1$. If we put $D = S \cup C$, the prime components of $G - D$ are those of $G - S$ except $\Gamma$, and those of $\Gamma - C$. Hence $c_p(G - D) = c_p(G - S) - 1 + c_p(\Gamma - C) > |S| - k - 1 + |C| + 1 = |D| - k$, which contradicts the maximality of $S$. Therefore, every prime component of $G - S$ is 1-fc and thus odd, which implies $c_p(G - S) = c_o(G - S)$. Moreover, it is easy to observe that if $n + k$ is even, then the three integers $|V \setminus S|$, $c_o(G - S)$ and $|S| - k$ have the same parity. Therefore $c_o(G - S) > |S| - k + 1 \geq 1$ and thus $S$ is a cutset.

Corollary 3.6. For a graph $G$ of order $n > k \geq 1$, the following three properties are equivalent.
(i) The graph $G$ is $k$-fc.

(ii) The graph $G$ satisfies Property $Q''_k$, is $k$-connected, and is not $(k-1)$-fc.

(iii) The graph $G$ satisfies Property $Q''_k$ and $n+k$ is even.

**Proof.** (i) $\implies$ (ii). Any $k$-fc graph $G$ of order $n > k$ is $k$-connected by Theorem 2.5, satisfies $Q''_k$ by Theorem 3.5 (since $Q'_k$ implies $Q''_k$), and is not $(k-1)$-fc for parity reasons.

(ii) $\implies$ (iii). Let $G$ satisfy (ii). By $Q''_k$, $c_p(G-D) \leq |D| - k + 1$ and thus $c_o(G-D) \leq |D| - k + 1$, for every subset $D$ of at least $k$ vertices of $V$. Since $G$ is not $(k-1)$-fc, it does not satisfy $Q_{k-1}$ by Theorem 3.5, and hence there exists a set $B$ of at least $k-1$ vertices of $V$ for which $c_o(G-B) > |B| - k + 1$. From what precedes, $|B| = k-1$. Since $G$ is $k$-connected, $G-B$ is connected and $c_o(G-B) \leq 1$. Therefore $c_o(G-B) = 1$, $|V|$ and $|B|$ have different parities and, since $|B| = k-1$, $n+k$ is even.

(iii) $\implies$ (i). If $G$ is not $k$-fc and $n+k$ is even, then, by Theorem 3.5, there exists a set $S$ of at least $k$ vertices for which $c_o(G-S) > |S| - k + 1$, in contradiction to $Q''_k$.

4. Factor-Criticality and Matching Extension

In 1980, Plummer introduced the concept of $q$-extendability [9]. An even graph $G$ is $q$-extendable if $G$ is connected, contains a set of $q$ independent edges, and every set of $q$ independent edges extends to (i.e., is a subset of) a perfect matching. Clearly, for $n$ and $k$ even, every $k$-fc graph is $\frac{k}{2}$-extendable and hence the class of $k$-fc graphs is intermediate between the class of $k$-hamiltonian graphs and that of $\frac{k}{2}$-extendable graphs. There are many results on matching extension that have been obtained recently (see e.g. [12]). Some of these results, saying that “if an even graph $G$ satisfies Property $P$, then $G$ is $q$-extendable”, can be improved to “if an even graph $G$ satisfies $P$, then $G$ is $2q$-fc” (and an analogous statement when $G$ is odd). This is the case when, in the proof of the $q$-extendability, we delete the set $X$ of the $2q$ endvertices of $q$ independent edges, and show that $G-X$ has a perfect matching without using the property that $\langle X \rangle$ itself has a perfect matching. The first example of such a proof can be found in [4]. Two other examples of simple adaptations of proofs and results on matching extension (see [10] and [11]) to $k$-factor-criticality are given below.

The **toughness** of a noncomplete graph $G$ is the number $\text{tough}(G) = \min\{|S| : S \text{ is a cutset of } G\}$. If $G$ is a clique, we put $\text{tough}(G) = +\infty$. 


Theorem 4.1. Let $G$ be a graph of order $n$, and let $k$ be an integer such that $2 \leq k < n$ and $n + k$ is even. If $\text{tough}(G) > \frac{k}{2}$, then $G$ is $k$-fc, and the value $\frac{k}{2}$ is sharp.

**Proof.** Suppose $G$ is not $k$-fc and let $X$ be a set of $k$ vertices of $G$ such that $G' = G - X$ has no perfect matching. By Tutte’s Theorem 3.1, there exists a set $S$ of vertices of $G'$ such that $c_0(G' - S) > |S|$. By parity, $c_0(G' - S) \geq |S| + 2$, and thus $c(G' - S) \geq s + 2$, where $s = |S|$. The set $S \cup X$ is a cutset of $G$, and $c(G - (S \cup X)) = c(G' - S)$. By the definition of toughness, $\text{tough}(G) \leq \frac{|S \cup X|}{c(G - (S \cup X))} \leq \frac{s + k}{s + 2} \leq \frac{k}{2}$ since $k \geq 2$. Therefore, if $\text{tough}(G) > \frac{k}{2}$, then $G$ is $k$-fc. The toughness of the graph $G$ obtained by joining all the vertices of a clique $K_k$ to all the vertices of two disjoint odd cliques $K_{2q+1}$ is equal to $\frac{k}{2}$, and $G$ is not $k$-fc, which proves the sharpness of the bound $\frac{k}{2}$.

The $t$-degree sum and the $t$-generalized independent minimum degree of $G$ are respectively $\sigma_t(G) = \min\{\sum_{w_i \in W} d(w_i) : W$ is an independent set of $t$ vertices of $G\}$ and $U_t = \min\{|\bigcup_{w_i \in W} N(w_i)| : W$ is an independent set of $t$ vertices of $G\}$. These two parameters are defined for $t$ at most equal to the independence number of $G$. For $t = 1$, $\sigma_1 = U_1 = \delta$.

Theorem 4.2. Let $G$ be a graph of order $n$ and connectivity $\kappa$, and let $k$ be an integer such that $0 \leq k \leq \kappa$ and $n + k$ is even. If for some integer $t$ with $1 \leq t \leq \kappa - k + 2$, $\sigma_t(G) \geq t(\frac{n + k}{2} - 1) + 1$ or $U_t(G) \geq n - \kappa + k - 1$, then $G$ is $k$-fc.

**Proof.** Let $G$ be a graph of order $n$ and connectivity $\kappa$, which is not $k$-fc for some integer $k \leq \kappa$ such that $n + k$ is even. Let $X$ be a set of $k$ vertices of $G$ such that $G' = G - X$ has no perfect matching. As in Theorem 4.1, by Tutte’s theorem, there is a set $S$ of vertices of $G'$ such that the number $c$ of components $C_i$ of $G' - S$ is at least $|S| + 2$. Let $s = |S|$. Since $G$ is $k$-connected, $G'$ is $(\kappa - k)$-connected and thus $s \geq \kappa - k$. On the other hand, the sets $X$, $S$ and $C_i$ are all disjoint and thus $|X| + |S| + c \leq n$, which implies, since $c \geq s + 2$, $s \leq \frac{n - k}{2} - 1$. For $1 \leq i \leq c$, let $w_i$ be a vertex of $C_i$. For any integer $t$ with $1 \leq t \leq \kappa - k + 2 \leq c$, the set $\{w_i : 1 \leq i \leq t\}$ is independent.

1. The degree in $G$ of each $w_i$ satisfies $d(w_i) \leq |X| + |S| + |C_i - \{w_i\}| = k + s + |C_i| - 1$. Therefore $\sigma_t \leq \sum_{i=1}^{t} d(w_i) \leq t(k + s - 1) + \sum_{i=1}^{t} |C_i|$. But
\[
\sum_{i=1}^{t} |G_i| = |V\setminus S\setminus X\setminus \bigcup_{i=t+1}^{c} C_i| = n - s - k - \sum_{i=t+1}^{c} |C_i| \leq n - s - k - (c - t) \leq n - k + t - 2s - 2
\]

and thus \(\sigma_t \leq tk + (t - 2)s + n - k - 2\).

If \(t \geq 2\), we get \(\sigma_k \leq tk + (t - 2)(\frac{n-k}{2} - 1) + n - k - 2 = t(\frac{n+k}{2} - 1)\).

Hence if for some \(t\) between 2 and \(\kappa - k + 2\), \(\sigma_t > t(\frac{n+k}{2} - 1)\), then \(G\) is \(k\)-fc.

For \(t = 1\) the condition \(\sigma_1(G) \geq t(\frac{n+k}{2} - 1) + 1\) reduces to \(\delta \geq \frac{n+k}{2}\) and it is known [2] that this implies that \(G\) is \(k\)-hamiltonian and thus \(k\)-fc.

2. The neighborhood in \(G\) of each \(w_i\) satisfies \(N(w_i) \subseteq X \cup S \cup C_i\). Therefore,
\[
U_t \leq |\bigcup_{i=1}^{t} N(w_i)| \leq |X| + |S| + \sum_{i=1}^{t} (|C_i| - 1) \leq k + s + (n - k - 2s - 2 + t) - t = n - s - 2 \leq n - \kappa + k - 2.
\]
Hence if for some \(t\) between 1 and \(\kappa - k + 2\), \(U_t \geq n - \kappa + k - 1\), then \(G\) is \(k\)-fc.

We finish with an example related to a property of the same kind as in [4] but for which the conclusion “\(G\) is \(k\)-extendable” cannot be replaced by “\(G\) is \(2k\)-fc”. Ryjáček proved in [13] that every even \((2k+1)\)-connected \(K_{1,k+3}\)-free graph such that the set of claw centers is independent, is \(k\)-extendable. The hypotheses do not imply that the graph is \(2k\)-fc as shown, for \(k = 1\), by the following construction. The graph \(G\) consists of four copies \(H_i\) of cliques \(K_p\) of odd order \(p \geq 3\), and four extra vertices \(x_i\), \(1 \leq i \leq 4\). In each \(H_i\), we select three vertices \(y_{ij}\) with \(1 \leq j \leq 4\) and \(j \neq i\). Each vertex \(x_i\) is adjacent to the three vertices \(y_{ji}\) with \(1 \leq j \leq 4\) and \(j \neq i\). The graph \(G\) is \(3\)-connected, \(K_{1,4}\)-free and the claw centers, which are the vertices \(x_i\), are independent. It is \(1\)-extendable but not \(2\)-fc since \(G - \{x_1, x_2\}\) has no perfect matching.

References


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