Discussiones Mathematicae
Graph Theory 16 (1996) 173–179

REMARKS ON 15-VERTEX (3,3)-RAMSEY GRAPHS NOT CONTAINING K_5

Sebastian Urbański

Department of Discrete Mathematics
Faculty of Mathematics and Computer Science
Adam Mickiewicz University Poznań, Poland

Abstract

The paper gives an account of previous and recent attempts to determine the order of a smallest graph not containing K_5 and such that every 2-coloring of its edges results in a monochromatic triangle. A new 14-vertex K_4-free graph with the same Ramsey property in the vertex coloring case is found. This yields a new construction of one of the only two known 15-vertex (3,3)-Ramsey graphs not containing K_5.

Keywords: Folkman numbers, K_n-free graphs, extremal graph theory, generalized Ramsey theory.

1991 Mathematics Subject Classification: 05C55, 05C35.

1. Introduction

Let G be a graph, and let k and l be positive integers. We write $G \to (k,l)^v$ ($G \to (k,l)^e$) if every red-blue coloring of the vertices (edges) of G forces a red complete subgraph K_k or a blue complete subgraph K_l in G. For $n > \max\{k,l\}$, let

$$\mathcal{G}^v(k,l;n) = \{G : G \to (k,l)^v \text{ and } K_n \not\subset G\}$$

and

$$\mathcal{G}^e(k,l;n) = \{G : G \to (k,l)^e \text{ and } K_n \not\subset G\}.$$

The graphs in $\mathcal{G}^v(k,l;n)$ are called *vertex-Folkman graphs* and the graphs in $\mathcal{G}^e(k,l;n)$ are called *edge-Folkman graphs*.

It is well known that $K_6 \to (3,3)^e$ and so $K_6 \in \mathcal{G}^e(3,3;n)$ for all $n > 6$. In 1967 Erdős and Hajnal [2] asked if $\mathcal{G}^e(3,3;6) \neq \emptyset$ and the following year Graham [6] answered this question showing that $K_8 - C_5 \in \mathcal{G}^e(3,3;6)$, where,

Research supported by KBN grant 2 P03A 023 09.

for $q \leq p$, $K_p - C_q$ is the graph obtained by deleting the edges of a cycle C_q from K_p. In 1970 Folkman [4] showed that for all k, l and $n > \max(k, l)$ the families $\mathcal{G}^v(k, l; n)$ and $\mathcal{G}^e(k, l; n)$ are nonempty. One can ask what the minimum number of vertices of a vertex- or edge-Folkman graph is. This problem leads to the notion of Folkman numbers. Let us denote

$$F^v(k, l; n) = \min\{|V(G)| : G \in \mathcal{G}^v(k, l; n)\}$$

and

$$F^e(k, l; n) = \min\{|V(G)| : G \in \mathcal{G}^e(k, l; n)\},$$

where $V(G)$ is the vertex set of a graph G. These numbers are called *vertex-Folkman numbers* and *edge-Folkman numbers*, respectively. Observe that for $n > k + l - 1$ we have $F^v(k, l; n) = k + l - 1$ as a trivial consequence of the pigeon-hole principle. Since the clique on $R(k, l)$ vertices is the smallest graph G with the property $G \to (k, l)^e$ (here $R(k, l)$ is the Ramsey number), obviously we have $F^e(k, l; n) = R(k, l)$ for every $n > R(k, l)$. Very little is known about the edge-Folkman numbers in the case $n \leq R(k, l)$.

An edge-Folkman number that is still unknown but has been bounded reasonably is $F^e(3, 3; 5)$. The first proof of existence of this number is due to Pósa (unpublished). Schauble [15] in 1969 showed that $F^e(3, 3; 5) \leq 42$. The next upper bound was obtained in 1971 by Graham and Spencer [7]. They proved that $F^e(3, 3; 5) \leq 23$ and conjectured that $F^e(3, 3; 5) = 23$, but as they admitted, without much evidence. Their bound was pushed down to 18 by Irving [11] in 1973. In 1979 Hadziivanov and Nenov [8] showed a 16-vertex graph from $\mathcal{G}^e(3, 3; 5)$ and in 1981 Nenov [14] presented the first 15-vertex graph with that property proving that $F^e(3, 3; 5) \leq 15$. The second one was found in 1984 by Hadziivanov and Nenov [9]. The last three papers (written in Russian) were not generally noticed at that time. In 1993 Erickson [3] found a 17-vertex graph in $\mathcal{G}^e(3, 3; 5)$ and conjectured that $F^e(3, 3; 5) = 17$. This was recently disproved by Bukor [1], who came up with the same 16-vertex graph as in [8]. The author found independently the 15-vertex graph discovered in [9], but the construction is different. This will be shown below.

As far as the lower bound is concerned, in 1972 Lin [12] showed that $F^e(3, 3; 5) \geq 10$ and his result was later improved by Nenov [13] to $F^e(3, 3; 5) \geq 11$ and by Hadziivanov and Nenov [10] to $F^e(3, 3; 5) \geq 12$.

Much less is known about the number $F^e(3, 3; 4)$. Frankl and Rödl [5] proved that $F^e(3, 3; 4) \leq 10^{12}$ and later Spencer [16] squeezed out from their proof the inequality $F^e(3, 3; 4) \leq 10^{10}$. No reasonable lower bound for this Folkman number is known.

2. CONSTRUCTIONS

There were two general lines of the search for small $(3,3)$-Ramsey graphs not containing K_5. The first one, originated in the construction of Graham, was based on the following fact proved explicitly in [9]. The join $H+G$ of two vertex disjoint graphs H and G is the graph with the vertex set $V(H) \cup V(G)$ and the edge set $E(H) \cup E(G) \cup \{\{u,v\} : v \in V(H), u \in V(G)\}$.

Proposition 1 (Hadziivanov, Nenov, 1984). *Let P be a path of order* 3. *If $\chi(G) > 2$ and the edges of $P+G$ are* 2*-colored without monochromatic triangle, then P is monochromatic.* ■

This fact was used by Hadziivanov and Nenov [9] to build the following 15-vertex graph $G_1 \in \mathcal{G}^e(3,3;5)$. Let C be a 5-cycle contained in K_5. Let G_0 be the graph obtained by elementary subdividing each edge of C as shown in Figure 1.

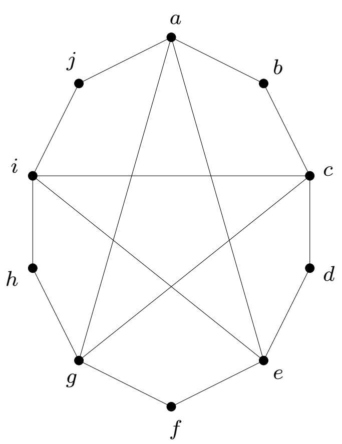

Figure 1

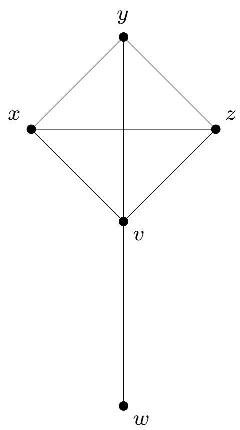

Figure 2

Observe that G_0 is a union of 3 edge-disjoint 5-cycles: $C_1 = \{a,b,c,d,e\}$, $C_2 = \{e,f,g,c,i\}$, $C_3 = \{g,h,i,j,a\}$. Consider the graph shown in Figure 2. It contains 3 paths of length 2: $P_1 = \{x,v,w\}$, $P_2 = \{y,v,w\}$, $P_3 = \{z,v,w\}$. Let G_1 be the union of the joins C_1+P_1, C_2+P_2 and C_3+P_3. One can easily check that there is no K_5 in G_1. We shall now prove that $G_1 \to (3,3)^e$. Suppose, on the contrary, that there exists a red-blue coloring of the edges of G_1 such that there is no monochromatic triangle. It follows from Proposition 1 that each path P_1, P_2 and P_3 is monochromatic. Thus, the edges $\{x,v\},\{z,v\},\{y,v\}$ have the same color, say red. Then the triangle $\{x,y,z\}$ cannot have a red edge, so it becomes blue. This contradiction proves that $G_1 \to (3,3)^e$ and, consequently, we have $G_1 \in \mathcal{G}^e(3,3;5)$.

The other method, going back to Pósa, constructs edge-Folkman graphs from vertex-Folkman graphs. Let $H+v$ denote the graph obtained from a graph H by adding a vertex v and all edges between v and H. The following result in case $k=l$ was proved in [11]. The idea of the proof below is basically taken from there.

Proposition 2. *Setting $m_1 = R(k-1,l)$ and $m_2 = R(k,l-1)$, if $H \in \mathcal{G}^v(m_1,m_2;n-1)$, then $H+K_1 \in \mathcal{G}^e(k,l;n)$.*
In particular,

$$F^e(k,l;n) \le F^v(m_1,m_2;n-1)+1.$$

Proof. Let $H \in \mathcal{G}^v(m_1,m_2;n-1)$ and $G = H+v$. Of course, $K_n \not\subset G$. Let us consider any red-blue coloring of the edges of G. For every vertex $x \in V(H)$ we say that x is red if the edge $\{x,v\}$ is red, and it is blue if $\{x,v\}$ is blue. Since $H \in \mathcal{G}^v(m_1,m_2;n-1)$, there are two possibilities:

either there exists a K_{m_1} on red vertices of H
or there exists a K_{m_2} on blue vertices of H.

Assume that the first case is true. Then the red K_{m_1} contains a K_{k-1} with all edges red (so that this $K_{k-1}+v$ creates a K_k with all edges red), or it contains K_l with all edges blue. If there is a K_{m_2} on the blue vertices of H, then this K_{m_2} either contains a K_k with all edges red or it contains a K_{l-1} with all edges blue (so that this $K_{l-1}+v$ creates a K_l with all edges blue). Hence, one way or another, every red-blue coloring of the edges of G forces a red K_k or a blue K_l. ■

We now present the other known 15-vertex graph G_2 belonging to $\mathcal{G}^e(3,3;5)$, constructed by this method. Figure 3 shows the graph F_1 from [14] which was the first 14-vertex graph discovered in the family $\mathcal{G}^v(3,3;4)$.

Claim 1. $F_1 \in \mathcal{G}^v(3,3;4)$.

Proof. One can very easily check that $K_4 \not\subset F_1$. Hence it is enough to prove that $F_1 \to (3,3)^v$. Suppose that there exists a red-blue coloring of the vertices of F_1 such that F_1 has no monochromatic triangle. Let F_0 denote

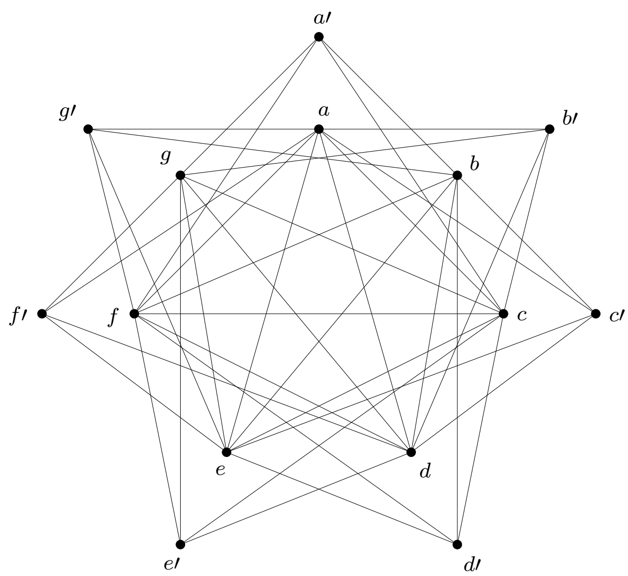

Figure 3. Graph F_1

the subgraph of F_1 induced by the vertices a, b, c, d, e, f, g. Since every 5 vertices of F_0 span a triangle, F_0 has at most 4 red vertices and at most 4 blue vertices. Without loss of generality, we may assume that it has precisely 3 red vertices and 4 blue vertices and that a and b are red. Now we consider four cases with respect to where the third red vertex might be.

(i) If c is the third red vertex, then a, c are red and d, g are blue, so we cannot color the vertex b'.

(ii) If d is red, then a, d are red and e, g are blue, and thus we cannot color the vertex f'.

(iii) If e is red, then a, e are red and d, g are blue so we have no color for the vertex f'.

(iv) Finally, if the vertex f (or g) is red, then we get the same situation as in case (ii) ((i) respectively) because of the symmetry of the graph F_1.

Thus no other vertex of F_0 can be red, a contradiction. Thus, such a coloring is impossible and $F_1 \in \mathcal{G}^v(3,3;4)$. ■

By Proposition 2, the join $G_2 = F_1 + K_1$ belongs to $\mathcal{G}^e(3,3;5)$, and this is the graph found by Nenov [14].

We shall now construct a 14-vertex graph $F_2 \in \mathcal{G}^v(3,3;4)$ different than Nenov's graph F_1 from Fig. 3. Let G_0 be the graph shown in Figure 1. We construct the required graph F_2 by adding four more vertices w, x, y, z and joining w to all vertices of G_0, x to all vertices of C_1, y to all vertices of C_2 and z to all vertices of C_3. Also we add the edges $\{x, y\}, \{y, z\}$ and $\{x, z\}$. Note that F_2 has 14 vertices.

Claim 2. $F_2 \in \mathcal{G}^v(3, 3; 4)$.

Proof. Let us first show that $K_4 \not\subset F_2$. Observe that $K_3 \not\subset G_0$ and hence $K_4 \not\subset G_0 + x$, $K_4 \not\subset G_0 + y$, $K_4 \not\subset G_0 + z$ and $K_4 \not\subset G_0 + w$. Moreover, $w \notin K_4$. Thus, if $K_4 \subset F_2$, then this K_4 must contain two or three vertices of the set $\{x, y, z\}$. The cycles C_1, C_2 and C_3 are edge-disjoint, so no two vertices of $\{x, y, z\}$ are in K_4. Thus, all x, y, z must be in K_4, but it is impossible because the cycles C_1, C_2, C_3 have no common vertex. Consequently, $K_4 \not\subset F_2$.

Assume that the vertices of F_2 are red-blue colored and there is no monochromatic triangle in F_2. Without loss of generality, we may assume that the vertex w is red. Each cycle C_1, C_2 and C_3 has at least two adjacent vertices of the same color. It must be blue since w is red. But then all x, y, z must be red and the triangle x, y, z becomes red. It is a contradiction proving that every red-blue coloring of vertices of F_2 forces a monochromatic triangle. Hence, the graph F_2 is the second known 14-vertex graph in $\mathcal{G}^v(3, 3; 4)$. ■

Note that the join $F_2 + K_1$ is isomorphic to graph G_1 described earlier. Thus, it turned out that both known 15-vertex $(3, 3)$-Ramsey graphs not containing K_5 can be viewed as a join of K_1 and a graph from $\mathcal{G}^v(3, 3; 4)$.

Open problem. Determine the precise value of the Folkman numbers $F^e(3, 3; 5)$ and $F^v(3, 3; 4)$, or tighten up the present estimates

$$11 \leq F^v(3, 3; 4) \leq 14,$$

$$12 \leq F^e(3, 3; 5) \leq 15.$$

(It follows from Proposition 2 that $F^e(3, 3; 5) \geq 12 \Rightarrow F^v(3, 3; 4) \geq 11$).

Acknowledgements

The author thanks Andrzej Ruciński for suggesting the problem and for many helpful remarks. Thanks are also due to an anonymous referee for several comments leading to an improvement of the paper.

References

[1] J. Bukor, *A note on the Folkman number* $F(3,3;5)$, Math. Slovaca **44** (1994) 479–480.

[2] P. Erdős and A. Hajnal, *Research problem* $2-5$, J. Combin. Theory **2** (1967) 104.

[3] M. Erickson, *An upper bound for the Folkman number* $F(3,3;5)$, J. Graph Theory **17** (1993) 679–681.

[4] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970) 19–24.

[5] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without* K_4, Graphs and Combinatorics **2** (1986) 135–144.

[6] R.L. Graham, *On edgewise* 2*-colored graphs with monochromatic triangles and containing no complete hexagon*, J. Combin. Theory **4** (1968) 300.

[7] R.L. Graham and J.H. Spencer, *On small graphs with forced monochromatic triangles*, in: Recent Trends in Graph Theory. Lecture Notes in Math. **186** (Springer-Verlag, Berlin, 1971) 137–141.

[8] N. Hadziivanov and N. Nenov, *On Graham-Spencer number*, C.R. Acad. Bulg. Sci. **32** (1979) 155–158.

[9] N. Hadziivanov and N. Nenov, *On Ramsey graphs*, God. Sofij. Univ. Fak. Mat. Mech. **78** (1984) 211–214.

[10] N. Hadziivanov and N. Nenov, *Every* $(3,3)$*-Ramsey graph without* 5*-cliques has more than* 11 *vertices*, Serdica **11** (1985) 341–356.

[11] R.W. Irving, *On a bound of Graham and Spencer for graph-coloring constant*, J. Combin. Theory **15** (1973) 200–203.

[12] S. Lin, *On Ramsey numbers and* K_r*-coloring of graphs*, J. Combin. Theory (B) **12** (1972) 82–92.

[13] N. Nenov, *New lower bound for Graham-Spencer number*, Serdica **6** (1980) 373–383.

[14] N. Nenov, *An example of* 15*-vertex* $(3,3)$*-Ramsey graph with the clique number* 4, C.R. Acad. Bulg. Sci. **34** (1981) 1487–1489.

[15] M. Schauble, *Zu einem Kantenfarbungsproblem*, Wiss. Z. Th. Ilemenau **15** (1969) 55–58.

[16] J. Spencer, *Three hundred million points suffice*, J. Combin. Theory (A) **49** (1988) 210–217. See erratum in **50** p. 323.

Received 27 June 1996
Revised 18 November 1996