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UNAVOIDABLE SET OF FACE TYPES FOR PLANAR MAPS

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Abstract

The type of a face f of a planar map is a sequence of degrees of vertices of f as they are encountered when traversing the boundary of f. A set $\mathcal{T}$ of face types is found such that in any normal planar map there is a face with type from $\mathcal{T}$. The set $\mathcal{T}$ has four infinite series of types as, in a certain sense, the minimum possible number. An analogous result is applied to obtain new upper bounds for the cyclic chromatic number of 3-connected planar maps.

Keywords: normal planar map, plane graph, type of a face, unavoidable set, cyclic chromatic number.

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1. Introduction

It is an old classical consequence of the famous Euler's polyhedral formula that a normal planar map contains a vertex of degree ≤ 5, a face of degree ≤ 5 and also a 3-valent vertex or a triangle.

A face of a map can be characterized by its type, a sequence of degrees of its vertices. Lebesgue [19] specified a set of small face types which intersects the face type set of any normal planar map. (For this Lebesgue's result and its application see Plummer and Toft [21].)

Kotzig [16] proved that each 3-connected normal planar map contains an edge of weight (the degree sum of its endvertices) at most 13; the condition of 3-connectedness can be abandonned, due to Borodin [1] (the same result was announced by Barnette, see Grünbaum [11]).

At present many results concerning the structure of planar maps are known. For example, Kotzig's result was generalized and strengthened in several directions, see Borodin [2,5], Grünbaum and Shephard [12], Ivančo

[13], Zaks [23]. Recently, unifying and strengthening Kotzig's results [18], Borodin [8] has proved that any planar triangulation without vertices of degree 4 contains either a triangle of weight (the degree sum of its incident vertices) at most 29 incident with a 3-valent vertex or a triangle whose weight does not exceed 17.

A sharp inequality for the number of triangles of weight at most 17 in planar maps with minimum degree 5 was found by Borodin [4]. Edges of small weights in planar maps of minimum degree 5 are investigated in a very recent paper by Borodin and Sanders [9]. Both the above mentioned papers complete the work contributed to by many authors, among others Grünbaum [10], Kotzig [17], Fisk (see [12]), Wernicke [22].

Many structural results on planar maps have been obtained by solving some colouring problems, see e.g. Borodin [1,3,6,7], Jendrol' and Skupień [14].

The main aim of this paper is to prove an analogue of Lebesgue's theorem, which is optimal in a certain sense.

2. Fundamentals

For integers p, q we denote by $[p, q]$ the set of all integers i, $p \le i \le q$, and by $[p, \infty)$ the set of all integers $\ge p$.

A finite sequence Q is said to be *equivalent* to a finite sequence P if Q can be obtained from P using rotation and/or mirror image. Thus, if $P = (p_1, \ldots, p_n)$, then $Q = (p_{1+i}, \ldots, p_{n+i})$ or $Q = (p_{n-i}, \ldots, p_{1-i})$ for some $i \in [0, n-1]$, where indices are taken modulo n. (We use this "modulo convention" throughout the whole paper.) Let P, P_1, P_2 be finite sequences and let $m \in [1, \infty)$. We denote by P_1P_2 the concatenation of P_1 and P_2 (in that order), by P^m the m-fold concatenation of P's and by $\operatorname{len}(P)$ the length of P.

Let M be a map on a 2-manifold, i.e. a 2-cell embedding of a graph, in which loops and multiple edges are allowed. $V(M)$, $E(M)$ and $F(M)$ are the vertex set, the edge set and the face set of M, respectively, deg c is the degree of $c \in V(M) \cup F(M)$. M is called *normal* if deg $c \ge 3$ for any $c \in V(M) \cup F(M)$. An *angle* of a face $f \in F(M)$ with *centre* $v \in V(M)$ is an alternating quintuple (u, d, v, e, w) of consecutive vertices and edges of f which are encountered when moving along the boundary of f, i.e., the curve consisting of all edges incident with f. The centre of an angle a will be denoted by $\dot{a}$. Let $A(f)$ be the set of all angles of f and $A(v)$ the set of all angles with centre v. Evidently, $|A(v)| = 2 \deg v$ and $|A(f)| = 2 \deg f$ for any $v \in V(M)$ and $f \in F(M)$. Due to the normality of M we know that

$A(v_1) \neq A(v_2)$ for any $v_1, v_2 \in V(M)$, $v_1 \neq v_2$ and $A(f_1) \neq A(f_2)$ for any $f_1, f_2 \in F(M)$, $f_1 \neq f_2$. Putting

$$A(M) := \{A(v) : v \in V(M)\} = \{A(f) : f \in F(M)\}$$

we see that there exists a natural bijection β_M between the sets $\{(a, v) \in A(M) \times V(M) : a \in A(v)\}$ and $\{(a, f) \in A(M) \times F(M) : a \in A(f)\}$. Let f be a face of degree n and let $(v_1, \ldots, v_n)$ be a sequence of vertices of f as they are encountered when traversing the boundary of f. Any sequence from the set $\tau(f)$ of all sequences equivalent to $(\deg v_1, \ldots, \deg v_n)$ is said to be a *type* of f.

Let $\mathcal{S}$ be the set consisting of all lexicographic minima of the set $\bigcup_{i=3}^{\infty}[3, \infty)^i$ (provided sequences of the same length are comparable only). We represent the set $\tau(f)$ by its representative in $\mathcal{S}$.

Let $\mathbb{M}$ be a class of normal maps on a 2-manifold. A set $\mathcal{T} \subseteq \mathcal{S}$ is said to be an *unavoidable* set of face types for $\mathbb{M}$ if for any $M \in \mathbb{M}$ there exists $T \in \mathcal{T}$ and $f \in F(M)$ such that $T \in \tau(f)$.

In 1940 Lebesgue [19] proved (in a dual form)

Theorem 1. *For the class of normal planar maps the following sequences form an unavoidable set of face types:*
$(3, i, j), i \in [3, 6], j \in [i, \infty)$, $(4, 4, i), i \in [4, \infty)$, $(3, 3, 3, i), i \in [3, \infty)$,
$(3, 7, i), i \in [7, 41]$, $(3, 8, i), i \in [8, 23]$, $(3, 9, i), i \in [9, 17]$, $(3, 10, i), i \in [10, 14]$,
$(3, 11, i), i \in [11, 13]$, $(4, 5, i), i \in [5, 19]$, $(4, 6, i), i \in [6, 11]$, $(4, 7, i), i \in [7, 9]$,
$(5, 5, i), i \in [5, 9]$, $(5, 6, i), i = 6, 7$,
$(3, 3, 4, i), i \in [4, 11]$, $(3, 3, 5, i), i \in [5, 7]$, $(3, 4, 3, i), i \in [4, 11]$,
$(3, 4, 4, i), i = 4, 5$, $(3, 4, 5, 4)$, $(3, 5, 3, i), i \in [5, 7]$,
$(3, 3, 3, 3, i), i \in [3, 5]$. ■

Note that in [19] an error occurred by omitting the types $(4, 4, i), i \in [4, \infty)$.

Let $\mathcal{T}$ be an unavoidable set of face types for $\mathbb{M}$. A sequence $S = (s_1, ..., s_n) \in \bigcup_{i=2}^{\infty}[3, \infty)^i$ such that either $(s_n, ..., s_1) = S$ or $s_j < s_{n+1-j}$ for $j = \min\{i \in [1, n] : s_i \neq s_{n+1-i}\}$ is a *$\mathcal{T}$-basic sequence* if the set $\mathcal{T} \cap \{S(i) : i \in [3, \infty)\}$ is infinite. Let $B(\mathcal{T})$ be the set of all $\mathcal{T}$-basic sequences. For $i \in [3, \infty)$ set

$$b_i(\mathcal{T}) := \mathrm{card}\{S \in B(\mathcal{T}) : \mathrm{len}(S) = i - 1\},$$

$$b_i^-(\mathcal{T}) = \mathrm{card}\{T \in \mathcal{T} - \{S(j) : S \in B(\mathcal{T}),\ \ j \in [3, \infty)\} : \mathrm{len}(T) = i\}.$$

The sequences $\{b_i(\mathcal{T})\}_{i=3}^{\infty}$ and $\{b_i^-(\mathcal{T})\}_{i=3}^{\infty}$ are called the *infinite* and the *finite characteristic* of $\mathcal{T}$, respectively. If $b_i(\mathcal{T}) = 0$ for all $i \in [p + 1, \infty)$ or

$b_i^-(\mathcal{T}) = 0$ for all $i \in [q+1, \infty)$, we present a corresponding characteristic simply as $(b_3(\mathcal{T}), \dots, b_p(\mathcal{T}))$ or $(b_3^-(\mathcal{T}), \dots, b_q^-(\mathcal{T}))$. For the Lebesgue's unavoidable set $\mathcal{L}$, we see that $B(\mathcal{L}) = \{(3, i) : i \in [3, 6]\} \cup \{(4, 4), (3, 3, 3)\}$ and that $\mathcal{L}$ has the infinite characteristic (5,1), and the finite characteristic (99,25,3).

An unavoidable set $\mathcal{T}$ is *good* if $\sum_{i=3}^{\infty} b_i^-(\mathcal{T})$ is finite. Two good unavoidable sets $\mathcal{T}$ and $\mathcal{T}'$ can be compared as follows: $\mathcal{T}$ is *more economical* than $\mathcal{T}'$ if $\sum_{i=3}^{\infty} b_i(\mathcal{T}) < \sum_{i=3}^{\infty} b_i(\mathcal{T}')$; this means that $\mathcal{T}$ contains a smaller number of (naturally structured) infinite subsets than $\mathcal{T}'$ (and a finite "rest"). Thus, we can pose

Problem 1. Find the minimum of $\sum_{i=3}^{\infty} b_i(\mathcal{T})$ for a good unavoidable set $\mathcal{T}$ of face types for normal planar maps.

We are going to show that the minimum of Problem 1 is equal to 4.

3. Main result

Theorem 2. *Let $\mathcal{T}$ be a good unavoidable set of face types for normal planar maps.*

(i) $\{(3, 4, 4)\} \cup \{(4, 4, i) : i \in [4, \infty)\} \cup \{(3, 3, 3, i) : i \in [3, \infty)\} \subseteq \mathcal{T}$.

(ii) *If* $(3, 3) \notin B(\mathcal{T})$, *then* $\{(3)^i : i \in [4, \infty)\} \subseteq B(\mathcal{T})$.

(iii) *If* $(3, 4) \notin B(\mathcal{T})$, *then* $\{(3, 3, 4), (4, 3, 4)\} \subseteq B(\mathcal{T})$.

Proof. Let $m \in [3, \infty)$, $n \in [1, \infty)$, $l \in [1, n]$ and let $P = (p_1, \dots, p_{2l}) \in [1, n]^{2l}$ be such a sequence that $p_i \neq p_j$ for any $(i, j) \in [1, l]^2 \cup [l+1, 2l]^2$, $i \neq j$. Let $G_m^n(P)$ be a planar graph with

$$V(G_m^n(P)) = \{x_i : i \in [1, mn]\} \cup \{y_0, y_1\},$$

$$E(G_m^n(P)) = \bigcup_{i=1}^{mn} \{x_i x_{i+1}\} \cup \bigcup_{i=0}^{m-1} \bigcup_{j=0}^{1} \{x_{in+k} y_j : k = p_{jl+1}, \dots, p_{jl+l}\}.$$

A plane embedding of $G_m^1(1, 1)$ (an m-sided bipyramid) has only faces of type $(4, 4m)$ and a plane embedding of $G_m^2(1, 2)$ (a dual of an m-sided antiprism) has only faces of type $(3, 3, 3, m)$; hence (i) follows from the finiteness of $\sum_{i=3}^{\infty} b_i^-(\mathcal{T})$.

A plane embedding of $G_m^{2n}(1, \dots, 2n)$, $n \geq 2$, has only faces of types $(3, 3, mn)$ and $(3)^{n+2}(mn)$, so that $(3, 3) \notin B(\mathcal{T})$ implies $(3)^{n+2}(mn) \in \mathcal{T}$ for all sufficiently large m and $(3)^{n+2} \in B(\mathcal{T})$.

Finally, a plane embedding of $G_m^4(1,2,3,1,3,4)$ has only faces of types $(3,4,3m)$ and $(4,3,4,3m)$, while a plane embedding of $G_m^6(1,2,4,5,1,3,4,6)$ has only faces of types $(3,4,4m)$ and $(3,3,4,4m)$. It means that if $(3,4) \notin B(\mathcal{T})$, then for every m large enough $(4,3,4,3m)$ as well as $(3,3,4,4m)$ belong to $\mathcal{T}$, so that $\{(3,3,4),(4,3,4)\} \subseteq B(\mathcal{T})$. ■

Corollary 3. $\sum_{i=3}^{\infty} b_i(\mathcal{T}) \geq 4$ *for any good unavoidable set* $\mathcal{T}$ *of face types for normal planar maps and, if the equality holds, then* $b_3(\mathcal{T}) = 3$, $b_4(\mathcal{T}) = 1$ *and* $B(\mathcal{T}) = \{(3,3),(3,4),(4,4),(3,3,3)\}$. ■

Thus our goal will be reached by finding an unavoidable set $\mathcal{T}$ of face types for normal planar maps with the infinite characteristic $(3,1)$ and with $\sum_{i=3}^{\infty} b_i^-(\mathcal{T})$ being finite.

One of well known corollaries of Euler's formula for a planar map M can be expressed as

$$\sum_{v \in V(M)} (6 - \deg\, v) + 2 \sum_{f \in F(M)} (3 - \deg\, f) = 12.$$

Thus, if we define the *basic charge* of a vertex $v \in V(M)$, of a face $f \in F(M)$ and of an angle $a \in A(M)$ by

$$b_v := \deg\, v - 6, \qquad b_f := 2\, \deg\, f - 6, \qquad b_a := \frac{\deg\, \dot{a} - 6}{2\, \deg\, \dot{a}},$$

then

$$b_v = \sum_{a \in A(v)} b_a,$$

$$\sum_{v \in V(M)} b_v + \sum_{f \in F(M)} b_f = \sum_{v \in V(M)} \sum_{a \in A(v)} b_a + \sum_{f \in F(M)} b_f = -12,$$

which, using the mentioned bijection β_M, can be rewritten as

$$\sum_{f \in F(M)} \sum_{a \in A(f)} b_a + \sum_{f \in F(M)} b_f = \sum_{f \in F(M)} (b_f + \sum_{a \in A(f)} b_a) = -12.$$

If the basic charges of vertices and faces are transformed to

$$b'_v := 0, \qquad b'_f := b_f + \sum_{a \in A(f)} b_a,$$

we see that

$$\sum_{v \in V(M)} b'_v + \sum_{f \in F(M)} b'_f = \sum_{f \in F(M)} b'_f = -12,$$

hence there exists a face f whose transformed charge b'_f is negative. For $T = (d_1, \dots, d_n) \in [3, \infty)^n$, $n \in [3, \infty)$, put

$$B'(T) := 2n - 6 + \sum_{i=1}^{n} \frac{d_i - 6}{d_i}.$$

Then, clearly, $b'_f = B'(T)$ for each $T \in \tau(f)$, and we can call $B'(T)$ the *transformed* charge of the face type T. The Lebesgue's set $\mathcal{L}$ consists just of face types with a negative transformed charge.

We modify the process of passing from basic charges to transformed charges in the following way: We define a rational *alternative* charge c_v of an angle $a \in A(M)$. Then we determine alternative charges of vertices and faces by

$$c_v := \sum_{a \in A(v)} c_a, \qquad c_f := b_f + \sum_{a \in A(f)} (b_a - c_a).$$

Due to the definition we have

$$c_f + \sum_{a \in A(f)} c_a = b_f + \sum_{a \in A(f)} b_a,$$

$$\sum_{v \in V(M)} c_v + \sum_{f \in F(M)} c_f = \sum_{v \in V(M)} b_v + \sum_{f \in F(M)} b_f = -12.$$

If all alternative vertex charges are non-negative, there exists a face $f \in F(M)$ with $c_f < 0$.

In the definition of the basic charge of an angle a the degree of $\dot{a}$ is involved only. To involve degrees of all the vertices of an angle in the definition of an alternative angle charge, we shall define, for $a = (v_{i-1}, e_{i-1}, v_i, e_i, v_{i+1})$,

$$c_a := c(\deg v_{i-1}, \deg v_i, \deg v_{i+1}),$$

where the mapping $c : [3, \infty)^3 \to \mathbb{Q}$ fulfills the condition

$$c(i, j, k) = c(k, j, i) \qquad \text{for any } (i, j, k) \in [3, \infty)^3.$$

If $T = (d_1, \dots, d_n) \in [3, \infty)^n$, $n \in [3, \infty)$, we define the *alternative* charge of the face type T by

$$C(T) := B'(T) - 2 \sum_{i=1}^{n} c(d_{i-1}, d_i, d_{i+1}).$$

Then, analogously as before, the alternative charge is an invariant rational on the set of all types of a fixed face.

Let v be a vertex of M with degree n and let $(e_1, \dots, e_n)$ be the sequence of edges incident with v as they are encountered when rotating around v. Let v_i be the vertex of M joined to v along e_i, $i = 1, \dots, n$. Then the alternative charge of v is $c_v = 2\sum_{i=1}^{n} c(\deg v_i, n, \deg v_{i+1})$. Set

$$s(d_1, \dots, d_n) := \sum_{i=1}^{n} c(d_i, n, d_{i+1}).$$

Thus, if the condition

$$(*) \qquad s(d_1, \dots, d_n) \geq 0 \qquad \text{for any } (d_1, \dots, d_n) \in [3, \infty)^n,\ n \in [3, \infty),$$

is fulfilled, then the set $\mathcal{T}$ of all face types T with $C(T) < 0$ is unavoidable for normal planar maps.

A degree $j \in [3, \infty)$ is called *absorbing* if there exists a pair $(i, k) \in [3, \infty)^2$ such that $c(i, j, k) < 0$, otherwise it is *non-absorbing.* Thus, to control $(*)$ it suffices to deal with absorbing n's and it is desirable to have only a small number of absorbing degrees. On the other hand, we need some absorbing degrees, since otherwise we would obtain as unavoidable a superset of the Lebesgue's set $\mathcal{L}$. For non-absorbing j's it is appropriate to define $c(i, j, k) := 0$, since the positivity of $c(i, j, k)$ could only enrich the unavoidable set.

First of all, it is clear that even degrees must be non-absorbing. To see this suppose $c(i, j, k) < 0$ for some $j \equiv 0 \pmod 2$; then, for $(d_1, \dots, d_j) = (i, k)^{j/2}$ we have

$$\sum_{l=1}^{j} c(d_l, j, d_{l+1}) = jc(i, j, k) < 0.$$

We need also

$$c(i, j, i) \geq 0 \qquad \text{for any } i, j \in [3, \infty);$$

otherwise, with $c(i, j, i) < 0$ and $(d_1, \dots, d_j) = (i)^j$, we would have

$$\sum_{l=1}^{j} c(d_l, j, d_{l+1}) = jc(i, j, i) < 0.$$

It could be a good idea to have non-absorbing all degrees large enough. Put

$$c_3(i) := \frac{1}{4} - \frac{3}{i}.$$

If $(3, i, i) \notin \mathcal{T}$ for some $i \in [3, \infty)$ (remember that we tend to have $b_3^-(\mathcal{T})$ finite), then $0 \leq C(3, i, i) = 4c_3(i) - 2c(i, 3, i) \leq 4c_3(i)$. As $c_3(i) < 0$ for

$i < 12$, the best we can do is to require that all degrees ≥ 12 be non-absorbing.

Now we have the following degrees as candidates to be absorbing: $3, 5, 7, 9, 11$. Since we want to obtain $\mathcal{T}$ with the infinite characteristic $(3, 1)$, by Corollary 3 $(3, 6) \notin B(\mathcal{T})$, which means that $(3, 6, i) \notin \mathcal{T}$ for a sufficiently large $i \geq 12$. As $C(3, 6, i) = -\frac{6}{i} - 2c(6, 3, i) \geq 0$, we see that $c(6, 3, i)$ must be negative and 3 is an absorbing degree.

It would be fine to be able to exclude from $\mathcal{T}$ all types which do not assure the existence of an edge of weight ≤ 13 (in order to cover Kotzig's result). One of these types is $(3, 11, 11)$. As $C(3, 11, 11) = -\frac{1}{11} - 2c(11, 3, 11) - 4c(3, 11, 11) \geq 0$, we obtain $c(3, 11, 11) \leq -\frac{1}{4}(\frac{1}{11} + 2c(11, 3, 11)) \leq -\frac{1}{44}$ and 11 is an absorbing degree, too.

As we shall see, it is possible to reach our goal by letting 5,7,9 be non-absorbing degrees.

Let i, j be non-absorbing degrees, $i \in [5, 10]$ and $j \in [12, \infty)$. We require $(3, j, j) \notin \mathcal{T}$ for j large enough. If, in the same time, $(3, i, j) \notin \mathcal{T}$, then we have $0 \leq 2C(3, i, j) + C(3, j, j) = 4c_3(i) + 8c_3(j) - (4c(i, 3, j) + 2c(j, 3, j)) \leq 4c_3(i) + 8c_3(j)$; the non-negativity of the sum in the brackets follows from $(*)$ for $(d_1, d_2, d_3) = (i, j, j)$. Putting

$$t_i := \left\lceil \frac{8i}{i-4} \right\rceil \qquad \text{for } \; i \in [5, 10]$$

we see that

$$c_3(i) + 2c_3(j) \geq 0 \Leftrightarrow j \geq t_i \qquad \text{for any } i \in [5, 10].$$

Thus we cannot expect nothing better than $(3, i, j) \notin \mathcal{T}$ for $i \in [5, 10]$ and $j \in [t_i, \infty)$.

For $i = 11$ and $j \in [12, \infty)$ the above procedure cannot be applied, since 11 is an absorbing degree. However, as we want to cover Kotzig's theorem, we put formally $t_{11} := 12$.

We define $c(i, 3, j)$ as follows:

$$\begin{aligned} c(i, 3, j) &:= 0 && \text{for } \; i, j = 3, 4, \\ c(3, 3, j) &:= 0 && \text{for } \; j \in [5, 11], \\ c(3, 3, j) &:= c_3(j) && \text{for } \; j \in [12, \infty), \end{aligned}$$

$$
\begin{aligned}
&c(4,3,j) := \tfrac{1}{2}c_3(t_j) && \text{for } \ j \in [5,11],\\
&c(4,3,j) := \tfrac{1}{2}c_3(j) && \text{for } \ j \in [12,\infty),\\
&c(i,3,j) := c_3(t_i) + c_3(t_j) && \text{for } \ i,j \in [5,11],\\
&c(i,3,j) := -c_3(j) && \text{for } i \in [5,11], \quad j \in [12, t_i - 1],\\
&c(i,3,j) := -c_3(t_i) && \text{for } i \in [5,11], \quad j \in [t_i, \infty),\\
&c(i,3,j) := c_3(i) + c_3(j) && \text{for } \ i,j \in [12,\infty).
\end{aligned}
$$

Let us check that $(*)$ is fulfilled for $n = 3$, i.e., that

$$s(i,j,k) = c(i,3,j) + c(j,3,k) + c(k,3,i) \geq 0 \qquad \text{for any } i,j,k \in [3,\infty).$$

For this purpose we put

$$S_1 := [3,4], \qquad S_2 := [5,11], \qquad S_3 := [12,\infty), \qquad S_4 := [5,\infty),$$

$$s_l := |S_l \cap (\{i\} \cup \{j\} \cup \{k\})| \qquad \text{for } l = 1,2,3;$$

for simplicity we shall write s instead of $s(i,j,k)$.

(1) If $s_1 \geq 2$, without loss of generality $i,j \in S_1$ and

$$
\begin{aligned}
s = c(i,3,k) + c(j,3,k) &= 0 && \text{for } k \in S_1,\\
&\geq \min\{0, \tfrac{1}{2}c_3(t_k), c_3(t_k)\} = 0 && \text{for } k \in S_2,\\
&\geq \min\{c_3(k), \tfrac{3}{2}c_3(k), 2c_3(k)\} = c_3(k) \geq 0 && \text{for } k \in S_3.
\end{aligned}
$$

(2) If $s_2 \geq 2$ and $i,j \in S_2$, then $s = c_3(t_i) + c(i,3,k) + c_3(t_j) + c(j,3,k) \geq 0$, since for any $p \in S_2$ we have

$$
\begin{aligned}
&c_3(t_p) + c(p,3,k)\\
&\geq c_3(t_p) + \min_{q=1,2,3} \min_{r \in S_q} c(p,3,r) = c_3(t_p) + \min\{0, \tfrac{1}{2}c_3(t_p), -c_3(t_p)\} = 0.
\end{aligned}
$$

(3) If $s_3 \geq 2$ and $i,j \in S_3$, then $s = c_3(i) + c(i,3,k) + c_3(j) + c(j,3,k) \geq 0$, as for $p \in S_3$ it holds

$$
\begin{aligned}
&c_3(p) + c(p,3,k)\\
&\geq c_3(p) + \min\{\min\{c_3(p), \tfrac{1}{2}c_3(p)\}, \min\{-c_3(p), \min_{q \in S_2 : t_q \leq p}(-c_3(t_q))\},\\
&\min_{q \in S_3} c(p,3,q)\} = c_3(p) + \min\{c_3(p), -c_3(p), c_3(p)\} = 0.
\end{aligned}
$$

(4) Let $s_1 = s_2 = s_3$ and $i \in S_1$, $j \in S_2$, $k \in S_3$. For $k \in [12, t_j - 1]$ we have $s \geq -2c_3(k) + \min\{2c_3(k), c_3(t_j) + c_3(k)\} = 0$, while the assumption $k \in [t_j, \infty)$ leads to $s \geq -2c_3(t_j) + \min\{2c_3(k), c_3(t_j) + c_3(k)\} = c_3(k) - c_3(t_j) \geq 0$.

To define $c(i,11,j)$, we set

$$\begin{array}{rll} c_{11}(i) &:= \frac{5}{44} & \text{for } i \in \{3\} \cup [6,\infty),\\ c_{11}(4) &:= \frac{3}{44}, & \\ c_{11}(5) &:= \frac{1}{4}B'(5,5,11) = \frac{3}{220}, & \\ c(i,11,j) &:= c_{11}(i) + c_{11}(j) & \text{for } (i,j) \in S_1^2 \cup S_4^2,\\ c(3,11,5) &:= \frac{17}{220}, & \\ c(3,11,11) &:= \frac{1}{4}B'(3,11,11) = -\frac{1}{44}, & \\ c(3,11,j) &:= \frac{1}{2}B'(3,11,j) = \frac{5}{22} - \frac{3}{j} & \text{for } j = 12, 13,\\ c(3,11,j) &:= -c_3(11) = \frac{1}{44} & \text{for } j \in [14,\infty),\\ c(4,11,5) &:= \frac{7}{220}, & \\ c(4,11,j) &:= \frac{1}{44}, & \text{for } j = 11, 12, 13,\\ c(i,11,j) &:= 0 & \text{for other pairs } (i,j) \in S_1 \times S_4. \end{array}$$

From these definitions we obtain

$$m := \min_{i,j\in[3,\infty)} c(i,11,j) = c(3,11,11) = c(3,11,12) = -\frac{1}{44},$$
$$(i \le j \wedge c(i,11,j) < 0) \Rightarrow (i,j) \in \{(3,11),(3,12),(3,13)\}.$$

To see that $(*)$ is true for $n = 11$ note that there exists $i \in [1,11]$ such that $(d_i, d_{i+1}) \in S_1^2 \cup S_4^2$, without loss of generality $i = 11$. Since then

$$\begin{aligned} s(d_1,\ldots,d_{11}) &= \sum_{i=1}^{10} c(d_i,11,d_{i+1}) + c_{11}(d_{11}) + c_{11}(d_1)\\ &= c_{11}(d_1) + \sum_{i=1}^{5} c(d_i,11,d_{i+1}) + c_{11}(d_{11})\\ &\quad + \sum_{i=1}^{5} c(d_{12-i},11,d_{11-i})\\ &= \breve{s}(d_1,d_2,d_3,d_4,d_5,d_6) + \breve{s}(d_{11},d_{10},d_9,d_8,d_7,d_6), \end{aligned}$$

where

$$\breve{s}(d_1,d_2,d_3,d_4,d_5,d_6) := c_{11}(d_1) + \sum_{j=1}^{5} c(d_j,11,d_{j+1}),$$

it suffices to show that $\breve{s}(d_1,d_2,d_3,d_4,d_5,d_6) \ge 0$ for any $(d_1,d_2,d_3,d_4,d_5,d_6) \in [3,\infty)^6$; we shall write $\breve{s}$ instead of $\breve{s}(d_1,d_2,d_3,d_4,d_5,d_6)$.

If $d_1 = 3$ or $d_1 \ge 6$, then $\breve{s} \ge \frac{5}{44} + 5m = 0$.

If $d_1 = 4$ and $d_2 \in \{3,11,12,13\}$, then $\breve{s} \ge \frac{3}{44} + \min\{\frac{2}{11}, \frac{1}{44}\} + 4m = 0$.

If $d_1 = 4$ and $d_2 \in [4,10] \cup [14,\infty)$, then $\breve{s} \geq \frac{3}{44} + 2 \cdot 0 + 3m = 0$.

Finally, suppose $d_1 = 5$. If there exists $j \in [1,5]$ such that $(d_j, d_{j+1}) \in [3,4]^2 \cup [6,\infty)^2$, then $\breve{s} \geq \frac{3}{220} + 2 \cdot \frac{3}{44} + 4m = \frac{13}{220}$. If there exists $j \in [1,5]$ such that $(d_j, d_{j+1}) = (5,3)$, then $\breve{s} \geq \frac{3}{220} + \frac{17}{220} + 4m = 0$. If there exists $j \in [1,5]$ such that $d_k = 5$ for any $k \in [1,j]$ and $d_{j+1} = 4$, then, since

$$\min_{p\in[3,\infty)} c(4,11,p) = 0 = \min_{p,q\in[3,\infty)} (c(p,11,4) + c(4,11,q)),$$

we have $\breve{s} \geq (2j-1) \cdot \frac{3}{220} + \frac{7}{220} + \max\{0, 3-j\} \cdot m \geq \frac{1}{22} + 2m = 0$. Of course, $(d_1, \ldots, d_6) = (5)^6$ gives $\breve{s} = \frac{3}{20}$.

Thus, since $(*)$ is fulfilled, the set $\mathcal{T}$ of types T with $C(T) < 0$ is an unavoidable set for normal planar maps. Which is its structure?

As for any $i \in [3,\infty)$ $C(3,3,i) = B'(3,3,i) - 4c(3,3,i) - 2c(3,i,3) \leq B'(3,3,i) = -1 - \frac{6}{i} < 0$ and $C(3,4,i) = B'(3,4,i) - 2c(4,3,i) - 2c(3,i,4) \leq B'(3,4,i) = -\frac{1}{2} - \frac{6}{i} < 0$, the types $(3,3,i)$, $i \in [3,\infty)$, and $(3,4,i)$, $i \in [4,\infty)$, are in $\mathcal{T}$.

Let $i \in [5,10]$. If $j \in [i,11]$, then $C(3,i,j) = 2c_3(i) - 2c_3(t_i) + 2c_3(j) - 2c_3(t_j) < 0$, since $c_3(k)$ is an increasing function of k and $i < t_i$, $j \leq t_j$. If $j \in [12, t_i - 1]$, then $C(3,i,j) = 2c_3(i) + 4c_3(j) \leq 2c_3(i) + 4c_3(t_i - 1) < 0$. Finally, for $j \in [t_i,\infty)$ we have $C(3,i,j) = 2c_3(i) + 2c_3(j) + 2c_3(t_i) \geq 2c_3(i) + 4c_3(t_i) \geq 0$. Thus we see that

$$(3,i,j) \in \mathcal{T} \Leftrightarrow j \in [i, t_i - 1] \qquad \text{for } i \in [5,10].$$

As $C(3,11,i) = 0$ for $i = 11,12,13$, and $C(3,11,i) = 2c_3(i) > 0$ for $i \in [14,\infty)$, there are no types $(3,11,i)$, $i \geq 11$, in $\mathcal{T}$.

For $i,j \in [12,\infty)$, $i \leq j$, we have $C(3,i,j) = 0$ and $(3,i,j) \notin \mathcal{T}$.

If $4 \leq i \leq j \leq k$, then $c(i,j,k) = c(j,k,i) = c(k,i,j) = 0$, hence $C(i,j,k) = B'(i,j,k) = 3(1 - \frac{2}{i} - \frac{2}{j} - \frac{2}{k})$ and we obtain the same types as are in $\mathcal{L}$, i.e., $(4,4,k)$, $k \in [4,\infty)$, and those determined by $9 \leq i + j \leq 11$ and $k < \frac{2ij}{ij-2i-2j}$.

We now pass to types of faces of degree ≥ 4. If $T = (d_1, \ldots, d_n)$, then

$$\begin{aligned} C(T) &= 2n - 6 + \sum_{i=1}^{n} \frac{d_i - 6}{d_i} - 2\sum_{i=1}^{n} c(d_{i-1}, d_i, d_{i+1}) \\ &= \sum_{i=1}^{n} (3 - \frac{6}{d_i} - 2c(d_{i-1}, d_i, d_{i+1})) - 6. \end{aligned}$$

Putting

$$a(i,j,k) := 3 - \frac{6}{j} - 2c(i,j,k),$$

$$\sigma(d_1,\dots,d_n) := \sum_{i=1}^{n} a(d_{i-1},d_i,d_{i+1}),$$

we obtain the following equivalence:

$$(d_1,\dots,d_n) \in \mathcal{T} \Leftrightarrow \sigma(d_1,\dots,d_n) \geq 6;$$

we shall use σ instead of $\sigma(d_1,\dots,d_n)$. Note that

$$a(i,j,k) \geq 3-\frac{6}{j}-2\sup_{p,q\in[3,\infty)} c(p,j,q) \geq 3-\frac{6}{3}-2\cdot\frac{1}{2} = 0 \text{ for any } (i,j,k)\in[3,\infty)^3.$$

Moreover, as

$$a(i,11,k) \geq 3-\frac{6}{11}-2\max_{p,q\in[3,\infty)} c(p,11,q) = \frac{27}{11}-2\cdot\frac{5}{22} = 2,$$

and $a(i,j,k) = 3-\frac{6}{j}$ for any $j \in [4,10]\cup[12,\infty)$, we have

$$\begin{aligned} a(i,j,k) &\geq \frac{5}{2} \text{ for } j \in [12,\infty),\\ &\geq \frac{3}{2} \text{ for } j \in [4,11],\\ &\geq \frac{9}{5} \text{ for } j \in [5,11]. \end{aligned}$$

Define

$$m(p,q) := |\{i \in [1,n] : p \leq d_i < q\}|, \qquad m(p) := |\{i \in [1,n] : d_i = p\}|.$$

(1) If $m(12,\infty) \geq 3$, then $\sigma \geq 3\cdot\frac{5}{2} = \frac{15}{2} > 6$ and $T \notin \mathcal{T}$.
(2) $m(12,\infty) = 2$
(21) $m(4,12) \geq 1$ yields $\sigma \geq 2\cdot\frac{5}{2}+\frac{3}{2} = \frac{13}{2} > 6$.
(22) $m(4,12) = 0$
(221) If $m(3) \geq 3$, there exists $i \in [1,n]$ such that $d_i = d_{i+1} = 3$.
However, as

$$a(p,3,3) = a(3,3,p) > 1-\sup_{q\in[3,\infty)} c(3,3,q) = \frac{1}{2} \qquad \text{for } p \in [3,\infty),$$

we obtain $a(d_{i-1},d_i,d_{i+1})+a(d_i,d_{i+1},d_{i+2}) > 2\cdot\frac{1}{2}$ and $\sigma > 1+2\cdot\frac{5}{2} = 6$.

(222) $m(3) = 2$
(2221) For $T = (3, 3, d_3, d_4)$, $12 \le d_3 \le d_4$, we have $\sigma > 6$, as in (221).
(2222) If $T = (3, d_2, 3, d_4)$, $12 \le d_2 \le d_4$, then $\sigma = 6$.
(3) $m(12, \infty) = 1$
(31) From $m(5, 12) \ge 2$ it follows $\sigma \ge \frac{5}{2} + 2 \cdot \frac{9}{5} = \frac{61}{10} > 6$.
(32) $m(5, 12) = 1$
(321) $m(4) \ge 1$ gives, due to $\sup\limits_{\min\{p,q\} \le 11} c(p, 3, q) = \frac{1}{2}$, $\sigma \ge \frac{5}{2} + \frac{9}{5} + \frac{3}{2} + \frac{1}{2} = \frac{63}{10} > 6$.
(322) Provided $m(4) = 0$ there exists $i \in [1, n]$ such that $\{d_i, d_{i+1}\} = \{3, p\}$ with $p \in [5, 11]$. As $c(p, 3, q) \le 0$ for any $q \in \{3\} \cup [12, \infty)$, we have $a(d_{j-1}, d_j, d_{j+1}) \ge 1$ for at least one $j \in [1, n]$ with $d_j = 3$ (more precisely, $j \in \{i, i + 1\}$).
(3221) $m(3) \ge 3$ means that $\sigma \ge \frac{5}{2} + \frac{9}{5} + 1 + 2 \cdot \frac{1}{2} = \frac{63}{10}$.
(3222) $m(3) = 2$
(32221) For $T = (3, d_2, 3, d_4)$, $d_2 \in [5, 11]$, $d_4 \in [12, \infty)$, we have the same lower bound for σ as in (3221), since $1 + 2 \cdot \frac{1}{2}$ can be replaced with $2 \cdot 1$.
(32222) If $T = (3, 3, d_3, d_4)$, $d_3 \in [5, 11]$, $d_4 \in [12, \infty)$, then $\sigma = \frac{15}{2} - \frac{6}{d_3} \ge \frac{63}{10}$.
(33) The assumption $m(5, 12) = 0$ leads to $a(d_{i-1}, d_i, d_{i+1}) > \frac{1}{2}$ for any $i \in [1, n]$.
(331) $m(4) \ge 2$ gives $\sigma > \frac{5}{2} + 2 \cdot \frac{3}{2} + \frac{1}{2} = 6$.
(332) $m(4) = 1$
(3321) If $m(3) \ge 3$, there exists $i \in [1, n]$ such that $d_i = 3$ and $\{d_{i-1}, d_{i+1}\} = \{3, 4\}$. As $a(3, 3, 4) = a(4, 3, 3) = 1$, we obtain $\sigma > \frac{5}{2} + \frac{3}{2} + 1 + 2 \cdot \frac{1}{2} = 6$.
(3322) $m(3) = 2$
(33221) For $T = (3, 3, 4, d_4)$, $d_4 \in [12, \infty)$, we have $\sigma = 6$.
(33222) For $T = (3, 4, 3, d_4)$, $d_4 \in [12, \infty)$, it holds $\sigma = 6$.
(333) $m(4) = 0$
(3331) $m(3) \ge 4$ and $T = (3)^{n-1}(d_n)$, $d_n \in [12, \infty)$, imply $\sigma = n+1+\frac{6}{d_n} \ge \frac{13}{2} > 6$.
(3332) For $m(3) = 3$ and $T = (3, 3, 3, d_4)$, $d_4 \in [12, \infty)$, it follows, from $\sigma = 5 + \frac{6}{d_4} \le \frac{11}{2} < 6$, that $T \in \mathcal{T}$.
(4) In the case $m(12, \infty) = 0$ we denote by $m^+(3)$ or $m^-(3)$ the number of those triples (d_{i-1}, d_i, d_{i+1}) for which $d_i = 3$ and the set $\{d_{i-1}\} \cup \{d_{i+1}\}$ does or does not contain 3, respectively. As $c(3, 3, p) = 0$ for $p \in [3, 11]$ and $c(q, 3, r) \le 2c_3(t_5) = \frac{7}{20}$ for $q, r \in [4, 11]$, we have $a(3, 3, p) = 1$, $a(q, 3, r) \ge \frac{3}{10}$ and

$$\sigma \ge \frac{3}{10} m^-(3) + m^+(3) + \frac{3}{2}(n - m^-(3) - m^+(3)) = \frac{3}{2} n - \frac{6}{5} m^-(3) - \frac{1}{2} m^+(3).$$

(41) If $\frac{6}{5}m^-(3) + \frac{1}{2}m^+(3) \le \frac{3}{2}n - 6$, then $\sigma \ge 6$ and $T \notin \mathcal{T}$.
(42) $\frac{6}{5}m^-(3) + \frac{1}{2}m^+(3) > \frac{3}{2}n - 6$
(421) For $n \ge 7$ we have

$$\frac{6}{5}m^-(3) + \frac{6}{5}m^+(3) > \frac{3}{2}n - 6,$$

$$m(3) = m^-(3) + m^+(3) > \frac{5}{6}\left(\frac{3}{2}n - 6\right) = \frac{n}{2} + \frac{3n-20}{4} > \frac{n}{2}.$$

As $m(3) > \frac{n}{2}$, $m^-(3)$ is upper bounded by $n - m(3) - 1$, which yields $2m^-(3) + m^+(3) \le n - 1$, hence

$$\frac{3}{5}\Big(n-1\Big) \ge \frac{6}{5}m^-(3) + \frac{3}{5}m^+(3) \ge \frac{6}{5}m^-(3) + \frac{1}{2}m^+(3) > \frac{3}{2}n - 6,$$

and, as a consequence, $n < 6$, a contradiction.
(422) $n = 6$
(4221) For $m(5,12) \ge 3$ it holds $\sigma \ge 3 \cdot \frac{9}{5} + 3 \cdot \frac{3}{10} = \frac{63}{10}$.
(4222) $m(5,12) = 2$
(42221) For $m(4) \ge 1$ we obtain $\sigma \ge 2 \cdot \frac{9}{5} + \frac{3}{2} + 3 \cdot \frac{3}{10} = 6$.
(42222) $m(4) = 0$ implies $m^+(3) \ge 3$ and $\sigma \ge 2 \cdot \frac{9}{5} + 3 \cdot 1 = \frac{33}{5} > 6$.
(4223) For the case $m(5,12) = 1$ note that $\max_{p \in [3,4], q \in [5,11]} c(p,3,q) = \frac{1}{2}c_3(t_5) = \frac{7}{80}$.
(42231) $m(4) \ge 1$ means that $\sigma \ge \frac{9}{5} + \frac{3}{2} + 4(1 - 2 \cdot \frac{7}{80}) = \frac{33}{5}$.
(42232) If $m(4) = 0$ and $T = (3)^5(d_6)$, $d_6 \in [5,11]$, then $\sigma = 8 - \frac{6}{d_6} \ge \frac{34}{5}$.
(4224) From $m(5,12) = 0$ it follows $\sigma = m(3) + \frac{3}{2}m(4) \ge m(3) + m(4) = 6$.
(423) The remaining case, $n = 4, 5$, was analysed, thanks to its finiteness, following from $m(12, \infty) = 0$, by a computer.

The described analysis led to

Theorem 4. *For the class of normal planar maps the following sequences form an unavoidable set of face types:*
$(3,i,j), i \in [3,4], j \in [i,\infty)$, $(4,4,i), i \in [4,\infty)$, $(3,3,3,i), i \in [3,\infty)$,
$(3,5,i), i \in [5,39]$, $(3,6,i), i \in [6,23]$, $(3,7,i), i \in [7,18]$, $(3,8,i), i \in [8,15]$,
$(3,9,i), i \in [9,14]$, $(3,10,i), i \in [10,13]$, $(4,5,i), i \in [5,19]$, $(4,6,i), i \in [6,11]$,
$(4,7,i), i \in [7,9]$, $(5,5,i), i \in [5,9]$, $(5,6,i), i = 6,7$,
$(3,3,4,i), i \in [4,11], (3,3,5,i), i \in [5,7], (3,4,3,i), i \in [4,11]$,
$(3,4,4,i), i \in [4,6], (3,4,5,i), i = 4,5, (3,5,3,i), i \in [5,11]$,
$(3,5,4,i), i \in [5,7], (3,5,i,5), i = 5,6, (3,6,3,i), i \in [6,11]$,
$(3,7,3,i), i = 7,8,9,11$,
$(3,3,3,3,i), i \in [3,5], (3,3,5,3,5)$. ∎

Corollary 5 (Borodin [1]). *In any normal planar map there exists an edge of weight at most* 13. ∎

Using Corollary 3 and Theorem 4 we see that the minimum of Problem 1 is in fact equal to 4. Having this in mind, the following could be of interest.

Problem 2. Determine the minimum $\sum_{i=3}^{\infty} b_i^-(\mathcal{T})$ for an unavoidable set $\mathcal{T}$ of face types for normal planar maps with the infinite characteristic $(3,1)$.

The set $\mathcal{T}$ from Theorem 4 has the finite characteristic $(114, 46, 4)$ so that the minimum of Problem 2 is at most 164.

4. Cyclic chromatic number of planar maps

If we do not insist on minimizing $\sum_{i=3}^{\infty} b_i^-(\mathcal{T})$ in a good unavoidable set $\mathcal{T}$, we can obtain unavoidable sets, which may be useful for a solution of other than structural type problems. E.g., by allowing 5 to be an absorbing degree it is possible to obtain an unavoidable set $\bar{\mathcal{T}}$ for normal planar graphs whose types do not contain large degrees except for types in "inexcludable" four infinite series. On the other hand, hexagonal types appear in $\bar{\mathcal{T}}$, while $b_6^-(\mathcal{T}) = 0$.

Theorem 6. *For the class of normal planar maps the following sequences form an unavoidable set of face types:*
$(3,i,j), i \in [3,4], j \in [i,\infty)$, $(4,4,i), i \in [4,\infty)$, $(3,3,3,i), i \in [3,\infty)$, $(3,5,i), i \in [5,23]$, $(3,6,i), i \in [6,23]$, $(3,7,i), i \in [7,18]$, $(3,8,i), i \in [8,15]$, $(3,9,i), i \in [9,14]$, $(3,10,i), i \in [10,13]$, $(4,5,i), i \in [5,11]$, $(4,6,i), i \in [6,11]$, $(4,7,i), i \in [7,9]$, $(5,5,i), i \in [5,7]$, $(5,6,i), i = 6,7$,
$(3,3,4,i), i \in [4,11]$, $(3,3,5,i), i \in [5,7]$, $(3,4,3,i), i \in [4,11]$, $(3,4,4,i), i \in [4,6]$, $(3,4,5,i), i \in [4,6]$, $(3,4,6,5)$, $(3,5,3,i), i \in [5,17]$, $(3,5,4,i), i \in [5,12]$, $(3,5,5,i), i \in [5,7]$, $(3,5,6,i), i = 5,6$, $(3,5,7,5)$, $(3,6,3,i), i \in [6,11]$, $(3,7,3,i), i = 7,8,9,11$, $(4,4,i,5), i = 4,5$, $(4,5,i,5), i = 4,5$,
$(3,3,3,3,i), i \in [3,5]$, $(3,3,3,i,5), i = 4,5$, $(3,3,4,3,5)$, $(3,3,5,3,i)$, $i = 5,6,7,8,9,11$, $(3,3,5,4,5)$, $(3,4,3,i,5), i = 4,5$, $(3,4,4,3,5)$, $(3,4,5,3,i), i = 5,6$, $(3,4,5,4,5)$, $(3,4,6,3,5)$, $(3,5,3,5,i), i \in [5,7]$, $(3,5,4,i,5), i = 4,5$, $(3,5,5,3,i), i = 6,7$,
$(3,i,3,5,3,5), i \in [3,5]$, $(3,5,3,5,3,i), i = 6,7$, $(3,5,3,5,4,5)$.

Proof. Put

$$\begin{aligned}
\bar{c}_5(i) &:= \tfrac{1}{5} && \text{for } i \in [3,4] \cup [12,\infty),\\
\bar{c}_5(i) &:= \tfrac{3}{40} && \text{for } i = 5,8,9,\\
\bar{c}_5(i) &:= \tfrac{3}{80} && \text{for } i = 6,10,11,\\
\bar{c}_5(7) &:= 0.
\end{aligned}$$

We define a new mapping $\bar{c} : [3, \infty)^3 \to \mathbb{Q}$ by $\bar{c}(i,j,k) := c(i,j,k)$ for $j \neq 5$ and

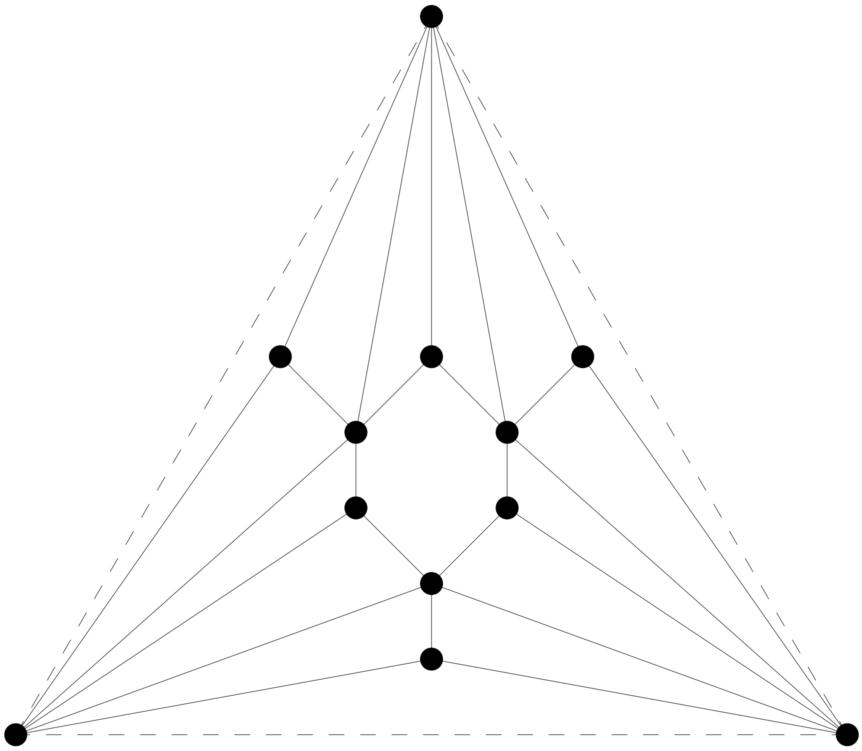
$$
\begin{array}{ll}
\bar{c}(i,5,j) = \bar{c}_5(i) + \bar{c}_5(j) & \text{for } (i,j) \in [3,6]^2 \cup [7,\infty)^2, \\
\bar{c}(i,5,7) := \frac{1}{10}, & \text{for } i = 3,4, \\
\bar{c}(i,5,j) := \frac{1}{40} & \text{for } i = 3,4,\ j \in [8,11], \\
\bar{c}(i,5,j) := -\frac{1}{10} & \text{for } i = 3,4,\ j \in [12,\infty), \\
\bar{c}(5,5,7) := \frac{3}{80}, & \\
\bar{c}(5,5,j) := -\frac{3}{80} & \text{for } j = 8,9, \\
\bar{c}(5,5,j) := 0 & \text{for } j \in [10,\infty), \\
\bar{c}(6,5,j) := 0 & \text{for } j \in [7,11], \\
\bar{c}(6,5,j) := \frac{1}{16} & \text{for } j \in [12,\infty).
\end{array}
$$

If the alternative charge of an angle $a = (v_{i-1}, e_i, v_i, e_i, v_{i+1})$ is determined by $\bar{c}_a := \bar{c}(\deg v_{i-1}, \deg v_i, \deg v_{i+1})$, an analysis analogous to that applied for Theorem 4 leads to the unavoidable set described in the statement of Theorem 6. ■

Let M be a map. The *weight* of a face $f \in F(M)$ with $T = (d_1, \ldots, d_n) \in \tau(f)$ is defined by $\mathrm{wt}(f) := \sum_{i=1}^{n} d_i$ and the weight of M by $\min_{f \in F(M)} \mathrm{wt(f)}$.

Corollary 7. *If a normal planar map M does not contain faces of types $(3,3,i)$, $(3,4,i)$, $(4,4,i)$, $(3,3,3,i)$, $i \geq 3$, then the weight of M is at most* 32. ■

One can wonder why hexagonal types appear in the statement of Theorem 6 in spite of the fact that by Euler's formula only faces of sizes ≤ 5 are necessarily present in a normal planar map. If we insert a configuration of Figure 1 into each face of an icosahedron map (dashed lines stand for edges of the original map), we obtain a map with faces only of types $(3,5,3,5,3,5)$, $(3,5,30)$ and $(3,30,30)$, from which the hexagonal type only occurs in the list of Theorem 6. Evidently, the above construction can be applied to any plane triangulation with minimum degree 5 (and even with minimum degree 4) to create a map with analogous face types property.



Let M be a 2-connected planar map. A *cyclic coloration* of M (introduced in Ore and Plummer [20]) is an assignment of colours to the vertices of M such that for any face all its vertices receive different colours. The *cyclic chromatic number* of M is the minimum number of colours in any cyclic coloration of M.

Plummer and Toft [21] obtained some upper bounds for the cyclic chromatic number of 3-connected planar maps. If we use in the proofs of Theorems 3.1, 3.2 and 3.3 of [21] our Theorem 6 (for our statements (i) – (vi), see below) or Theorem 4 (for (vii) – (viii)) instead of Lebesgue's result, by the same method we obtain the following theorem which improves the corresponding Theorems 3.1 and 3.3 of [21].

Theorem 8. *Let M be a* 3*-connected planar map with maximum face degree d and cyclic chromatic number n. Then*

(i) *if* $d \geq 24$*, then* $n \leq d+3$*;*
(ii) *if* $d \geq 19$*, then* $n \leq d+4$*;*
(iii) *if* $d \geq 16$*, then* $n \leq d+5$*;*
(iv) *if* $d \geq 15$*, then* $n \leq d+6$*;*
(v) *if* $d \geq 14$*, then* $n \leq d+7$*;*
(vi) *if* $d \geq 10$*, then* $n \leq d+8$*;*
(vii) *if* $d \leq 9$*, then* $n \leq d+7$*;*
(viii) *if* $d \leq 8$*, then* $n \leq d+6$*.* ■

Note that Theorem 8(i) was announced (but probably not published yet) by Borodin, cf. Jensen and Toft [15, Chapter 1].

References

[1] O.V. Borodin, *On the total coloring of planar graphs*, J. reine angew. Math. **394** (1989) 180–185.

[2] O.V. Borodin, *Precise lower bound for the number of edges of minor weight in planar maps*, Math. Slovaca **42** (1992) 129–142.

[3] O.V. Borodin, *A structural theorem on planar maps and its application to coloring*, Diskret. Mat. **4** (1992) 60–65. (Russian)

[4] O.V. Borodin, *Structural properties of planar maps with the minimal degree* 5, Math. Nachr. **158** (1992) 109–117.

[5] O.V. Borodin, *Joint extension of two theorems of Kotzig on* 3*-polytopes*, Combinatorica **13** (1993) 121–125.

[6] O.V. Borodin, *The structure of edge neighbourhoods in plane graphs and the simultaneous coloring of vertices, edges and faces*, Mat. Zametki **53** (1993) 35–47. (Russian)

[7] O.V. Borodin, *Simultaneous coloring of edges and faces of plane graphs*, Discrete Math. **128** (1994) 21–33.

[8] O.V. Borodin, *Triangles with restricted degree sum of their boundary vertices in plane graphs*, Discrete Math. **137** (1995) 45–51.

[9] O.V. Borodin and D. P. Sanders, *On light edges and triangles in planar graphs of minimum degree five*, Math. Nachr. **170** (1994) 19–24.

[10] B. Grünbaum, *Acyclic coloring of planar graphs*, Israel J. Math. **14** (1973) 390–408.

[11] B. Grünbaum, *New views on some old questions of combinatorial geometry*, in.: Proc. Internat. Colloq. Combin. Theory (Rome, 1973) Accad. Naz. Lincei Rome (1976) 451–468.

[12] B. Grünbaum and G.C. Shephard, *Analogues for tilings of Kotzig's theorem on minimal weights of edges*, Ann. Discrete Math. **12** (1982) 129–140.

[13] J. Ivančo, *The weight of a graph*, Ann. Discrete Math. **51** (1992) 113–116.

[14] S. Jendroľ and Z. Skupień, *On the vertex/edge distance colourings of planar graphs*, submitted.

[15] T.R. Jensen and B. Toft, Graph Coloring Problems (John Wiley&Sons, Inc. New York, 1995).

[16] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat.-Fyz. Časopis Sloven. Akad. Vied **5** (1955) 101–103. (Slovak)

[17] A. Kotzig, *On the theory of Euler polyhedra*, Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963) 20–31. (Russian)

[18] A. Kotzig, *Extremal polyhedral graphs*, Ann. New York Acad. Sci. **319** (1979) 569–570.

[19] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (1940) 19–43.

[20] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, in: Recent Progress in Combinatorics (Proceedings of the Third Waterloo Conference on Combinatorics), Academic Press, New York (1969) 287–293.

[21] M.D. Plummer and B. Toft, *Cyclic coloration of* 3*-polytopes*, J. Graph Theory **11** (1987) 507–515.

[22] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.

[23] J. Zaks, *Extending Kotzig's theorem*, Israel J. Math. **45** (1983) 281–296.

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