UNAVOIDABLE SET OF FACE TYPES
FOR PLANAR MAPS

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Abstract

The type of a face \( f \) of a planar map is a sequence of degrees of vertices of \( f \) as they are encountered when traversing the boundary of \( f \). A set \( T \) of face types is found such that in any normal planar map there is a face with type from \( T \). The set \( T \) has four infinite series of types as, in a certain sense, the minimum possible number. An analogous result is applied to obtain new upper bounds for the cyclic chromatic number of 3-connected planar maps.

Keywords: normal planar map, plane graph, type of a face, unavoidable set, cyclic chromatic number.

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1. Introduction

It is an old classical consequence of the famous Euler’s polyhedral formula that a normal planar map contains a vertex of degree \( \leq 5 \), a face of degree \( \leq 5 \) and also a 3-valent vertex or a triangle.

A face of a map can be characterized by its type, a sequence of degrees of its vertices. Lebesgue [19] specified a set of small face types which intersects the face type set of any normal planar map. (For this Lebesgue’s result and its application see Plummer and Toft [21].)

Kotzig [16] proved that each 3-connected normal planar map contains an edge of weight (the degree sum of its endvertices) at most 13; the condition of 3-connectedness can be abandoned, due to Borodin [1] (the same result was announced by Barnette, see Grünbaum [11]).

At present many results concerning the structure of planar maps are known. For example, Kotzig’s result was generalized and strengthened in several directions, see Borodin [2,5], Grünbaum and Shephard [12], Ivančo
M. Horňák and S. Jendrol’ [13], Zaks [23]. Recently, unifying and strengthening Kotzig’s results [18], Borodin [8] has proved that any planar triangulation without vertices of degree 4 contains either a triangle of weight (the degree sum of its incident vertices) at most 29 incident with a 3-valent vertex or a triangle whose weight does not exceed 17.

A sharp inequality for the number of triangles of weight at most 17 in planar maps with minimum degree 5 was found by Borodin [4]. Edges of small weights in planar maps of minimum degree 5 are investigated in a very recent paper by Borodin and Sanders [9]. Both the above mentioned papers complete the work contributed to by many authors, among others Grünbaum [10], Kotzig [17], Fisk (see [12]), Wernicke [22].

Many structural results on planar maps have been obtained by solving some colouring problems, see e.g. Borodin [1,3,6,7], Jendrol’ and Skupień [14].

The main aim of this paper is to prove an analogue of Lebesgue’s theorem, which is optimal in a certain sense.

2. Fundamentals

For integers $p, q$ we denote by $[p, q]$ the set of all integers $i$, $p \leq i \leq q$, and by $[p, \infty)$ the set of all integers $i \geq p$.

A finite sequence $Q$ is said to be equivalent to a finite sequence $P$ if $Q$ can be obtained from $P$ using rotation and/or mirror image. Thus, if $P = (p_1, \ldots, p_n)$, then $Q = (p_{i+1}, \ldots, p_{n+i})$ or $Q = (p_{n-i}, \ldots, p_{1-i})$ for some $i \in [0, n-1]$, where indices are taken modulo $n$. (We use this “modulo convention” throughout the whole paper.) Let $P, P_1, P_2$ be finite sequences and let $m \in [1, \infty)$. We denote by $P_1 P_2$ the concatenation of $P_1$ and $P_2$ (in that order), by $P^m$ the $m$-fold concatenation of $P$’s and by $\text{len}(P)$ the length of $P$.

Let $M$ be a map on a 2-manifold, i.e. a 2-cell embedding of a graph, in which loops and multiple edges are allowed. $V(M)$, $E(M)$ and $F(M)$ are the vertex set, the edge set and the face set of $M$, respectively, $\deg c$ is the degree of $c \in V(M) \cup F(M)$. $M$ is called normal if $\deg c \geq 3$ for any $c \in V(M) \cup F(M)$. An angle of a face $f \in F(M)$ with centre $v \in V(M)$ is an alternating quintuple $(u, d, v, e, w)$ of consecutive vertices and edges of $f$ which are encountered when moving along the boundary of $f$, i.e., the curve consisting of all edges incident with $f$. The centre of an angle $a$ will be denoted by $\dot{a}$. Let $A(f)$ be the set of all angles of $f$ and $A(v)$ the set of all angles with centre $v$. Evidently, $|A(v)| = 2 \deg v$ and $|A(f)| = 2 \deg f$ for any $v \in V(M)$ and $f \in F(M)$. Due to the normality of $M$ we know that
$A(v_1) \neq A(v_2)$ for any $v_1, v_2 \in V(M)$, $v_1 \neq v_2$ and $A(f_1) \neq A(f_2)$ for any $f_1, f_2 \in F(M)$, $f_1 \neq f_2$. Putting

$$A(M) := \{A(v) : v \in V(M)\} = \{A(f) : f \in F(M)\}$$

we see that there exists a natural bijection $\beta_M$ between the sets $\{(a, v) \in A(M) \times V(M) : a \in A(v)\}$ and $\{(a, f) \in A(M) \times F(M) : a \in A(f)\}$. Let $f$ be a face of degree $n$ and let $(v_1, \ldots, v_n)$ be a sequence of vertices of $f$ as they are encountered when traversing the boundary of $f$. Any sequence from the set $\tau(f)$ of all sequences equivalent to $(\deg v_1, \ldots, \deg v_n)$ is said to be a type of $f$.

Let $\mathcal{S}$ be the set consisting of all lexicographic minima of the set $\bigcup_{i=3}^{\infty}[3, \infty]^i$ (provided sequences of the same length are comparable only). We represent the set $\tau(f)$ by its representative in $\mathcal{S}$.

Let $\mathbb{M}$ be a class of normal maps on a 2-manifold. A set $\mathcal{T} \subseteq \mathcal{S}$ is said to be an unavoidable set of face types for $\mathbb{M}$ if for any $M \in \mathbb{M}$ there exists $T \in \mathcal{T}$ and $f \in F(M)$ such that $T \in \tau(f)$.

In 1940 Lebesgue [19] proved (in a dual form)

**Theorem 1.** For the class of normal planar maps the following sequences form an unavoidable set of face types:

$$(3, i, j), i \in [3, 6], j \in [i, \infty), (4, 4, i), i \in [4, \infty), (3, 3, 3, i), i \in [3, \infty), (3, 7, i), i \in [7, 41], (3, 8, i), i \in [8, 23], (3, 9, i), i \in [9, 17], (3, 10, i), i \in [10, 14], (3, 11, i), i \in [11, 13], (4, 5, i), i \in [5, 19], (4, 6, i), i \in [6, 11], (4, 7, i), i \in [7, 9], (5, 5, i), i \in [5, 9], (5, 6, i), i = 6, 7, (3, 3, 4, i), i \in [4, 11], (3, 3, 5, i), i \in [5, 7], (3, 4, 3, i), i \in [4, 11], (3, 4, 4, i), i = 4, 5, (3, 4, 5, 4), (3, 5, 3, i), i \in [5, 7], (3, 3, 3, 3, i), i \in [3, 5].$$

Note that in [19] an error occurred by omitting the types $(4, 4, i), i \in [4, \infty)$. Let $\mathcal{T}$ be an unavoidable set of face types for $\mathbb{M}$. A sequence $S = (s_1, \ldots, s_n) \in \bigcup_{i=3}^{\infty}[3, \infty]^i$ such that either $(s_n, \ldots, s_1) = S$ or $s_j < s_{n+1-j}$ for $j = \min\{i \in [1, n] : s_i \neq s_{n+1-i}\}$ is a $\mathcal{T}$-basic sequence if the set $\mathcal{T} \cap \{S(i) : i \in [3, \infty)\}$ is infinite. Let $B(\mathcal{T})$ be the set of all $\mathcal{T}$-basic sequences. For $i \in [3, \infty)$ set

$$b_i(\mathcal{T}) := \text{card}\{S \in B(\mathcal{T}) : \text{len}(S) = i - 1\},$$

$$b_i^-(\mathcal{T}) = \text{card}\{T \in \mathcal{T} - \{S(j) : S \in B(\mathcal{T}), j \in [3, \infty)\} : \text{len}(T) = i\}.$$ 

The sequences $\{b_i(\mathcal{T})\}_{i=3}^{\infty}$ and $\{b_i^-(\mathcal{T})\}_{i=3}^{\infty}$ are called the infinite and the finite characteristic of $\mathcal{T}$, respectively. If $b_i(\mathcal{T}) = 0$ for all $i \in [p + 1, \infty)$ or
\(b^{-}_i(T) = 0\) for all \(i \in [q+1, \infty)\), we present a corresponding characteristic simply as \((b_3(T), \ldots, b_p(T))\) or \((b^{-}_3(T), \ldots, b^{-}_p(T))\). For the Lebesgue’s unavoidable set \(L\), we see that \(B(L) = \{(3, i) : i \in [3,6]\} \cup \{(4,4), (3,3,3)\}\) and that \(L\) has the infinite characteristic \((5,1)\), and the finite characteristic \((99,25,3)\).

An unavoidable set \(T\) is \textit{good} if \(\sum_{i=3}^{\infty} b^{-}_i(T)\) is finite. Two good unavoidable sets \(T\) and \(T'\) can be compared as follows: \(T\) is \textit{more economical} than \(T'\) if \(\sum_{i=3}^{\infty} b_i(T) < \sum_{i=3}^{\infty} b_i(T')\); this means that \(T\) contains a smaller number of (naturally structured) infinite subsets than \(T'\) (and a finite “rest”). Thus, we can pose

**Problem 1.** Find the minimum of \(\sum_{i=3}^{\infty} b_i(T)\) for a good unavoidable set \(T\) of face types for normal planar maps.

We are going to show that the minimum of Problem 1 is equal to 4.

### 3. Main result

**Theorem 2.** Let \(T\) be a good unavoidable set of face types for normal planar maps.

(i) \(\{(3,4,4)\} \cup \{(4,4,i) : i \in [4,\infty)\} \cup \{(3,3,3,i) : i \in [3,\infty)\} \subseteq T\).

(ii) If \((3,3) \notin B(T)\), then \(\{(3)^i : i \in [4,\infty)\} \subseteq B(T)\).

(iii) If \((3,4) \notin B(T)\), then \(\{(3,3,4),(4,3,4)\} \subseteq B(T)\).

**Proof.** Let \(m \in [3,\infty), n \in [1,\infty), l \in [1,n]\) and let \(P = (p_1, \ldots, p_{2l}) \in [1,n]^{2l}\) be such a sequence that \(p_i \neq p_j\) for any \((i,j) \in [1,l]^2 \cup [l+1,2l]^2\), \(i \neq j\). Let \(G^n_m(P)\) be a planar graph with

\[
V(G^n_m(P)) = \{x_i : i \in [1,mn]\} \cup \{y_0, y_1\},
\]

\[
E(G^n_m(P)) = \bigcup_{i=1}^{mn} \{x_ix_{i+1}\} \cup \bigcup_{i=0}^{m-1} \bigcup_{j=0}^{1} \{x_{im+k}y_j : k = p_{ji+1}, \ldots, p_{ji+l}\}.
\]

A plane embedding of \(G^n_m(1,1)\) (an \(m\)-sided bipyramid) has only faces of type \((4,4m)\) and a plane embedding of \(G^n_m(1,2)\) (a dual of an \(m\)-sided antiprism) has only faces of type \((3,3,3,m)\); hence (i) follows from the finiteness of \(\sum_{i=3}^{\infty} b^{-}_i(T)\).

A plane embedding of \(G^n_m(1,\ldots,2n), n \geq 2\), has only faces of types \((3,3,mn)\) and \((3)^{n+2}(mn)\), so that \((3,3) \notin B(T)\) implies \((3)^{n+2}(mn) \in T\) for all sufficiently large \(m\) and \((3)^{n+2} \in B(T)\).
Finally, a plane embedding of $G^4_m(1, 2, 3, 1, 3, 4)$ has only faces of types $(3, 4, 3m)$ and $(4, 3, 4, 3m)$, while a plane embedding of $G^6_m(1, 2, 4, 5, 1, 3, 4, 6)$ has only faces of types $(3, 4, 4m)$ and $(3, 3, 4, 4m)$. It means that if $(3, 4) \notin B(T)$, then for every $m$ large enough $(4, 3, 4, 3m)$ as well as $(3, 3, 4, 4m)$ belong to $T$, so that $\{(3, 3, 4), (4, 3, 4)\} \subseteq B(T)$.

**Corollary 3.** $\sum_{i=3}^{\infty} b_i(T) \geq 4$ for any good unavoidable set $T$ of face types for normal planar maps and, if the equality holds, then $b_3(T) = 3$, $b_4(T) = 1$ and $B(T) = \{(3, 3), (3, 4), (4, 4), (3, 3, 3)\}$. ■

Thus our goal will be reached by finding an unavoidable set $T$ of face types for normal planar maps with the infinite characteristic $(3, 1)$ and with $\sum_{i=3}^{\infty} b_i^+(T)$ being finite.

One of well known corollaries of Euler’s formula for a planar map $M$ can be expressed as

$$\sum_{v \in V(M)} (6 - \deg v) + 2 \sum_{f \in F(M)} (3 - \deg f) = 12.$$ 

Thus, if we define the basic charge of a vertex $v \in V(M)$, of a face $f \in F(M)$ and of an angle $a \in A(M)$ by

$$b_v := \deg v - 6, \quad b_f := 2 \deg f - 6, \quad b_a := \frac{\deg \hat{a} - 6}{2 \deg \hat{a}},$$

then

$$\sum_{v \in V(M)} b_v + \sum_{f \in F(M)} b_f = \sum_{v \in V(M)} \sum_{a \in A(v)} b_a + \sum_{f \in F(M)} b_f = -12,$$

which, using the mentioned bijection $\beta_M$, can be rewritten as

$$\sum_{f \in F(M)} \sum_{a \in A(f)} b_a + \sum_{f \in F(M)} b_f = \sum_{f \in F(M)} (b_f + \sum_{a \in A(f)} b_a) = -12.$$ 

If the basic charges of vertices and faces are transformed to

$$b'_v := 0, \quad b'_f := b_f + \sum_{a \in A(f)} b_a,$$

we see that

$$\sum_{v \in V(M)} b'_v + \sum_{f \in F(M)} b'_f = \sum_{f \in F(M)} b'_f = -12,$$
hence there exists a face $f$ whose transformed charge $b'_f$ is negative. For $T = (d_1, \ldots, d_n) \in [3, \infty)^n$, $n \in [3, \infty)$, put

$$B'(T) := 2n - 6 + \sum_{i=1}^{n} \frac{d_i - 6}{d_i}.$$  

Then, clearly, $b'_f = B'(T)$ for each $T \in \tau(f)$, and we can call $B'(T)$ the transformed charge of the face type $T$. The Lebesgue’s set $L$ consists just of face types with a negative transformed charge.

We modify the process of passing from basic charges to transformed charges in the following way: We define a rational alternative charge $c_v$ of an angle $a \in A(M)$. Then we determine alternative charges of vertices and faces by

$$c_v := \sum_{a \in A(v)} c_a, \quad c_f := b_f + \sum_{a \in A(f)} (b_a - c_a).$$

Due to the definition we have

$$c_f + \sum_{a \in A(f)} c_a = b_f + \sum_{a \in A(f)} b_a,$$

$$\sum_{v \in V(M)} c_v + \sum_{f \in F(M)} c_f = \sum_{v \in V(M)} b_v + \sum_{f \in F(M)} b_f = -12.$$

If all alternative vertex charges are non-negative, there exists a face $f \in F(M)$ with $c_f < 0$.

In the definition of the basic charge of an angle $a$ the degree of $\dot{a}$ is involved only. To involve degrees of all the vertices of an angle in the definition of an alternative angle charge, we shall define, for $a = (v_{i-1}, e_{i-1}, v_i, e_i, v_{i+1})$,

$$c_a := c(\deg v_{i-1}, \deg v_i, \deg v_{i+1}),$$

where the mapping $c : [3, \infty)^3 \to \mathbb{Q}$ fulfills the condition

$$c(i, j, k) = c(k, j, i) \quad \text{for any} \ (i, j, k) \in [3, \infty)^3.$$

If $T = (d_1, \ldots, d_n) \in [3, \infty)^n$, $n \in [3, \infty)$, we define the alternative charge of the face type $T$ by

$$C(T) := B'(T) - 2 \sum_{i=1}^{n} c(d_{i-1}, d_i, d_{i+1}).$$

Then, analogously as before, the alternative charge is an invariant rational on the set of all types of a fixed face.
Let $v$ be a vertex of $M$ with degree $n$ and let $(e_1, \ldots, e_n)$ be the sequence of edges incident with $v$ as they are encountered when rotating around $v$. Let $v_i$ be the vertex of $M$ joined to $v$ along $e_i$, $i = 1, \ldots, n$. Then the alternative charge of $v$ is $c_v = 2 \sum_{i=1}^{n} c(\deg v_i, n, \deg v_{i+1})$. Set

$$s(d_1, \ldots, d_n) := \sum_{i=1}^{n} c(d_i, n, d_{i+1}).$$

Thus, if the condition

\[ (*) \quad s(d_1, \ldots, d_n) \geq 0 \quad \text{for any } (d_1, \ldots, d_n) \in [3, \infty)^n, \quad n \in [3, \infty), \]

is fulfilled, then the set $T$ of all face types $T$ with $C(T) < 0$ is unavoidable for normal planar maps.

A degree $j \in [3, \infty)$ is called absorbing if there exists a pair $(i, k) \in [3, \infty)^2$ such that $c(i, j, k) < 0$, otherwise it is non-absorbing. Thus, to control $(*)$ it suffices to deal with absorbing $n$’s and it is desirable to have only a small number of absorbing degrees. On the other hand, we need some absorbing degrees, since otherwise we would obtain as unavoidable a superset of the Lebesgue’s set $L$. For non-absorbing $j$’s it is appropriate to define $c(i, j, k) := 0$, since the positivity of $c(i, j, k)$ could only enrich the unavoidable set.

First of all, it is clear that even degrees must be non-absorbing. To see this suppose $c(i, j, k) < 0$ for some $j \equiv 0 \pmod{2}$; then, for $(d_1, \ldots, d_j) = (i, j, k)_{/2}$ we have

$$\sum_{l=1}^{j} c(d_l, j, d_{l+1}) = jc(i, j, k) < 0.$$  

We need also

$$c(i, j, i) \geq 0 \quad \text{for any } i, j \in [3, \infty);$$

otherwise, with $c(i, j, i) < 0$ and $(d_1, \ldots, d_j) = (i)_{j}^2$, we would have

$$\sum_{l=1}^{j} c(d_l, j, d_{l+1}) = jc(i, j, i) < 0.$$  

It could be a good idea to have non-absorbing all degrees large enough. Put

$$c_3(i) := \frac{1}{4} \cdot \frac{3}{7}.$$  

If $(3, i, i) \notin T$ for some $i \in [3, \infty)$ (remember that we tend to have $b^-_3(T)$ finite), then $0 \leq C(3, i, i) = 4c_3(i) - 2c(i, 3, i) \leq 4c_3(i)$. As $c_3(i) < 0$ for
Now we have the following degrees as candidates to be absorbing: 3, 5, 7, 9, 11. Since we want to obtain $T$ with the infinite characteristic $(3, 1)$, by Corollary 3 $(3, 6) \notin B(T)$, which means that $(3, 6, i) \notin T$ for a sufficiently large $i \geq 12$. As $C(3, 6, i) = -\frac{6}{i} - 2c(6, 3, i) \geq 0$, we see that $c(6, 3, i)$ must be negative and 3 is an absorbing degree.

It would be fine to be able to exclude from $T$ all types which do not assure the existence of an edge of weight $\leq 13$ (in order to cover Kotzig’s result). One of these types is $(3, 11, 11)$. As $C(3, 11, 11) = -\frac{11}{i} - 2c(11, 3, 11) - 4c(3, 11, 11) \geq 0$, we obtain $c(3, 11, 11) \leq -\frac{11}{4}(\frac{11}{i} + 2c(11, 3, 11)) \leq -\frac{1}{4}$ and 11 is an absorbing degree, too.

As we shall see, it is possible to reach our goal by letting 5, 7, 9 be non-absorbing degrees.

Let $i, j$ be non-absorbing degrees, $i \in [5, 10]$ and $j \in [12, \infty)$. We require $(3, j, j) \notin T$ for $j$ large enough. If, in the same time, $(3, i, j) \notin T$, then we have $0 \leq 2C(3, i, j) + C(3, j, j) = 4c(3, j) - 4c(i, 3, j) + 2c(j, 3, j) \leq 4c(i) + 8c(j)$; the non-negativity of the sum in the brackets follows from (*). Putting

$$t_i := \left\lceil \frac{8i}{i - 4} \right\rceil \quad \text{for } i \in [5, 10]$$

we see that

$$c_3(i) + 2c_3(j) \geq 0 \iff j \geq t_i \quad \text{for any } i \in [5, 10].$$

Thus we cannot expect nothing better than $(3, i, j) \notin T$ for $i \in [5, 10]$ and $j \in [t_i, \infty)$.

For $i = 11$ and $j \in [12, \infty)$ the above procedure cannot be applied, since 11 is an absorbing degree. However, as we want to cover Kotzig’s theorem, we put formally $t_{11} := 12$.

We define $c(i, 3, j)$ as follows:

$$c(i, 3, j) := 0 \quad \text{for } i, j = 3, 4,$$

$$c(3, 3, j) := 0 \quad \text{for } j \in [5, 11],$$

$$c(3, 3, j) := c_3(j) \quad \text{for } j \in [12, \infty).$$
\[c(4, 3, j) := \frac{1}{2}c_3(t_j) \quad \text{for } j \in [5, 11],\]
\[c(4, 3, j) := \frac{1}{2}c_3(j) \quad \text{for } j \in [12, \infty),\]
\[c(i, 3, j) := c_3(t_i) + c_3(t_j) \quad \text{for } i, j \in [5, 11],\]
\[c(i, 3, j) := -c_3(j) \quad \text{for } i \in [5, 11], \quad j \in [12, t_i - 1],\]
\[c(i, 3, j) := -c_3(t_i) \quad \text{for } i \in [5, 11], \quad j \in [t_i, \infty),\]
\[c(i, 3, j) := c_3(i) + c_3(j) \quad \text{for } i, j \in [12, \infty).\]

Let us check that (⋆) is fulfilled for \(n = 3\), i.e., that
\[s(i, j, k) = c(i, 3, j) + c(j, 3, k) + c(k, 3, i) \geq 0 \quad \text{for any } i, j, k \in [3, \infty).\]

For this purpose we put
\[S_1 := [3, 4], \quad S_2 := [5, 11], \quad S_3 := [12, \infty), \quad S_4 := [5, \infty),\]
\[s_l := |S_l \cap (\{i\} \cup \{j\} \cup \{k\})| \quad \text{for } l = 1, 2, 3;\]
for simplicity we shall write \(s\) instead of \(s(i, j, k)\).

1. If \(s_1 \geq 2\), without loss of generality \(i, j \in S_1\) and
\[s = c(i, 3, k) + c(j, 3, k) = 0 \quad \text{for } k \in S_1,\]
\[\geq \min\{0, \frac{1}{2}c_3(t_k), c_3(t_k)\} = 0 \quad \text{for } k \in S_2,\]
\[\geq \min\{c_3(k), \frac{3}{2}c_3(k), 2c_3(k)\} = c_3(k) \geq 0 \quad \text{for } k \in S_3.\]

2. If \(s_2 \geq 2\) and \(i, j \in S_2\), then
\[s = c_3(t_i) + c(i, 3, k) + c_3(t_j) + c(j, 3, k) \geq 0,
\text{since for any } p \in S_2 \text{ we have}
\[c_3(t_p) + c(p, 3, k) \geq c_3(t_p) + \min_{q=1,2,3} \min_{r \in S_q} c(p, 3, r) = c_3(t_p) + \min\{0, \frac{1}{2}c_3(t_p), -c_3(t_p)\} = 0.\]

3. If \(s_3 \geq 2\) and \(i, j \in S_3\), then
\[s = c_3(i) + c(i, 3, k) + c_3(j) + c(j, 3, k) \geq 0,
\text{as for } p \in S_3 \text{ it holds}
\[c_3(p) + c(p, 3, k) \geq c_3(p) + \min\{c_3(p), \frac{1}{2}c_3(p)\}, \min\{c_3(p), \frac{1}{2}c_3(p)\},
\min\{c_3(p) - c_3(t_q), c_3(p) - c_3(t_q)\},
\min\{c(p, 3, q)\} = c_3(p) + \min\{c_3(p), c_3(p)\} = 0.\]

4. Let \(s_1 = s_2 = s_3\) and \(i \in S_1, \ j \in S_2, \ k \in S_3\). For \(k \in [12, t_j - 1]\) we have
\[s \geq -2c_3(k) + \min\{2c_3(k), c_3(t_j) + c_3(k)\} = 0,\]
while the assumption \(k \in [t_j, \infty)\) leads to
\[s \geq -2c_3(t_j) + \min\{2c_3(k), c_3(t_j) + c_3(k)\} = c_3(k) - c_3(t_j) \geq 0.\]
To define \( c(i, 11, j) \), we set
\[
\begin{align*}
  c_{11}(i) &:= \begin{cases} 
    \frac{5}{44} & \text{for } i \in \{3\} \cup [6, \infty), \\
    \frac{3}{44} & \text{for } i = 4 \\
    \frac{1}{4} B'(5,5,11) = \frac{3}{220} & \text{for } i = 5
  \end{cases} \\
  c_{11}(4) &:= \frac{3}{44}, \\
  c_{11}(5) &:= \frac{1}{4} B'(5,5,11) = \frac{3}{220}, \\
  c(i, 11, j) &:= c_{11}(i) + c_{11}(j) \quad \text{for } (i, j) \in S_1^2 \cup S_2^2, \\
  c(3, 11, 5) &:= \frac{17}{220}, \\
  c(3, 11, 11) &:= \frac{1}{4} B'(3,11,11) = -\frac{1}{44}, \\
  c(3, 11, j) &:= \frac{1}{4} B'(3,11, j) = \frac{5}{22} - \frac{3}{7} \quad \text{for } j = 12, 13, \\
  c(3, 11, j) &:= -c_{3}(11) = \frac{1}{44} \quad \text{for } j \in [14, \infty), \\
  c(4, 11, 5) &:= \frac{7}{220}, \\
  c(4, 11, j) &:= \frac{1}{44}, \quad \text{for } j = 11, 12, 13, \\
  c(i, 11, j) &:= 0 \quad \text{for other pairs } (i, j) \in S_1 \times S_4.
\end{align*}
\]

From these definitions we obtain
\[
m := \min_{i,j \in [3,\infty)} \frac{c(i, 11, j)}{c(3, 11, 11)} = \frac{c(3, 11, 12)}{c(3, 11, 11)} = -\frac{1}{44},
\]
\[(i \leq j \wedge c(i, 11, j) < 0) \Rightarrow (i, j) \in \{(3, 11), (3, 12), (3, 13)\}.
\]

To see that \((*)\) is true for \( n = 11 \) note that there exists \( i \in [1, 11] \) such that \( (d_i, d_{i+1}) \in S_1^2 \cup S_2^2 \), without loss of generality \( i = 11 \). Since then
\[
s(d_1, \ldots, d_{11}) = \sum_{i=1}^{10} c(d_i, 11, d_{i+1}) + c_{11}(d_{11}) + c_{11}(d_1)
\]
\[
= c_{11}(d_1) + \sum_{i=1}^{5} c(d_i, 11, d_{i+1}) + c_{11}(d_{11})
\]
\[
+ \sum_{i=1}^{5} c(d_{12-i}, 11, d_{11-i})
\]
\[
= \bar{s}(d_1, d_2, d_3, d_4, d_5, d_6) + \bar{s}(d_{11}, d_{10}, d_9, d_8, d_7, d_6),
\]
where
\[
\bar{s}(d_1, d_2, d_3, d_4, d_5, d_6) := c_{11}(d_1) + \sum_{j=1}^{5} c(d_j, 11, d_{j+1}),
\]
it suffices to show that \( \bar{s}(d_1, d_2, d_3, d_4, d_5, d_6) \geq 0 \) for any \( (d_1, d_2, d_3, d_4, d_5, d_6) \in [3, \infty)^6 \); we shall write \( \bar{s} \) instead of \( \bar{s}(d_1, d_2, d_3, d_4, d_5, d_6) \).

If \( d_1 = 3 \) or \( d_1 \geq 6 \), then \( \bar{s} \geq \frac{5}{44} + 5m = 0 \).
If \( d_1 = 4 \) and \( d_2 \in \{3, 11, 12, 13\} \), then \( \bar{s} \geq \frac{3}{44} + \min\{\frac{2}{44}, \frac{1}{44}\} + 4m = 0 \).
If \( d_1 = 4 \) and \( d_2 \in [4, 10] \cup [14, \infty) \), then \( \tilde{s} \geq \frac{3}{17} + 2 \cdot 0 + 3m = 0 \).

Finally, suppose \( d_1 = 5 \). If there exists \( j \in [1, 5] \) such that \( (d_j, d_{j+1}) \in [3, 4] \cup [6, \infty]^2 \), then \( \tilde{s} \geq \frac{3}{220} + 2 \cdot \frac{3}{17} + 4m = \frac{13}{220} \). If there exists \( j \in [1, 5] \) such that \( (d_j, d_{j+1}) = (5, 3) \), then \( \tilde{s} \geq \frac{3}{220} + \frac{17}{220} + 4m = 0 \). If there exists \( j \in [1, 5] \) such that \( d_k = 5 \) for any \( k \in [1, j] \) and \( d_{j+1} = 4 \), then, since

\[
\min_{p \in [3, \infty)} c(4, 11, p) = 0 = \min_{p, q \in [3, \infty)} (c(p, 11, 4) + c(4, 11, q)),
\]

we have \( \tilde{s} \geq (2j - 1) \cdot \frac{3}{220} + \frac{7}{220} + \max\{0, 3 - j\} \cdot m \geq \frac{1}{22} + 2m = 0 \). Of course, \( (d_1, \ldots, d_6) = (5) \) gives \( \tilde{s} = \frac{3}{20} \).

Thus, since (*) is fulfilled, the set \( T \) of types \( T \) with \( C(T) < 0 \) is an unavoidable set for normal planar maps. Which is its structure?

As for any \( i \in [3, \infty) \) \( C(3, 3, i) = B'(3, 3, i) - 4c(3, 3, i) - 2c(3, i, 3) \leq B'(3, 3, i) = -1 - \frac{6}{i} < 0 \) and \( C(3, 4, i) = B'(3, 4, i) - 2c(4, 3, i) - 2c(3, i, 4) \leq B'(3, 4, i) = -\frac{1}{2} - \frac{6}{i} < 0 \), the types \( (3, 3, i) \), \( i \in [3, \infty) \), and \( (3, 4, i) \), \( i \in [4, \infty) \), are in \( T \).

Let \( i \in [5, 10] \). If \( j \in [i, 11] \), then \( C(3, i, j) = 2c_3(i) - 2c_3(t_i) + 2c_3(j) - 2c_3(t_j) < 0 \), since \( c_3(k) \) is an increasing function of \( k \) and \( i < t_i, j < t_j \). If \( j \in [12, t_i - 1] \), then \( C(3, i, j) = 2c_3(i) + 4c_3(j) \leq 2c_3(i) + 4c_3(t_i - 1) < 0 \).

Finally, for \( j \in [t_i, \infty) \) we have \( C(3, i, j) = 2c_3(i) + 2c_3(j) + 2c_3(t_i) \geq 2c_3(i) + 4c_3(t_i) \geq 0 \). Thus we see that

\[
(3, i, j) \in T \iff j \in [i, t_i - 1] \quad \text{for } i \in [5, 10].
\]

As \( C(3, 11, i) = 0 \) for \( i = 11, 12, 13 \), and \( C(3, 11, i) = 2c_3(i) > 0 \) for \( i \in [14, \infty) \), there are no types \( (3, 11, i) \), \( i \geq 11 \), in \( T \).

For \( i, j \in [12, \infty) \), \( i \leq j \), we have \( C(3, i, j) = 0 \) and \( (3, i, j) \notin T \).

If \( 4 \leq i \leq j \leq k \), then \( c(i, j, k) = c(j, k, i) = c(k, i, j) = 0 \), hence \( C(i, j, k) = B'(i, j, k) = 3(1 - \frac{2}{i} - \frac{2}{j} - \frac{2}{k}) \) and we obtain the same types as are in \( \mathcal{L} \), i.e., \( (4, 4, k) \), \( k \in [4, \infty) \), and those determined by \( 9 \leq i + j \leq 11 \) and \( k < \frac{2d}{i + j - 2i - 2j} \).

We now pass to types of faces of degree \( \geq 4 \). If \( T = (d_1, \ldots, d_n) \), then

\[
C(T) = 2n - 6 + \sum_{i=1}^{n} \frac{d_i - 6}{d_i} - 2 \sum_{i=1}^{n} c(d_{i-1}, d_i, d_{i+1})
= \sum_{i=1}^{n} (3 - \frac{6}{d_i} - 2c(d_{i-1}, d_i, d_{i+1})) - 6.
\]

Putting

\[
a(i, j, k) := 3 - \frac{6}{j} - 2c(i, j, k),
\]
\[
\sigma(d_1, \ldots, d_n) := \sum_{i=1}^{n} a(d_{i-1}, d_i, d_{i+1}),
\]
we obtain the following equivalence:

\[(d_1, \ldots, d_n) \in T \iff \sigma(d_1, \ldots, d_n) \geq 6;\]

we shall use \(\sigma\) instead of \(\sigma(d_1, \ldots, d_n)\). Note that

\[a(i, j, k) \geq 3 - \frac{6}{j} - 2 \cdot \frac{1}{2} = 0\]

for any \((i, j, k) \in [3, \infty)^3\).

Moreover, as

\[a(i, 11, k) \geq 3 - \frac{6}{11} - 2 \cdot \max_{p, q \in [3, \infty)} c(p, 11, q) = \frac{27}{11} - 2 \cdot \frac{5}{22} = 2,\]

and \(a(i, j, k) = 3 - \frac{6}{j}\) for any \(j \in [4, 10] \cup [12, \infty)\), we have

\[a(i, j, k) \geq \frac{5}{2} \text{ for } j \in [12, \infty),\]

\[\geq \frac{3}{2} \text{ for } j \in [4, 11],\]

\[\geq \frac{9}{5} \text{ for } j \in [5, 11].\]

Define

\[m(p, q) := |\{i \in [1, n] : p \leq d_i < q\}|, \quad m(p) := |\{i \in [1, n] : d_i = p\}|.\]

(1) If \(m(12, \infty) \geq 3\), then \(\sigma \geq 3 \cdot \frac{5}{2} = \frac{15}{2} > 6\) and \(T \notin \mathcal{T}\).

(2) \(m(12, \infty) = 2\)

(21) \(m(4, 12) \geq 1\) yields \(\sigma \geq 2 \cdot \frac{5}{2} + \frac{3}{2} = \frac{13}{2} > 6\).

(22) \(m(4, 12) = 0\)

(221) If \(m(3) \geq 3\), there exists \(i \in [1, n]\) such that \(d_i = d_{i+1} = 3\). However, as

\[a(p, 3, 3) = a(3, 3, p) > 1 - \sup_{q \in [3, \infty)} c(3, 3, q) = \frac{1}{2} \quad \text{for } p \in [3, \infty),\]

we obtain \(a(d_{i-1}, d_i, d_{i+1}) + a(d_i, d_{i+1}, d_{i+2}) > 2 \cdot \frac{1}{2} \text{ and } \sigma > 1 + 2 \cdot \frac{5}{2} = 6.\)
Unavoidable Set of Face Types...

(222) \( m(3) = 2 \)

(2221) For \( T = (3, d_3, d_4) \), \( 12 \leq d_3 \leq d_4 \), we have \( \sigma > 6 \), as in (221).

(2222) If \( T = (3, d_2, 3, d_4) \), \( 12 \leq d_2 \leq d_4 \), then \( \sigma = 6 \).

(3) \( m(12, \infty) = 1 \)

(31) From \( m(5, 12) \geq 2 \) it follows \( \sigma \geq \frac{5}{2} + 2 \cdot \frac{9}{5} = \frac{61}{10} > 6 \).

(32) \( m(5, 12) = 1 \)

(321) \( m(4) \geq 1 \) gives, due to \( \sup_{\min(p,q) \leq 11} c(p, 3, q) = \frac{1}{2} \), \( \sigma \geq \frac{5}{2} + \frac{9}{5} + \frac{3}{2} + \frac{1}{2} = \frac{63}{10} > 6 \).

(322) Provided \( m(4) = 0 \) there exists \( i \in [1, n] \) such that \( \{d_i, d_{i+1}\} = \{3, p\} \) with \( p \in [5, 11] \). As \( c(p, 3, q) \leq 0 \) for any \( q \in \{3\} \cup [12, \infty) \), we have \( a(d_{j-1}, d_j, d_{j+1}) \geq 1 \) for at least one \( j \in [1, n] \) with \( d_j = 3 \) (more precisely, \( j \in \{i, i + 1\} \)).

(3221) \( m(3) \geq 3 \) means that \( \sigma \geq \frac{5}{2} + \frac{9}{5} + 1 + 2 \cdot \frac{1}{2} = \frac{63}{10} \).

(3222) \( m(3) = 2 \)

(32221) For \( T = (3, d_2, 3, d_4) \), \( d_2 \in [5, 11], d_4 \in [12, \infty) \), we have the same lower bound for \( \sigma \) as in (3221), since \( 1 + 2 \cdot \frac{1}{2} \) can be replaced with \( 2 \cdot 1 \).

(32222) If \( T = (3, d_3, d_4) \), \( d_3 \in [5, 11], d_4 \in [12, \infty) \), then \( \sigma = \frac{15}{2} - \frac{6}{d_3} \geq \frac{63}{10} \).

(33) The assumption \( m(5, 12) = 0 \) leads to \( a(d_{i-1}, d_i, d_{i+1}) > \frac{1}{2} \) for any \( i \in [1, n] \).

(331) \( m(4) \geq 2 \) gives \( \sigma > \frac{5}{2} + 2 \cdot \frac{3}{2} + \frac{1}{2} = 6 \).

(332) \( m(4) = 1 \)

(3321) If \( m(3) \geq 3 \), there exists \( i \in [1, n] \) such that \( d_i = 3 \) and \( \{d_{i-1}, d_{i+1}\} = \{3, 4\} \). As \( a(3, 3, 4) = a(4, 3, 3) = 1 \), we obtain \( \sigma > \frac{5}{2} + \frac{3}{2} + 1 + 2 \cdot \frac{1}{2} = 6 \).

(3322) \( m(3) = 2 \)

(33221) For \( T = (3, 3, 4, d_4) \), \( d_4 \in [12, \infty) \), we have \( \sigma = 6 \).

(33222) For \( T = (3, 4, 3, d_4) \), \( d_4 \in [12, \infty) \), it holds \( \sigma = 6 \).

(333) \( m(4) = 0 \)

(3331) \( m(3) \geq 4 \) and \( T = (3)^{n-1}(d_n), d_n \in [12, \infty) \), imply \( \sigma = n + 1 + \frac{6}{d_n} \geq \frac{13}{2} > 6 \).

(3332) For \( m(3) = 3 \) and \( T = (3, 3, 3, d_4) \), \( d_4 \in [12, \infty) \), it follows, from \( \sigma = 5 + \frac{6}{d_4} \leq \frac{11}{2} < 6 \), that \( T \in T \).

(4) In the case \( m(12, \infty) = 0 \) we denote by \( m^+(3) \) or \( m^-(3) \) the number of those triples \( (d_{i-1}, d_i, d_{i+1}) \) for which \( d_i = 3 \) and the set \( \{d_{i-1}\} \cup \{d_{i+1}\} \) does or does not contain 3, respectively. As \( c(3, 3, p) = 0 \) for \( p \in [3, 11] \) and \( c(q, 3, r) \leq 2c_3(t_3) = \frac{7}{20} \) for \( q, r \in [4, 11] \), we have \( a(3, 3, p) = 1 \), \( a(q, 3, r) \geq \frac{3}{10} \) and

\[
\sigma \geq \frac{3}{10} m^-(3) + m^+(3) + \frac{3}{2}(n - m^-(3) - m^+(3)) = \frac{3}{2} n - \frac{6}{5} m^-(3) - \frac{1}{2} m^+(3).
\]
(41) If \( \frac{6}{5}m^{-}(3) + \frac{1}{2}m^{+}(3) \leq \frac{3}{2}n - 6 \), then \( \sigma \geq 6 \) and \( T \notin T \).

(42) \( \frac{6}{5}m^{-}(3) + \frac{1}{2}m^{+}(3) > \frac{3}{2}n - 6 \)

(421) For \( n \geq 7 \) we have

\[
\frac{6}{5}m^{-}(3) + \frac{3}{5}m^{+}(3) > \frac{3}{2}n - 6,
\]

\[
m(3) = m^{-}(3) + m^{+}(3) > \frac{5}{6} \left( \frac{3}{2}n - 6 \right) = \frac{n}{2} + \frac{3n - 20}{4} > \frac{n}{2}.
\]

As \( m(3) > \frac{n}{2} \), \( m^{-}(3) \) is upper bounded by \( n - m(3) - 1 \), which yields \( 2m^{-}(3) + m^{+}(3) \leq n - 1 \), hence

\[
\frac{3}{5}(n - 1) \geq \frac{6}{5}m^{-}(3) + \frac{3}{5}m^{+}(3) \geq \frac{6}{5}m^{-}(3) + \frac{1}{2}m^{+}(3) > \frac{3}{2}n - 6,
\]

and, as a consequence, \( n < 6 \), a contradiction.

(422) \( n = 6 \)

(4221) For \( m(5, 12) \geq 3 \) it holds \( \sigma \geq 3 \cdot \frac{9}{5} + 3 \cdot \frac{3}{10} = \frac{63}{10} \).

(4222) \( m(5, 12) = 2 \)

(42221) For \( m(4) \geq 1 \) we obtain \( \sigma \geq 2 \cdot \frac{9}{5} + \frac{3}{2} + 3 \cdot \frac{3}{10} = 6 \).

(42222) \( m(4) = 0 \) implies \( m^{+}(3) \geq 3 \) and \( \sigma \geq 2 \cdot \frac{9}{5} + 3 \cdot 1 = \frac{33}{5} > 6 \).

(4223) For the case \( m(5, 12) = 1 \) note that \( \max_{p \in [3, 4], q \in [5, 11]} c(p, 3, q) = \frac{1}{2}c_3(t_5) = \frac{7}{50} \).

(42231) \( m(4) \geq 1 \) means that \( \sigma \geq \frac{9}{5} + \frac{3}{2} + 4(1 - 2 \cdot \frac{7}{50}) = \frac{33}{5} \).

(42232) If \( m(4) = 0 \) and \( T = (3)^5(d_6), d_6 \in [5, 11] \), then \( \sigma = 8 - \frac{6}{d_6} \geq \frac{34}{5} \).

(4224) From \( m(5, 12) = 0 \) it follows \( \sigma = m(3) + \frac{2}{5}m(4) \geq m(3) + m(4) = 6 \).

(423) The remaining case, \( n = 4, 5 \), was analysed, thanks to its finiteness, following from \( m(12, \infty) = 0 \), by a computer.

The described analysis led to

**Theorem 4.** For the class of normal planar maps the following sequences form an unavoidable set of face types:

\[
(3, i, j), i \in [3, 4], j \in [i, \infty), (4, 4, i), i \in [4, \infty), (3, 3, 3, i), i \in [3, \infty),
\]

\[
(3, 5, i), i \in [5, 39], (3, 6, i), i \in [6, 23], (3, 7, i), i \in [7, 18], (3, 8, i), i \in [8, 15],
\]

\[
(3, 9, i), i \in [9, 14], (3, 10, i), i \in [10, 13], (4, 5, i), i \in [5, 19], (4, 6, i), i \in [6, 11],
\]

\[
(4, 7, i), i \in [7, 9], (5, 5, i), i \in [5, 9], (5, 6, i), i = 6, 7,
\]

\[
(3, 3, 4, i), i \in [4, 11], (3, 3, 5, i), i \in [5, 7], (3, 4, 3, i), i \in [4, 11],
\]

\[
(3, 4, 4, i), i \in [4, 6], (3, 4, 5, i), i = 4, 5, (3, 5, 3, i), i \in [5, 11],
\]

\[
(3, 5, 4, i), i \in [5, 7], (3, 5, 5, i), i = 5, 6, (3, 6, 3, i), i \in [6, 11],
\]

\[
(3, 7, 3, i), i = 7, 8, 9, 11,
\]

\[
(3, 3, 3, 3, i), i \in [3, 5], (3, 3, 5, 3, 5).
\]

\[\blacksquare\]
**Corollary 5** (Borodin [1]). In any normal planar map there exists an edge of weight at most 13.

Using Corollary 3 and Theorem 4 we see that the minimum of Problem 1 is in fact equal to 4. Having this in mind, the following could be of interest.

**Problem 2.** Determine the minimum \( \sum_{i=3}^{\infty} b_i^-(T) \) for an unavoidable set \( T \) of face types for normal planar maps with the infinite characteristic (3, 1).

The set \( T \) from Theorem 4 has the finite characteristic (114, 46, 4) so that the minimum of Problem 2 is at most 164.

4. Cyclic chromatic number of planar maps

If we do not insist on minimizing \( \sum_{i=3}^{\infty} b_i^-(T) \) in a good unavoidable set \( T \), we can obtain unavoidable sets, which may be useful for a solution of other than structural type problems. E.g., by allowing 5 to be an absorbing degree it is possible to obtain an unavoidable set \( T \) for normal planar graphs whose types do not contain large degrees except for types in “inexcludable” four infinite series. On the other hand, hexagonal types appear in \( T \), while \( b_6^-(T) = 0 \).

**Theorem 6.** For the class of normal planar maps the following sequences form an unavoidable set of face types:

- \((3, i, j), i \in [3, 4], j \in [i, \infty), (4, 4, i), i \in [4, \infty), (3, 3, 3, i), i \in [3, \infty), (3, 5, i), i \in [5, 23), (3, 6, i), i \in [6, 23), (3, 7, i), i \in [7, 18), (3, 8, i), i \in [8, 15)\)
- \((3, 9, i), i \in [9, 14), (3, 10, i), i \in [10, 13), (4, 5, i), i \in [5, 11), (4, 6, i), i \in [6, 11), (4, 7, i), i \in [7, 9], (5, 5, i), i \in [5, 7], (5, 6, i), i = 6, 7\)
- \((3, 3, 4, i), i \in [4, 11), (3, 3, 5, i), i \in [5, 7), (3, 4, 3, i), i \in [4, 11), (3, 4, 4, i), i \in [4, 6], (3, 4, 5, i), i \in [4, 6], (3, 4, 6, 5), (3, 5, 3, i), i \in [5, 17), (3, 5, 4, i), i \in [5, 12), (3, 5, 5, i), i \in [5, 7), (3, 5, 6, i), i = 5, 6, (3, 5, 7, 5), (3, 6, 3, i), i \in [6, 11), (3, 7, 3, i), i \in [7, 9, 11, (4, 4, i, 5), i = 4, 5, (4, 5, i, 5), i = 4, 5\)
- \((3, 3, 3, 3, i), i \in [3, 5), (3, 3, 3, i, 5), i = 4, 5, (3, 3, 4, 3, 5), (3, 3, 5, 3, i), i = 5, 6, 7, 8, 9, 11, (3, 3, 5, 4, 5), (3, 4, 3, i, 5), i = 4, 5, (3, 4, 4, 3, 5), (3, 4, 5, 3, i), i = 5, 6, (3, 4, 5, 4, 5), (3, 4, 6, 3, 5), (3, 5, 3, 5, i), i \in [5, 7], (3, 5, 4, i, 5), i = 4, 5, (3, 5, 5, 3, i), i = 6, 7, (3, 5, 3, 5, 3, i), i = 6, 7, (3, 5, 3, 5, 4, 5)\).

**Proof.** Put

\[
\hat{c}_5(i) := \begin{cases} 
\frac{1}{4} & \text{for } i \in [3, 4] \cup [12, \infty), \\
\frac{3}{20} & \text{for } i = 5, 8, 9, \\
\frac{3}{22} & \text{for } i = 6, 10, 11, \\
0 & \text{otherwise.}
\end{cases}
\]
We define a new mapping \( \bar{c} : [3, \infty)^3 \to \mathbb{Q} \) by \( \bar{c}(i, j, k) := c(i, j, k) \) for \( j \neq 5 \) and

\[
\begin{align*}
\bar{c}(i, 5, j) &= \bar{c}_5(i) + \bar{c}_5(j) & \text{for } (i, j) \in [3, 6]^2 \cup [7, \infty)^2, \\
\bar{c}(i, 5, 7) &= \frac{1}{10}, & \text{for } i = 3, 4, \\
\bar{c}(i, 5, j) &= -\frac{1}{50} & \text{for } i = 3, 4, j \in [8, 11], \\
\bar{c}(5, 5, 7) &= \frac{3}{50}, & \text{for } j = 8, 9, \\
\bar{c}(5, 5, j) &= -\frac{3}{50} & \text{for } j \in [10, \infty), \\
\bar{c}(6, 5, j) &= 0 & \text{for } j \in [7, 11], \\
\bar{c}(6, 5, j) &= \frac{1}{15} & \text{for } j \in [12, \infty).
\end{align*}
\]

If the alternative charge of an angle \( a = (v_{i-1}, e_i, v_i, e_i, v_{i+1}) \) is determined by \( \bar{c}_a := \bar{c} \deg v_{i-1}, \deg v_i, \deg v_{i+1} \), an analysis analogous to that applied for Theorem 4 leads to the unavoidable set described in the statement of Theorem 6.

Let \( M \) be a map. The weight of a face \( f \in F(M) \) with \( T = (d_1, \ldots, d_n) \in \tau(f) \) is defined by \( \text{wt}(f) := \sum_{i=1}^n d_i \) and the weight of \( M \) by \( \min_{f \in F(M)} \text{wt}(f) \).

**Corollary 7.** If a normal planar map \( M \) does not contain faces of types \((3, 3, i), (3, 4, i), (4, 4, i), (3, 3, 3, i), i \geq 3, \) then the weight of \( M \) is at most 32.

One can wonder why hexagonal types appear in the statement of Theorem 6 in spite of the fact that by Euler’s formula only faces of sizes \( \leq 5 \) are necessarily present in a normal planar map. If we insert a configuration of Figure 1 into each face of an icosahedron map (dashed lines stand for edges of the original map), we obtain a map with faces only of types \((3, 5, 3, 5, 3, 5), (3, 5, 30) \) and \((3, 30, 30), \) from which the hexagonal type only occurs in the list of Theorem 6. Evidently, the above construction can be applied to any plane triangulation with minimum degree 5 (and even with minimum degree 4) to create a map with analogous face types property.
Let $M$ be a 2-connected planar map. A cyclic coloration of $M$ (introduced in Ore and Plummer [20]) is an assignment of colours to the vertices of $M$ such that for any face all its vertices receive different colours. The cyclic chromatic number of $M$ is the minimum number of colours in any cyclic coloration of $M$.

Plummer and Toft [21] obtained some upper bounds for the cyclic chromatic number of 3-connected planar maps. If we use in the proofs of Theorems 3.1, 3.2 and 3.3 of [21] our Theorem 6 (for our statements (i) – (vi), see below) or Theorem 4 (for (vii) – (viii)) instead of Lebesgue’s result, by the same method we obtain the following theorem which improves the corresponding Theorems 3.1 and 3.3 of [21].

**Theorem 8.** Let $M$ be a 3-connected planar map with maximum face degree $d$ and cyclic chromatic number $n$. Then
(i) if \( d \geq 24 \), then \( n \leq d + 3 \);
(ii) if \( d \geq 19 \), then \( n \leq d + 4 \);
(iii) if \( d \geq 16 \), then \( n \leq d + 5 \);
(iv) if \( d \geq 15 \), then \( n \leq d + 6 \);
(v) if \( d \geq 14 \), then \( n \leq d + 7 \);
(vi) if \( d \geq 10 \), then \( n \leq d + 8 \);
(vii) if \( d \leq 9 \), then \( n \leq d + 7 \);
(viii) if \( d \leq 8 \), then \( n \leq d + 6 \).

Note that Theorem 8(i) was announced (but probably not published yet) by Borodin, cf. Jensen and Toft [15, Chapter 1].

References


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