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GENERALIZED DOMINATION, INDEPENDENCE AND IRREDUDANCE IN GRAPHS

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Abstract

The purpose of this paper is to present some basic properties of $\mathcal{P}$-dominating, $\mathcal{P}$-independent, and $\mathcal{P}$-irredundant sets in graphs which generalize well-known properties of dominating, independent and irredundant sets, respectively.

Keywords: hereditary property of graphs, generalized domination, independence and irredundance numbers.

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In this paper we will consider finite undirected graphs with no multiple edges, and with no loops. For a graph G we will refer to $V(G)$ (or V) and $E(G)$ (or E) as the vertex and edge set, respectively.

A nonempty subset D of the vertex set V of a graph G is a *dominating* set if every vertex in $V - D$ is adjacent to a member of D. If $u \in D$ and $v \in V - D$, and $uv \in E$, we say that u *dominates* v and v is *dominated* by u.

The minimum (maximum) of the cardinalities of the minimal dominating sets in G is called the *upper domination number* of G and it is denoted by $\gamma(G)$ $(\Gamma(G))$.

We write $H \leq G$ if H is an induced subgraph of G. We use the notation $G[A]$ for the subgraph of G induced by $A \subseteq V(G)$.

A set $S \subseteq V(G)$ is said to be *independent* if $G[S]$ is totally disconnected (i.e., $G[S]$ is an edgeless graph). Obviously, each maximal independent set is a minimal dominating set. If S is a maximal independent set of G, then $G[S \cup \{v\}]$ contains as a subgraph K_2, i.e., the subgraph which is forbidden for the property "*to be totally disconnected*".

For $v \in V$, we denote by $N(v)$ a set of vertices adjacent to v (neighbours of v) and by $N(A)$ a set of neighbours of vertices of A. By $N[v]$ and $N[A]$ we denote $N(v) \cup \{v\}$ and $N(A) \cup A$, respectively.

A set $R \subseteq V(G)$ is called *irredundant in* G, if for each vertex $v \in R$, $N[v] - N[R - \{v\}] \neq \emptyset$.

This definition fits intuitive ideas of redundancy, for in the context of communication network, any vertex that may receive a communication from some vertex x in R, may also be informed from some vertex in $R - \{x\}$, i.e., x may be removed from R without affecting the totality of accesible vertices. It is apparent that irredundance is a hereditary property and that any independent set of vertices is also an irredundant set.

The minimum (maximum) of the cardinalities of the maximal irredundant sets of G is called the *lower* (*upper*) *irredundance number* and it is denoted by $ir(G)$, $(IR(G))$.

The study of domination in graphs has been initiated by Ore [6], for a survey see a special volume of the *Discrete Mathematics* **86** (1990). Applications of minimum dominating sets have been suggested by many authors. The determination of the domination number is an NP-complete problem (see [4]). It should be noted that bounds on $\gamma(G)$ do exist through the parameters which are also difficult to determine.

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Let $\mathcal{I}$ denote the class of all finite simple graphs. A *graph property* is a nonempty isomorphism-closed subclass of $\mathcal{I}$. (We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$).

A property $\mathcal{P}$ of graphs is said to be *induced hereditary* if whenever $G \in \mathcal{P}$ and $H \leq G$, then also $H \in \mathcal{P}$. For hereditary properties with respect to other partial order on $\mathcal{I}$ we refer the reader to [1].

Any induced hereditary property $\mathcal{P}$ of graphs is uniquely determined by the set of all its forbidden induced subgraphs

$$\boldsymbol{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H)\}.$$

Let us denote by $\mathbb{M}$ the set of all induced hereditary properties of graphs. According to [1] we list below some of the induced hereditary properties.

$$\mathcal{O} = \{G \in \mathcal{I} : G \text{ is totally disconnected } \},\ \boldsymbol{C}(\mathcal{P}) = \{K_2\};$$
$$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\},\ \boldsymbol{C}(\mathcal{S}_k) = \{H :\mid V(H) \mid= k+2 = \Delta(H)+1\};$$
$$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},\ \boldsymbol{C}(\mathcal{I}_k) = \{K_{k+2}\}.$$

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Let $\mathcal{P} \in \mathbb{M}$ and $G = (V, E)$ be a graph. Two vertices u and v of G are called *$\mathcal{P}$-adjacent* if there is an induced subgraph H' of G containing u and v such that $H' \simeq H \in \boldsymbol{C}(\mathcal{P})$.

For a vertex $v \in V$, by $N_{\mathcal{P}}(v)$ we denote the *$\mathcal{P}$-neighbourhood* of v, i.e., $N_{\mathcal{P}}(v) = \{u \in V : u \textit{ is } \mathcal{P}\text{-adjacent to } v\}$ and $N_{\mathcal{P}}[v] = N_{\mathcal{P}}(v) \cup \{v\}$. For a set $X \subseteq V$, let $N_{\mathcal{P}}(X) = \bigcup_{v \in X} N_{\mathcal{P}}(v)$ and $N_{\mathcal{P}}[X] = N_{\mathcal{P}}(X) \cup X$. Especially, $N(v) = N_{\mathcal{O}}(v)$.

Next, for a vertex $v \in V(G)$ we denote the set of all forbidden subgraphs containing v by $\boldsymbol{C}_{G,\mathcal{P}}(v) = \{H' \leq G : v \in V(H'), H' \simeq H \in \boldsymbol{C}(\mathcal{P})\}$.

The number $|\boldsymbol{C}_{G,\mathcal{P}}(v)|$ is called *$\mathcal{P}$-degree* of v in G and is denoted by $\deg_{G,\mathcal{P}}(v)$.

If $\deg_{G,\mathcal{P}}(v) = 1$, then v is said to be *$\mathcal{P}$-pendant* in G and if $\deg_{G,\mathcal{P}}(v) = 0$, then v is said to be *$\mathcal{P}$-isolated* in G.

A set $D \subseteq V$ is said to be *$\mathcal{P}$-dominating* in G if $N_{\mathcal{P}}(v) \cap D \neq \emptyset$ for any $v \in V - D$.

A set $D \subseteq V$ is said to be *strongly $\mathcal{P}$-dominating* in G if for each $v \in V - D$ there is $H' \leq G$ containing v such that $H' \simeq H \in \boldsymbol{C}(\mathcal{P})$ and $V(H') - \{v\} \subseteq D$.

The minimum (maximum) of the cardinalities of the minimal $\mathcal{P}$-dominating sets in G is called the *lower*, (*upper*) *$\mathcal{P}$-domination number* of G and it is denoted by $\gamma_{\mathcal{P}}(G)$, $(\Gamma_{\mathcal{P}}(G))$, respectively.

The minimum (maximum) of the cardinalities of the minimal strongly $\mathcal{P}$-dominating sets in G is called the *lower* (*upper*) *strong $\mathcal{P}$-dominating number* and it is denoted by $\gamma'_{\mathcal{P}}(G)$, $(\Gamma'_{\mathcal{P}}(G))$, respectively.

If $\mathcal{P} = \mathcal{I}_{n-2}$, then the $\mathcal{I}_{n-2}$-dominating sets are also called K_n-dominating sets in G (see [5]).

A set $R \subseteq V$ is called *$\mathcal{P}$-irredundant* if for every vertex $v \in R$, $N_{\mathcal{P}}[v] - N_{\mathcal{P}}[R - \{v\}] \neq \emptyset$.

The minimum (maximum) of the cardinalities of the maximal $\mathcal{P}$-irredundant sets is called the *lower* (*upper*) *$\mathcal{P}$-irredundance number of* G and is denoted by $ir_{\mathcal{P}}(G)$ $(IR_{\mathcal{P}}(G))$, respectively.

A set $S \subseteq V(G)$ is *$\mathcal{P}$-independent* in G if $G[S] \in \mathcal{P}$. A set $S \subseteq V(G)$ is said to be *strongly $\mathcal{P}$-independent* in G if for every $v \in S$, $N_{\mathcal{P}}(v) \cap S = \emptyset$.

The minimum (maximum) of the cardinalities of the maximal strongly $\mathcal{P}$-independent sets in G, is called the *strong $\mathcal{P}$-independence number of G* and it is denoted by $i'_{\mathcal{P}}(G)$, $(\alpha'_{\mathcal{P}}(G))$.

The minimum (maximum) of the cardinalities of the maximal $\mathcal{P}$-independent sets in G, is called the *$\mathcal{P}$-independence number of G* and it is denoted by $i_{\mathcal{P}}(G)$, $(\alpha_{\mathcal{P}}(G))$.

Notice, that if $\mathcal{P} = \mathcal{O}$, then $\mathcal{P}$-dominating and strongly $\mathcal{P}$-dominating sets in G are dominating sets, $\mathcal{P}$-independent and strongly $\mathcal{P}$-independent sets are independent sets, also $\mathcal{P}$-irredundant sets are irredundant sets in an ordinary sense.

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The following theorem generalizes a clasical result of Ore [6].

Theorem 1. *Let D be a $\mathcal{P}$-dominating set of a graph G. Then D is a minimal $\mathcal{P}$-dominating set of G if and only if for each vertex $d \in D$, d has one of the following properties:*

(i) *there exists a vertex $v \in V - D$ such that $N_{\mathcal{P}}(v) \cap D = \{d\}$,*

(ii) $N_{\mathcal{P}}(d) \cap D = \emptyset$.

Proof. Suppose that D is a minimal $\mathcal{P}$-dominating set of G. Then for each vertex $d \in D$, the set $D - \{d\}$ is not a $\mathcal{P}$-dominating set of G. Hence, there is a vertex $v \in V - (D - \{d\})$ that is $\mathcal{P}$-adjacent to no vertex of $D - \{d\}$. If $v = d$, d is $\mathcal{P}$-adjacent to no vertex of D, while if $v \in V - D$, then since D is a $\mathcal{P}$-dominating set of $G, N_{\mathcal{P}}(v) \cap D = \{d\}$.

Conversely, if every vertex $d \in D$ has at least one of the properties (i) or (ii), then $D - \{d\}$ is not a $\mathcal{P}$-dominating set of G. ■

Theorem 2. *If G is a graph without $\mathcal{P}$-isolated vertices, then there exists a minimum $\mathcal{P}$-dominating set of vertices of G in which every vertex has property* (i).

Proof. Among all the $\mathcal{P}$-dominating sets of G with cardinality equal to $\gamma_{\mathcal{P}}(G)$, let D be chosen so that D contains the maximum possible numbers of vertices which are $\mathcal{P}$-adjacent to some vertex of D in G. Suppose there exists a vertex $d \in D$, that d has no property (i). However, by Theorem 1, d has the property (ii). This implies that d is $\mathcal{P}$-adjacent to no vertex of D. Since G is a graph without isolated vertices, then there exists a vertex $w \in N_{\mathcal{P}}(d)$ and $w \in V(G) - (D - \{d\})$. The vertex w is $\mathcal{P}$-adjacent to some vertex of $D - \{d\}$. Let $D' = (D - \{d\}) \cup \{w\}$. Necessarilly D' is a $\mathcal{P}$-dominating set of G with $|D'| = \gamma_{\mathcal{P}}(G)$ and the set D' contains more vertices than the set D which are $\mathcal{P}$-adjacent to some vertices of D'. This contradicts our choice of D. ■

Now we shall establish some properties of $\mathcal{P}$-dominating, strongly $\mathcal{P}$-dominating, $\mathcal{P}$-independent and strongly $\mathcal{P}$-independent sets, and $\mathcal{P}$-irredudant sets.

Proposition 3. *If $D \subseteq V(G)$ is a minimal strongly $\mathcal{P}$-dominating set in G, then D is $\mathcal{P}$-dominating in G.*

Proposition 3 implies the following inequality.

For any graph G,

$$\gamma_{\mathcal{P}}(G) \leq \gamma'_{\mathcal{P}}(G). \tag{1}$$

Proposition 4. *Let G be a graph. If X is a maximal $\mathcal{P}$-independent set in G, then X is a minimal strongly $\mathcal{P}$-dominating set in G.*

Proof. For each vertex $v \in V - X$ a subgraph $G[X \cup \{v\}]$ has no property $\mathcal{P}$. Hence, there exists an induced subgraph H' of $G, H' \simeq H$, $H \in \boldsymbol{C}(\mathcal{P})$, such that $V(H') \cap X = V(H') - \{v\}$. It implies that X is the strongly $\mathcal{P}$-dominating set. Moreover, for each vertex $x \in X$ the set $X - \{x\}$ is not strongly $\mathcal{P}$-dominating. It follows from the fact that there is no induced subgraph $H' \simeq H \in \boldsymbol{C}(\mathcal{P})$ containig the vertex x and $V(H') \subseteq X$. Thus, X is a minimal strongly $\mathcal{P}$-dominating set. ■

From Proposition 4, we obtain the following inequalities.

For any graph G,

$$\gamma'_{\mathcal{P}}(G) \leq i_{\mathcal{P}}(G) \leq \alpha_{\mathcal{P}}(G) \leq \Gamma'_{\mathcal{P}}(G). \tag{2}$$

Proposition 5. *Let G be a graph. If X is a maximal strongly $\mathcal{P}$-indepedent set, then X is a minimal $\mathcal{P}$-dominating set.*

Proof. Let X be a maximal strongly $\mathcal{P}$-independent set in G. Suppose there exists a vertex $v \in V - X$ such that each induced subgraph H' of G such that $v \in V(H')$, $H' \simeq H \in \boldsymbol{C}(\mathcal{P})$ has no common vertices with the set X, thus $X \cup \{v\}$ is strongly $\mathcal{P}$-independent, a contradiction. Hence, for each vertex $v \in V - X$ there is $H' \leq G, H' \simeq H, v \in V(H'), H \in \boldsymbol{C}(\mathcal{P})$ such that $N_{\mathcal{P}}(v) \cap X \neq \emptyset$. Hence, X is $\mathcal{P}$-dominating. Moreover, by the definition of a strongly $\mathcal{P}$-independent set, for each $x \in X, N_{\mathcal{P}}(x) \cap (X - \{x\}) = \emptyset$, thus, X is a minimal $\mathcal{P}$-dominating set in G. ∎

Proposition 5 implies the following property.

For any graph G,

$$\gamma_{\mathcal{P}}(G) \leq i'_{\mathcal{P}}(G) \leq \alpha'_{\mathcal{P}}(G) \leq \Gamma_{\mathcal{P}}(G). \tag{3}$$

Proposition 6. *Let G be a graph without $\mathcal{P}$-isolated vertices. If S is a maximal strongly $\mathcal{P}$-independet set in G, then $V - S$ is strongly $\mathcal{P}$-dominating.*

Proof. By the definition of the strongly $\mathcal{P}$-independet set, for each vertex $v \in S$ there is a subgraph $H', H' \leq G$ such that $v \in V(H'), H' \simeq H \in \boldsymbol{C}(\mathcal{P})$ and $V(H') \cap (V - S) = V(H') - \{v\}$. ∎

Therefore, we obtain.

Let G be a graph without $\mathcal{P}$-isolated vertices. Then

$$\gamma'_{\mathcal{P}}(G) \leq |V(G)| - i'_{\mathcal{P}}(G). \tag{4}$$

Proposition 7. *Let G be a graph. If D is a minimal $\mathcal{P}$-dominating set, then D is maximal $\mathcal{P}$-irredundant.*

Proof. Let D be a minimal $\mathcal{P}$-dominating. By Theorem 1, every vertex $d \in D$ has one of the properties (i) or (ii).
Assume d has the property (i). Thus there exists vertex $v \in V - D$ such that $N_{\mathcal{P}}(v) \cap D = \{d\}$, then $v \in N_{\mathcal{P}}[d]$ and $v \notin N_{\mathcal{P}}[D - \{d\}]$. It implies that $v \in (N_{\mathcal{P}}[d] - N_{\mathcal{P}}[D - \{d\}])$.
Suppose that d has the property (ii) and d has no property (i). Therefore, $d \notin N_{\mathcal{P}}[D - \{d\}]$ and $d \in (N_{\mathcal{P}}[d] - N_{\mathcal{P}}[D - \{d\}])$. Thus, D is an irredundant set in G. Moreover, $N_{\mathcal{P}}(D) = V(G)$ and hence for each $v \in V - D$, the set $D \cup \{v\}$ is not $\mathcal{P}$-irredundant. Hence, D is a maximal $\mathcal{P}$-irredundant set. ∎

From this theorem we have.

For any graph G,

$$ir_{\mathcal{P}}(G) \leq \gamma_{\mathcal{P}}(G) \leq \Gamma_{\mathcal{P}}(G) \leq IR_{\mathcal{P}}(G). \tag{5}$$

Theorem 8. *For any graph G we have the following inequalities*:

$$ir_{\mathcal{P}}(G) \leq \gamma_{\mathcal{P}}(G) \leq i'_{\mathcal{P}}(G) \leq \alpha'_{\mathcal{P}}(G) \leq \Gamma_{\mathcal{P}}(G) \leq IR_{\mathcal{P}}(G). \tag{6}$$

$$ir_{\mathcal{P}}(G) \leq \gamma_{\mathcal{P}}(G) \leq \gamma'_{\mathcal{P}}(G) \leq i_{\mathcal{P}}(G) \leq \alpha_{\mathcal{P}}(G) \leq \Gamma'_{\mathcal{P}}(G). \tag{7}$$

Proof. (6) is obtained from (3) and (5) and (7) from (1), (2), (5). ■

Remark 1. Notice that the inequalities (6) are generalizations of results of Cokayne and Hedetniemi [3].

Remark 2. We know that some of the inequalities are strict for some properties and some graphs.

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