

ON GENERALIZED LIST COLOURINGS OF GRAPHS

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Abstract

Vizing [15] and Erdős et al. [8] independently introduce the idea of considering list-colouring and k -choosability. In the both papers the choosability version of Brooks' theorem [4] was proved but the choosability version of Gallai's theorem [9] was proved independently by Thomassen [14] and by Kostochka et al. [11]. In [3] some extensions of these two basic theorems to (\mathcal{P}, k) -choosability have been proved.

In this paper we prove some extensions of the well-known bounds for the \mathcal{P} -chromatic number to the (\mathcal{P}, k) -choice number and then an extension of Brooks' theorem.

Keywords: hereditary property of graphs, list colouring, vertex partition number.

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1. INTRODUCTION AND NOTATION

All graphs considered in this paper are simple, i.e., finite, undirected, loopless and without multiple edges. The notation $(H \leq G) H \subseteq G$ means that H is (an induced) subgraph of G . We say that G contains H whenever G contains (an induced) subgraph isomorphic to H .

For undefined concepts we refer the reader to [6, 10].

We denote by \mathcal{I} the class of all finite simple graphs. A *graph property* is a nonempty isomorphism-closed subclass of \mathcal{I} . (We also say that a graph has the property \mathcal{P} if $G \in \mathcal{P}$.) A property \mathcal{P} of graphs is called (*induced*) *hereditary* if it is closed under (induced) subgraphs, i.e., if $(H \leq G) H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property \mathcal{P} is called *additive* if it is closed under disjoint unions of graphs, i.e., if every graph has the property \mathcal{P} provided all of its connected components have this property.

Many known properties of graphs are both hereditary and additive. According to [1], [2] we list some of them to introduce the necessary notions which will be used in the paper.

In what follows it is convenient to work with an arbitrary nonnegative integer k .

$$\begin{aligned} \mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\}, \\ \mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \leq k \\ &\quad \text{for each } H \subseteq G\}, \\ \mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \\ &\quad \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\ \mathcal{SP} &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_4\}. \end{aligned}$$

We have $\mathcal{D}_1 = \mathcal{T}_1$ to be the class of all forests, \mathcal{T}_2 and \mathcal{T}_3 the class of all outerplanar and all planar graphs, respectively and \mathcal{SP} the class of all series-parallel graphs.

Let us denote by (\mathbb{M}) , (\mathbb{L}) and (\mathbb{M}^a) , (\mathbb{L}^a) the set of all (induced) hereditary and additive (induced) hereditary properties, respectively. It can be proved that (\mathbb{L}, \subseteq) is a proper sublattice of (\mathbb{M}, \subseteq) . For more details, see [1].

A hereditary property $\mathcal{P} \in \mathbb{M}$ is said to be nontrivial if $\mathcal{P} \neq \mathcal{I}$. Let $\delta(\mathcal{P}) = \min\{\delta(H) : H \in \mathcal{C}(\mathcal{P})\}$, where $\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H-v) \in \mathcal{P} \text{ for any } v \in V(H)\}$.

A \mathcal{P} -*partition* (*colouring*) of a graph G is a partition (V_1, \dots, V_n) of $V(G)$ such that the subgraph $G[V_i]$ induced by the set V_i has the property \mathcal{P} for

each $i = 1, \dots, n$. If (V_1, \dots, V_n) is a \mathcal{P} -partition of a graph G , then the corresponding vertex colouring f is defined by $f(v) = i$ whenever $v \in V_i$, for $i = 1, \dots, n$. The smallest integer n for which G has a \mathcal{P} -partition is called the \mathcal{P} -chromatic (or \mathcal{P} -vertex-partition) number of G and is denoted by $\chi_{\mathcal{P}}(G)$. The \mathcal{O} -chromatic number is the ordinary chromatic number. See [1], [2] for a survey and more details.

Let G be a graph and let $L(v)$ be a list of colours (say, positive integers) prescribed for the vertex v , and $\mathcal{P} \in \mathbb{M}$. A (\mathcal{P}, L) -colouring is a graph \mathcal{P} -colouring $f(v)$ with the additional requirement that for all $v \in V(G)$, $f(v) \in L(v)$. If G admits a (\mathcal{P}, L) -colouring, then G is said to be (\mathcal{P}, L) -colourable. The graph G is (\mathcal{P}, k) -choosable if it is (\mathcal{P}, L) -colourable for every list assignment L of G satisfying $|L(v)| = k$ for every $v \in V(G)$. The \mathcal{P} -choice number $\text{ch}_{\mathcal{P}}(G)$ is the smallest natural number k such that G is (\mathcal{P}, k) -choosable.

For a nontrivial property $\mathcal{P} \in \mathbb{M}$ a graph G is said to be (\mathcal{P}, L) -critical if G has no (\mathcal{P}, L) -colouring but $G - v$ is (\mathcal{P}, L) -colourable for all $v \in V(G)$. A graph G is said to be (vertex) (\mathcal{P}, k) -choice critical if $\text{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\text{ch}_{\mathcal{P}}(G - v) < k$ for all vertices v of G . According to the previous definitions, it follows immediately that if G is $(\mathcal{P}, k+1)$ -choice critical, then G is (\mathcal{P}, L) -critical for some list assignment with $|L(v)| = k$ for all $v \in V(G)$.

The idea of considering (\mathcal{O}, L) -colouring and (\mathcal{O}, k) -choosability has been introduced independently by Vizing [15] and Erdős, Rubin and Taylor [8]. In both papers the choosability version of Brooks' theorem [4] was proved. The choosability version of Gallai's theorem [9] was proved independently by Thomassen [14] and by Kostochka et al. [11]. In [3] some extensions of these two basic theorems to (\mathcal{P}, k) -choosability, have been proved.

The aim of this paper is to prove some extensions of the well-known bounds for the \mathcal{P} -chromatic number [2], [3], [4] to the (\mathcal{P}, k) -choice number and an extension of Brooks' theorem for this number.

2. RESULTS

Let us recall two results which we will use later.

Lemma 1 [3]. *Let $\mathcal{P} \in \mathbb{M}$ and G be (\mathcal{P}, L) critical. Then $d_G(v) \geq \delta(\mathcal{P}) |L(v)|$ for any vertex v of G .*

Theorem 1 [3]. *Let $\mathcal{P} \in \mathbb{M}^a$ and G be a connected graph other than*

- (i) *a complete graph of order $n\delta(\mathcal{P}) + 1$, $n \geq 0$,*
- (ii) *a $\delta(\mathcal{P})$ -regular graph belonging to $\mathcal{C}(\mathcal{P})$,*
- (iii) *an odd cycle if $\mathcal{P} = \mathcal{O}$.*

Then

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil.$$

Theorem 2. Let $\mathcal{P} \in \mathbb{M}$, $\delta(\mathcal{P}) \geq 1$, and let f be a real valued function on graphs $G \in \mathcal{I}$ with properties

- (a) $f(H) \leq f(G)$, for any induced subgraph H of G ,
- (b) $f(G) \geq \delta(G)$, for each graph.

Then

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lfloor \frac{f(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Proof. Obviously, the theorem holds for G with $\text{ch}_{\mathcal{P}}(G) = 1$. Let $\text{ch}_{\mathcal{P}}(G) = t + 1 \geq 2$, and let H be a $(\mathcal{P}, t + 1)$ -choice critical induced subgraph of G . Thus there exists a list assignment with $|L(v)| = t$ for all $v \in V(G)$ and H is (\mathcal{P}, L) -critical. Lemma 1 implies $\delta(H) \geq t\delta(\mathcal{P})$, and (a) yields that $f(H) \leq f(G)$. Thus, by (b) we have $t\delta(\mathcal{P}) \leq \delta(H) \leq f(H) \leq f(G)$. Hence,

$$\text{ch}_{\mathcal{P}}(G) = t + 1 \leq \left\lfloor \frac{f(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Corollary 1. Let $\rho(G) = \max\{\delta(H) : H \leq G\}$. Then

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lfloor \frac{\rho(G)}{\delta(\mathcal{P})} \right\rfloor + 1.$$

Proof. It easy to see that $\rho(G)$ satisfies (a) and (b) of the above theorem.

Corollary 2. Let $\mathcal{P} = \mathcal{D}_r$ and $G \in \mathcal{D}_k$, $r \leq k$. Then

$$\text{ch}_{\mathcal{P}}(G) \leq \left\lfloor \frac{k}{r + 1} \right\rfloor + 1.$$

Proof. The statement follows by the previous corollary because $\rho(G) \leq k$ for every k -degenerate graph G and $\delta(\mathcal{D}_r) = r + 1$.

Corollary 3. Let $G \in \mathcal{T}_3$. Then $\text{ch}_{\mathcal{D}_1}(G) \leq 3$.

Proof. The statement follows by the previous corollary because any planar graph is 5-degenerate.

Corollary 3 implies the well-known result of Chartrand and Kronk [5]: *The vertex-arboricity of any planar graph is at most 3.*

Similarly as above, we have a corresponding result for series-parallel and outerplanar graphs which are 2-degenerate. This Corollary implies Dirac's [7] result: $\mathcal{SP} \subset \mathcal{D}_1^2$.

Corollary 4. *Let $G \in \mathcal{SP}$. Then $\text{ch}_{\mathcal{D}_1}(G) \leq 2$.*

In [12] Mihók obtained an extension of Brooks' theorem. From the previous results and the following Mihók's lemma we derive a choosability version of his result.

Lemma 2 [12]. *Let G be a connected graph with $\delta(G) \geq k \geq 1$, other than K_{k+1} . Then G contains as a subgraph every tree on $k + 2$ vertices, except for $T = K_{1,k+1}$ if G is k -regular.*

Theorem 3. *Let $\mathcal{P} \in \mathbf{L}^a$ with $\delta(\mathcal{P}) \geq 1$. If a connected graph G does not contain a given tree T on $k + 2$ vertices, $k \geq 3$, as a subgraph, then G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$ -choosable unless $G = K_{k+1}$.*

Proof. Assume that G is a connected graph $\neq K_{k+1}$ and G does not contain a given tree T on $k + 2$ vertices. If $T = K_{1,k+1}$, then by Theorem 1, G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$ -choosable. If $T \neq K_{1,k+1}$, then by Lemma 2, G must be $(k - 1)$ -degenerate, otherwise G contains T . Thus by Corollary 1, G is $(\mathcal{P}, \lceil k/\delta(\mathcal{P}) \rceil)$ -choosable, too.

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