

UNIQUELY PARTITIONABLE GRAPHS

JOZEF BUCKO

Department of Geometry and Algebra, P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: bucko@duro.upjs.sk

MARIETJIE FRICK¹

Department of Mathematics, Applied Mathematics and Astronomy
University of South Africa, P.O. Box 392, Pretoria, 0001 South Africa
e-mail: frickm@risc5.unisa.ac.za

PETER MIHÓK² AND ROMAN VASKY²

Department of Geometry and Algebra, P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovak Republic
e-mail: mihok@kosice.upjs.sk
e-mail: vasky@duro.upjs.sk

Abstract

Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of a graph G is a partition of the vertex set $V(G)$ into subsets V_1, \dots, V_n such that the subgraph $G[V_i]$ induced by V_i has property \mathcal{P}_i ; $i = 1, \dots, n$. A graph G is said to be uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable if G has exactly one $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition. A property \mathcal{P} is called hereditary if every subgraph of every graph with property \mathcal{P} also has property \mathcal{P} . If every graph that is a disjoint union of two graphs that have property \mathcal{P} also has property \mathcal{P} , then we say that \mathcal{P} is additive. A property \mathcal{P} is called degenerate if there exists a bipartite graph that does not have property \mathcal{P} . In this paper, we prove that if $\mathcal{P}_1, \dots, \mathcal{P}_n$ are degenerate, additive, hereditary properties of graphs, then there exists a uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.

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1. NOTATION AND BACKGROUND

All graphs considered in this paper are finite and simple. In general, we follow the notation and terminology of [15].

We denote the set of all mutually nonisomorphic graphs by \mathcal{I} . Each nonempty subset $\mathcal{P} \subseteq \mathcal{I}$ is also said to be a *property of graphs*. A property \mathcal{P} is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. A property \mathcal{P} is *additive* if $G_1, G_2 \in \mathcal{P}$ implies that the disjoint union $G_1 \cup G_2$ is also in \mathcal{P} . We shall denote the set of all hereditary properties by \mathbb{L} , and the set of all additive, hereditary properties by \mathbb{L}^a . We list some additive, hereditary properties in Table 1. (We use the notation of [6] for most of them).

Table 1

The property	The graphs which have the property
\mathcal{O}	$G \in \mathcal{I}$; G is totally disconnected
\mathcal{S}_k	$G \in \mathcal{I}$; $\Delta(G) \leq k$
\mathcal{W}_k	$G \in \mathcal{I}$; the length of the longest path in G does not exceed k
\mathcal{D}_k	$G \in \mathcal{I}$; G is k -degenerate i.e., $\delta(H) \geq k$ for $H \subseteq G$
\mathcal{T}_k	$G \in \mathcal{I}$; G contains no subgraph homeomorphic to K_{k+2} or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}$
$\overline{\mathcal{I}}_k$	$G \in \mathcal{I}$; G does not contain K_{k+2} as a subgraph

Any hereditary property \mathcal{P} is uniquely determined by the set

$$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} \mid G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}$$

of *minimal forbidden subgraphs* (see [6], [14], [16], [18]), or by the set of so-called \mathcal{P} -*maximal* graphs

$$\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} \mid G + e \notin \mathcal{P} \text{ for every } e \in \overline{G}\},$$

(see [6], [24], [29]).

The *join* of two vertex disjoint graphs G_1 and G_2 is obtained by joining every vertex of G_1 to every vertex of G_2 , and is denoted by $G_1 + G_2$.

A graph G is said to be \mathcal{P} -*strict* if $G \in \mathcal{P}$ and $G + K_1 \notin \mathcal{P}$.

Let \mathcal{P} be a hereditary property, $\mathcal{P} \neq \mathcal{I}$. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of \mathcal{P} . Clearly, every \mathcal{P} -maximal graph G with $|V(G)| \geq c(\mathcal{P}) + 1$ is \mathcal{P} -strict.

For any property \mathcal{P} we define the minimum degree of \mathcal{P} as

$$\delta(\mathcal{P}) = \min\{\delta(G) \mid G \in \mathbf{F}(\mathcal{P})\},$$

and the chromatic number of \mathcal{P} as

$$\chi(\mathcal{P}) = \min\{\chi(G) \mid G \in \mathbf{F}(\mathcal{P})\}.$$

Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of a graph G is a partition $\{V_1, \dots, V_n\}$ of $V(G)$ such that the subgraph $G[V_i]$ induced by V_i has property \mathcal{P}_i for $i = 1, \dots, n$. The property $\mathcal{R} = \mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$ is defined as the set of all graphs that have a $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition.

If $\mathcal{P}_1 = \dots = \mathcal{P}_n$, the property $\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$ will be denoted by \mathcal{P}^n . For example, the class of all n -colourable graphs is denoted by \mathcal{O}^n .

If there exist properties \mathcal{P} and \mathcal{Q} such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$, then \mathcal{R} is said to be a *reducible* property and \mathcal{P} , and \mathcal{Q} are said to *divide* \mathcal{R} ; otherwise \mathcal{R} is called *irreducible* (see e.g., [6], [20], [23]). Different generalizations of regular colouring of the vertices of graphs (see e.g. [1], [2], [8], [9], [10], [11], [12], [20], [21], [22], [25], [26], [31]) can be expressed using the notion of reducible properties.

We shall need the following two lemmas concerning reducible properties.

Lemma 1. *If \mathcal{P}_1 and \mathcal{P}_2 are (additive) hereditary properties of graphs, then the property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is also (additive) hereditary.*

Lemma 2. *Let \mathcal{P}_1 and \mathcal{P}_2 be hereditary properties of graphs and let G be a $\mathcal{P}_1 \circ \mathcal{P}_2$ -maximal graph. If $\{V_1, V_2\}$ is any $(\mathcal{P}_1, \mathcal{P}_2)$ -partition of G , then*

$$G = G[V_1] + G[V_2]$$

and the graph $G[V_i]$ are \mathcal{P}_i -maximal, $i = 1, 2$.

Proof. Suppose that there exists an edge $e = (x, y)$ such that $x \in V_1$ and $y \in V_2$ and $e \notin E(G)$. Then the graph $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$, contradicting our assumption that $G \in \mathbf{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$. This proves that $G = G[V_1] + G[V_2]$.

Now suppose $G[V_1]$ is not \mathcal{P}_1 -maximal. Then $G[V_1] + e \in \mathcal{P}_1$ for some $e \in E(\overline{G[V_1]})$. But then, again, $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$. This contradiction proves that $G[V_1]$ is \mathcal{P}_1 -maximal. Likewise, $G[V_2]$ is \mathcal{P}_2 -maximal. ■

A graph $G \in \mathcal{P}_1 \circ \dots \circ \mathcal{P}_n$ is said to be *uniquely* $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -*partitionable* if G has exactly one $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition. The set of all uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs will be denoted by $\mathbf{U}(\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n)$, e.g., $\mathbf{U}(\mathcal{O}^n)$ denotes the set of all uniquely n -colourable graphs (see [4], [19], [17]); $\mathbf{U}(\mathcal{S}_k^n)$ denotes the set of all uniquely $(m, k)^\Delta$ -colourable graphs (see [12], [13], [32]); $\mathbf{U}(\mathcal{W}_k^n)$ has been studied in [12], [3] and $\mathbf{U}(\mathcal{D}_k^n)$ in [5], [27], and $\mathbf{U}(\mathcal{I}_k^n)$ in [7], [12]. The basic properties of $\mathbf{U}(\mathcal{P}^n)$ have been investigated in [5], [23]. Another generalization of uniquely colourable graphs was introduced by X. Zhu in [33].

The notion of *degenerate* hereditary property appeared with regards to the famous Erdős-Simonovits formula

$$\text{ext}(n, \mathcal{P}) = \frac{\chi(\mathcal{P}) - 2}{\chi(\mathcal{P}) - 1} \binom{n}{2} + o(n^2),$$

where

$$\text{ext}(n, \mathcal{P}) = \max\{|E(G)| \mid G \in \mathcal{P} \text{ and } |V(G)| = n\},$$

A property $\mathcal{P} \in \mathbb{L}^a$ is said to be *degenerate* if $\chi(\mathcal{P}) = 2$, i.e., if $\mathbf{F}(\mathcal{P})$ contains some bipartite graph (see [29], [30]). Obviously, $\mathcal{O}, \mathcal{S}_k, \mathcal{Q}_k, \mathcal{O}_k, \mathcal{D}_k$ and \mathcal{I}_k are degenerate properties of graphs, but the property \mathcal{I}_k is not degenerate.

In [23] it is proved that if the property \mathcal{P} is a reducible property of graphs, then $\mathbf{U}(\mathcal{P}^n) = \emptyset$ and we also proved that $\mathbf{U}(\mathcal{P}^n) \neq \emptyset$ for every degenerate property \mathcal{P} , which means that every degenerate property is irreducible. In Section 4 of this paper, we generalize this result by proving that $\mathbf{U}(\mathcal{P}_1 \circ \dots \circ \mathcal{P}_n) \neq \emptyset$ if $\mathcal{P}_1, \dots, \mathcal{P}_n$ are degenerate, additive, hereditary properties.

In Section 2 we present some basic properties of uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs, generalizing results known to hold for uniquely colourable graphs.

In Section 3 we provide a necessary and sufficient condition for one hereditary property to be divisible by another. This result is used to prove our main result, Theorem 3, which gives a necessary and sufficient condition for the existence of uniquely $\mathcal{P} \circ \mathcal{Q}$ -partitionable graphs, when \mathcal{P} and \mathcal{Q} are additive, hereditary properties and \mathcal{Q} is degenerate.

2. BASIC PROPERTIES OF UNIQUELY PARTITIONABLE GRAPHS

The results on uniquely \mathcal{P}^n -partitionable graphs obtained in [23] can be directly generalized to obtain the properties of uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graphs presented in the following two theorems.

Theorem 1. *Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be hereditary properties of graphs, $n \geq 2$. If G is a uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph and $\{V_1, \dots, V_n\}$ is the unique $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of $V(G)$, then*

1. $G \notin \mathcal{P}_1 \circ \dots \circ \mathcal{P}_{j-1} \circ \mathcal{P}_{j+1} \circ \dots \circ \mathcal{P}_n$ for $j = 1, \dots, n$,
2. the subgraphs $G[V_i]$ are \mathcal{P}_i -strict, $i = 1, 2, \dots, n$,
3. if $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, then $V_{i_1} \cup \dots \cup V_{i_k}$ induces a uniquely $(\mathcal{P}_{i_1}, \dots, \mathcal{P}_{i_n})$ -partitionable subgraph of G ,
4. $\delta(G) \geq \max_j \sum_{i=1, i \neq j}^n \delta(\mathcal{P}_i)$,
5. $|V(G)| \geq \sum_{i=1}^n (c(\mathcal{P}_i) + 2) - 1$,
6. the graph $G = G[V_1] + \dots + G[V_n]$ is uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable.

Theorem 2. *Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be hereditary properties of graphs. If $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$ and $\mathbf{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n) \neq \emptyset$, then G is an induced subgraph of some uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.*

3. DIVISIBILITY AND UNIQUELY $(\mathcal{P}, \mathcal{Q})$ -PARTITIONABLE GRAPHS

Lemma 3. *If \mathcal{P} and \mathcal{Q} are properties of graphs such that one of the following holds:*

1. \mathcal{P} divides \mathcal{Q}
2. \mathcal{Q} divides \mathcal{P}
3. there exists a property \mathcal{S} such that \mathcal{S} divides both \mathcal{P} and \mathcal{Q} ,

then $\mathbf{U}(\mathcal{P} \circ \mathcal{Q}) = \emptyset$.

Proof. 1. Suppose $\mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$ for some property \mathcal{Q}^* . Let $G \in \mathcal{P} \circ \mathcal{Q}$ and let $\{V_1, V_2\}$ be a $(\mathcal{P}, \mathcal{Q})$ -partition of G , with $V_1, V_2 \neq \emptyset$. Since $G[V_2] \in \mathcal{Q} = \mathcal{P} \circ \mathcal{Q}^*$, there exists a partition $\{V_{21}, V_{22}\}$ of $G[V_2]$, with $V_{21}, V_{22} \neq \emptyset$, such that $G[V_{21}] \in \mathcal{P}$ and $G[V_{22}] \in \mathcal{Q}^*$. But then $G[V_1 \cup V_{22}] \in \mathcal{P} \circ \mathcal{Q}^* = \mathcal{Q}$, and thus $\{V_{21}, V_1 \cup V_{22}\}$ is a $(\mathcal{P}, \mathcal{Q})$ -partition of G different from $\{V_1, V_2\}$, which implies that G is not uniquely $(\mathcal{P}, \mathcal{Q})$ -partitionable.

Cases (2) and (3) can be proved in an analogous way. ■

If \mathcal{P} and \mathcal{Q} are additive hereditary properties and \mathcal{Q} is also degenerate, then converse of Lemma 1 also holds. In order to prove this, we introduce the concept of an extendible set.

Let \mathcal{P} and \mathcal{Q} be hereditary properties of graphs and let $G \in \mathcal{P}$. If S is a subset of $V(G)$ such that $G[S] \in \mathcal{Q}$ and for every graph $T \in \mathcal{Q}$ the graph $T + (G - S) \in \mathcal{P}$, then S is said to be a $(\mathcal{Q}, \mathcal{P})$ -*extendible set* of G .

We shall need the following lemma.

Lemma 4. *Let \mathcal{P} and \mathcal{Q} be hereditary properties of graphs. If H is a graph with property \mathcal{P} that has no $(\mathcal{Q}, \mathcal{P})$ -extendible set, then there exists a \mathcal{P} -strict graph G such that G has no $(\mathcal{Q}, \mathcal{P})$ -extendible set.*

Proof. Let H be a graph with property \mathcal{P} such that H has no $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let G be a \mathcal{P} -strict graph such that $H \subseteq G$. Suppose, to the contrary, that G contains a $(\mathcal{Q}, \mathcal{P})$ -extendible set S . Let $S' = S \cap V(H)$. Let T be any graph with property \mathcal{Q} . Then

$$T + (G - S) \in \mathcal{P}.$$

Since $T + (H - S') \subseteq T + (H - S)$ and \mathcal{P} is hereditary, this implies that $T + (H - S') \in \mathcal{P}$, so that S' is an extendible set of H . \blacksquare

We have the following connection between divisibility and the existence of an extendible set.

Theorem 3. *Let \mathcal{P} and \mathcal{Q} be hereditary properties of graphs. Then \mathcal{Q} divides \mathcal{P} if and only if every \mathcal{P} -maximal graph contains a $(\mathcal{Q}, \mathcal{P})$ -extendible set.*

Proof. Suppose \mathcal{Q} divides \mathcal{P} . Then there is a property \mathcal{P}^* such that $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$. Let $G \in \mathcal{P}$ and let $\{V_1, V_2\}$ be a $(\mathcal{Q}, \mathcal{P}^*)$ -partition of G . Let T be any graph with property \mathcal{Q} . Then $\{V(T), V_2\}$ is a $(\mathcal{Q}, \mathcal{P}^*)$ -partition of $T + G[V_2]$, and hence $T + G[V_2] \in \mathcal{P}$. Since $G[V_2] = G - V_1$, this proves that V_1 is a $(\mathcal{Q}, \mathcal{P})$ -extendible set of G .

To prove the converse, suppose every \mathcal{P} -maximal graph contains a $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let

$$\mathcal{S}(G) = \{S \subseteq V(G) \mid S \text{ is an extendible set of } G\}$$

and put

$$\mathcal{P}' = \{G - S \mid G \in \mathbf{M}(\mathcal{P}), S \in \mathcal{S}(G)\}.$$

Now let \mathcal{P}^* be the property consisting of all subgraphs of graphs in \mathcal{P}' . Then \mathcal{P}^* is a hereditary property. We shall prove that $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$.

Suppose $G \in \mathbf{M}(\mathcal{P})$. Then, by our assumption, G has a $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let S be such a set. Then $G - S \in \mathcal{P}'$, by the definition

of \mathcal{P}^* . Thus $\{S, G - S\}$ is a $(\mathcal{Q}, \mathcal{P}^*)$ -partition of G , so that $G \in \mathcal{Q} \circ \mathcal{P}^*$. This proves that $\mathbf{M}(\mathcal{P}) \subseteq \mathcal{Q} \circ \mathcal{P}^*$. But $\mathcal{Q} \circ \mathcal{P}^*$ is a hereditary property by Lemma 2, and hence $\mathcal{P} \subseteq \mathcal{Q} \circ \mathcal{P}^*$.

Now suppose $G \in \mathbf{M}(\mathcal{Q} \circ \mathcal{P}^*)$. Let $\{V_1, V_2\}$ be a $(\mathcal{Q}, \mathcal{P}^*)$ -partition of G . Then it follows from Lemma 2 that $G[V_1] \in \mathbf{M}(\mathcal{Q})$, $G[V_2] \in \mathbf{M}(\mathcal{P}^*)$ and $G = G[V_1] + G[V_2]$. By the definition of \mathcal{P}^* there exists a \mathcal{P} -maximal graph F and a $(\mathcal{Q}, \mathcal{P})$ -extendible set S of F such that $G[V_2] \subseteq F - S$. But then, since $G[V_1] \in \mathcal{Q}$, we have $G[V_1] + F - S \in \mathcal{P}$. But $G \subseteq G[V_1] + F - S$, and hence $G \in \mathcal{P}$. This proves that $\mathcal{Q} \circ \mathcal{P}^* \subseteq \mathcal{P}$. ■

Theorem 4. *Let $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$ and let \mathcal{Q} be a degenerate property. Then $U(\mathcal{Q} \circ \mathcal{P}) \neq \emptyset$ if and only if \mathcal{Q} does not divide \mathcal{P} .*

Proof. If $U(\mathcal{Q} \circ \mathcal{P}) \neq \emptyset$ then, by Lemma 3, \mathcal{Q} does not divide \mathcal{P} .

To prove the converse, suppose \mathcal{Q} does not divide \mathcal{P} . Then it follows from Theorem 3 and Lemma 4 that there exists a \mathcal{P} -strict graph H that contains no $(\mathcal{Q}, \mathcal{P})$ -extendible set. Let

$$Z = \{S \mid S \subseteq V(H) \text{ and } H[S] \in \mathcal{Q}\}.$$

Then, for every $S \in Z$, there exists a \mathcal{Q} -strict graph $T(S)$ such that

$$T(S) + (H - S) \notin \mathcal{P}.$$

Now let

$$T = \cup_{S \in Z} T(S).$$

Since \mathcal{Q} is a degenerate property, there is an integer q such that $K_{q,q} \notin \mathcal{Q}$. Let

$$G_1 = qT, \quad G_2 = qH, \quad \text{and} \quad G = G_1 + G_2.$$

Since \mathcal{P} and \mathcal{Q} are additive properties, $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{Q}$, and thus $G \in \mathcal{P} \circ \mathcal{Q}$.

Now let $\{W_1, W_2\}$ be any $(\mathcal{P}, \mathcal{Q})$ -partition of G . Suppose each of the q copies of H in G_2 has at least one vertex in W_1 . Then

$$|V(G_2) \cap W_1| \geq q.$$

Now let H_0 be a specific copy of H in G_2 , and let $S_0 = V(H_0) \cap W_1$. Then $H_0[S_0] \in \mathcal{Q}$ and hence, by the definition of T , we have

$$T + H_0 - S_0 \notin \mathcal{P}.$$

Since $V(H_0) - S_0 \in W_2$, it follows that none of the q copies of T in G_1 has all its vertices in W_2 . Thus

$$|V(G_1) \cap W_1| \geq q.$$

But then $K_{q,q} \subseteq G[W_1]$. This contradiction proves that at least one of the q copies of H in G_2 has all its vertices in W_2 . Since H is \mathcal{P} -strict, it follows that $W_2 \cap V(G_1) = \emptyset$. But G_1 is \mathcal{Q} -strict, and hence $W_1 = G(V_1)$, which implies that $\{V(G_1), V(G_2)\}$ is the only $(\mathcal{Q}, \mathcal{P})$ -partition of G . Thus $G \in \mathcal{U}(\mathcal{Q} \circ \mathcal{P})$. \blacksquare

4. CONSTRUCTION OF UNIQUELY $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -PARTITIONABLE GRAPHS FOR DEGENERATE PROPERTIES

Uniquely (\mathcal{P}^n) -partitionable graphs have been proved to exist for several specific degenerate properties \mathcal{P} (see [5], [23], [28]). The following theorem generalizes those results.

Theorem 5. *Let $\mathcal{P}_1, \dots, \mathcal{P}_n$, be degenerate, additive, hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable graph.*

Proof. We may assume, without loss of generality, that the properties $\mathcal{P}_1, \dots, \mathcal{P}_n$ are ordered in such a way that $\mathcal{P}_i \not\subseteq \mathcal{P}_j$ if $i < j$ and, if $\mathcal{P}_i = \mathcal{P}_j$ and $i < k < j$, then $\mathcal{P}_i = \mathcal{P}_k$. Then there exist graphs H_1, \dots, H_n such that H_i is \mathcal{P}_i -strict for $i = 1, \dots, n$ and, if $i < j$, then $H_i \notin \mathcal{P}_j$ unless $\mathcal{P}_i = \mathcal{P}_j$. Since $\mathcal{P}_1, \dots, \mathcal{P}_n$ are degenerate properties, there exists an integer q such that $K_{q,q} \notin \mathcal{P}_i$ for $i = 1, \dots, n$. Now let

$$G_i = (n(q-1) + 1)H_i \text{ for } i = 1, \dots, n.$$

and

$$G = G_1 + \dots + G_n.$$

We shall prove, by induction on n , that the graph G thus constructed is uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable.

The result is true for $n = 1$.

Now let $n \geq 2$. Put

$$V_i = V(G_i), \quad i = 1, \dots, n.$$

Let $\{W_1, \dots, W_n\}$ be any $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partition of G . Since $|V(G_i)| \geq n(q-1) + 1$ for each $i = 1, \dots, n$, we have that, for each $i \in \{1, \dots, n\}$

$$|V_i \cap W_j| \geq q \text{ for at least one } j \in \{1, \dots, n\}.$$

Now suppose two different members of $\{W_1, \dots, W_n\}$ each contain at least q vertices of V_1 . Then there are at least $n+1$ sets of the form $V_i \cap W_j$ whose cardinality is at least q . Then, by Dirichlet's principle, there exist integers $i, r, s \in \{1, \dots, n\}$ with $r \neq s$ such that

$$|W_i \cap V_r| \geq q \text{ and } |W_i \cap V_s| \geq q,$$

and thus

$$K_{q,q} \subseteq G[W_i].$$

This contradiction proves that only one of the W_i , say W_t , contains at least q vertices of V_1 . Since $G[V_1]$ contains $n(q-1)+1$ copies of H_1 , at least one of these copies has all its vertices in W_t . Our assumption on the ordering of the properties $\mathcal{P}_1, \dots, \mathcal{P}_n$, implies that $H_1 \notin \mathcal{P}_i$ for $i = 2, \dots, n$ unless $\mathcal{P}_i = \mathcal{P}_1$. We may therefore assume, without loss of generality, that $t = 1$. Since H_1 is \mathcal{P}_1 -strict, it then follows that

$$W_1 \cap V_i = \emptyset \text{ for } i = 2, \dots, n.$$

By our induction hypothesis, the graph $G_2 + \dots + G_n$ is uniquely $(\mathcal{P}_2, \dots, \mathcal{P}_n)$ -partitionable, so that $\{V_2, \dots, V_n\}$ is the only $(\mathcal{P}_2, \dots, \mathcal{P}_n)$ -partition of $G_2 + \dots + G_n$. Thus, for each $i \in \{2, \dots, n\}$, we have that $W_i \supseteq V_i$ and hence, since G_i is \mathcal{P}_i -strict, $W_i = V_i$. This implies that $\{W_1, \dots, W_n\} = \{V_1, \dots, V_n\}$, and hence G is uniquely $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ -partitionable. ■

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