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MAXIMAL GRAPHS WITH RESPECT TO HEREDITARY PROPERTIES

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Abstract

A property of graphs is a non-empty set of graphs. A property $\mathcal{P}$ is called hereditary if every subgraph of any graph with property $\mathcal{P}$ also has property $\mathcal{P}$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. We say that a graph G has property $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ if the vertex set of G can be partitioned into n sets $V_1, \ldots, V_n$ such that the subgraph of G induced by V_i has property $\mathcal{P}_i$; $i = 1, \ldots, n$. A hereditary property $\mathcal{R}$ is said to be reducible if there exist two hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$. If $\mathcal{P}$ is a hereditary property, then a graph G is called $\mathcal{P}$- maximal if G has property $\mathcal{P}$ but $G+e$ does not have property $\mathcal{P}$ for every $e \in E(\overline{G})$. We present some general results on maximal graphs and also investigate $\mathcal{P}$-maximal graphs for various specific choices of $\mathcal{P}$, including reducible hereditary properties.

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1. Introduction and Notation

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For undefined concepts we refer the reader to [6] and [4].

Since we have in general no reason to distinguish between isomorphic copies of a graph, we use the notation $\mathcal{I}$ to denote the set of all mutually non-isomorphic graphs, considered as unlabelled graphs. Therefore, by saying that H is a subgraph of G, we mean that H is isomorphic to a subgraph of G.

If $\mathcal{P}$ is a non-empty subset of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$. We shall use the terms *set of graphs* and *property of graphs* interchangeably. A property $\mathcal{P}$ is called *additive* if for all graphs $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{P}$ we have that the disjoint union $G_1 \cup G_2 \in \mathcal{P}$. A property $\mathcal{P}$ is *hereditary* if it is closed with respect to the relation $\subseteq$ to be a *subgraph*.

In the sequel we shall concentrate on the following concrete hereditary properties — we use the notation of [4] for most of them:

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is totally disconnected}\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \le k\},\\
\mathcal{W}_k &= \{G \in \mathcal{I} : \text{ the length of the longest path in } G \text{ is at most } k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},\\
\to H &= \{G \in \mathcal{I} : G \text{ is homomorphic to } H\}.
\end{aligned}$$

It is easy to verify that $\mathcal{O}_k \subseteq \mathcal{S}_k \subseteq \mathcal{D}_k \subseteq \mathcal{I}_k$, $\mathcal{O}_k \subseteq \mathcal{W}_k \subseteq \mathcal{D}_k$ and $\mathcal{O}_1 = \mathcal{S}_1 = \mathcal{W}_1$.

Let $\mathcal{P}$ be a hereditary property, $\mathcal{P} \neq \mathcal{I}$. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$ (for more details see [4]). Obviously $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$. It is also easy to see that $c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{W}_k) = c(\mathcal{D}_k) = c(\mathcal{I}_k) = k$.

For a hereditary property $\mathcal{P}$ we define the set of *minimal forbidden subgraphs* of $\mathcal{P}$ by

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}.$$

A direct consequence of this definition is

Lemma 1. *Let $\mathcal{P}$ be a hereditary property. Then $G \in \mathcal{P}$ if and only if no subgraph of G is in $\boldsymbol{F}(\mathcal{P})$.*

Thus any hereditary property is uniquely determined by its set of minimal forbidden subgraphs. An alternative way is to characterize $\mathcal{P}$ by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs. To be more accurate, let us define the set of $\mathcal{P}$-*maximal graphs* by

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}$$

and the set of $\mathcal{P}$-*maximal graphs of order* n by

$$\boldsymbol{M}(n, \mathcal{P}) = \{G \in \mathcal{P} : |V(G)| = n \text{ and } G \in \boldsymbol{M}(\mathcal{P})\}.$$

From these definitions it is evident that $\boldsymbol{M}(\mathcal{P}) = \bigcup_{n \geq 1} \boldsymbol{M}(n, \mathcal{P})$.

Let n be a positive integer and let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partition of a graph* G is a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, \ldots, n$ the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$, we shall call a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition a $(\mathcal{P}^n)$-partition.

The property $\mathcal{R} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition and is called the *product* of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$ we write $\mathcal{R} = \mathcal{P}^n$. If $\mathcal{R} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, we call $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ a *factorization* of $\mathcal{R}$, and we say $\mathcal{R}$ is *divisible* by $\mathcal{P}_i$, $i = 1, \ldots, n$.

The next lemma follows immediately from our definitions.

Lemma 2 [4]**.** *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be additive hereditary properties of graphs and let $\mathcal{R} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. Then $\mathcal{R}$ is additive and hereditary and the completeness of $\mathcal{R}$ is $c(\mathcal{R}) = c(\mathcal{P}_1) + \cdots + c(\mathcal{P}_n) + n - 1$.*

A hereditary property (an additive hereditary property) $\mathcal{R}$ is called *reducible* (see also [2], [4], [5] and [14]) if there are two hereditary properties (additive hereditary properties respectively) $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise $\mathcal{R}$ is called *irreducible*.

A graph G is said to be *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partitionable* if G has a unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition (permutations of partitions allowed). Note that, if G is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1, \ldots, V_n\}$ is the unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of G, then $V_i \neq \emptyset$ for $i = 1, \ldots, n$. It is shown in [13] that, if $\mathcal{P}$ is a reducible property, then there are no uniquely $(\mathcal{P}^n)$-partitionable graphs.

A vertex of a graph G that has degree equal to $|V(G)| - 1$ is called a *universal* vertex of G.

We say that a graph G is the *join* of n graphs $G_1, \ldots, G_n$ and write $G = G_1 + \cdots + G_n$ if $V(G) = \bigcup_{i=1}^{n} V(G_i)$ and $E(G) = \{xy : xy \in E(G_i)$ for some i, or $x \in V(G_i)$ and $y \in V(G_j);\ i \neq j\}$.

If a graph G is a join of two non-empty graphs, we say that G is *decomposable*; otherwise, G is *indecomposable*. In [5] we show for various properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ that the existence of indecomposable $\mathcal{P}_i$-maximal graphs ensures the existence of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs. The same idea is used in [1], but with each $\mathcal{P}_i = \mathcal{W}_k$.

We shall show in this paper that, if $\mathcal{P}$ is any hereditary property such that there exists an indecomposable $\mathcal{P}$-maximal graph, then $\mathcal{P}$ is an irreducible property. We also present some general results on graphs that are maximal with respect to hereditary properties, and investigate graphs that are maximal with respect to the specific properties in our list, and some reducible properties that are products of these properties.

2. General Results

The definition of $\mathcal{P}$-maximal graphs and the completeness of $\mathcal{P}$ immediately yield the following:

Lemma 3. *Let $\mathcal{P}$ be a hereditary property. Then*

1. $\boldsymbol{M}(n, \mathcal{P}) = \{K_n\}$ *for each n with $1 \leq n \leq c(\mathcal{P}) + 1$.*
2. *If $G, H \in \boldsymbol{M}(n, \mathcal{P})$ and $G \neq H$ then G is not contained in H.*

The next lemma describes the relationship between comparable hereditary properties of graphs and the corresponding sets of $\mathcal{P}$-maximal graphs.

Lemma 4 [17]**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if for every positive integer n and every graph $G \in \boldsymbol{M}(n, \mathcal{P}_1)$ there is a graph $G' \in \boldsymbol{M}(n, \mathcal{P}_2)$ such that $G \subseteq G'$.*

The following result is again an easy consequence of the definitions.

Lemma 5. *Let $\mathcal{P}$ be a hereditary property such that $\mathbf{F}(\mathcal{P})$ contains no bipartite graph. Then any $\mathcal{P}$-maximal graph with chromatic number two is a complete bipartite graph.*

If G is a graph with property $\mathcal{P}$, such that the graph $G + K_1$ does not belong to $\mathcal{P}$, then G is called $\mathcal{P}$-*strict*. The next result states that almost all $\mathcal{P}$-maximal graphs are $\mathcal{P}$-strict.

Lemma 6. *If $\mathcal{P}$ is a hereditary property and $G \in \boldsymbol{M}(n, \mathcal{P})$, $n \geq c(\mathcal{P}) + 1$, then G is a $\mathcal{P}$-strict graph.*

Proof. Let us consider two cases. Firstly, if $n = c(\mathcal{P})+1$, then $G = K_{c(\mathcal{P})+1}$ and evidently $G + K_1 = K_{c(\mathcal{P})+2}$ does not belong to $\mathcal{P}$. Thus G is $\mathcal{P}$-strict. Secondly, if $n > c(\mathcal{P}) + 1$, then G is not complete so that there is a vertex $v \in V(G)$ which is not universal. Thus G is a proper subgraph of $(G-v)+K_1$, which implies that $(G - v) + K_1 \notin \mathcal{P}$. Hence $G + K_1 \notin \mathcal{P}$ and again G is $\mathcal{P}$-strict. ∎

3. Maximal Graphs with respect to Reducible Properties

It is natural to expect that graphs maximal with respect to a reducible hereditary property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ can be derived from $\mathcal{P}_1$-maximal and $\mathcal{P}_2$-maximal graphs. This fact is exactly expressed in the following lemma.

Lemma 7. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be hereditary properties of graphs. A graph G belongs to $\boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$ if and only if for each $(\mathcal{P}_1, \mathcal{P}_2)$-partition $\{V_1, V_2\}$ of G the following holds: $G[V_1] \in \boldsymbol{M}(\mathcal{P}_1)$, $G[V_2] \in \boldsymbol{M}(\mathcal{P}_2)$ and $G = G[V_1]+G[V_2]$.*

Proof. Suppose $G \in \boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$ and $\{V_1, V_2\}$ is any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G. If there is a missing edge e with one end in V_1 and the other in V_2, then it is evident that $\{V_1, V_2\}$ is also a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $G + e$, and therefore $G + e$ belongs to $\mathcal{P}_1 \circ \mathcal{P}_2$, which contradicts the $\mathcal{P}_1 \circ \mathcal{P}_2$-maximality of G. Hence $G = G[V_1] + G[V_2]$.

If $G[V_1]$ is not $\mathcal{P}_1$-maximal, then there exists a graph $G^* \in \boldsymbol{M}(|V_1|, \mathcal{P}_1)$ such that $G[V_1] \subset G^*$. But then the graph $G^* + G[V_2]$ belongs to $\mathcal{P}_1 \circ \mathcal{P}_2$, it has the same order as G and G is a proper subgraph of $G^* + G[V_2]$. This is again a contradiction. Hence $G[V_1] \in \boldsymbol{M}(\mathcal{P}_1)$. In a similar manner we can show that $G[V_2] \in \boldsymbol{M}(\mathcal{P}_2)$.

Assume now that for any $(\mathcal{P}_1, \mathcal{P}_2)$-partition $\{V_1, V_2\}$ of G we have that $G[V_1] \in \boldsymbol{M}(\mathcal{P}_1)$, $G[V_2] \in \boldsymbol{M}(\mathcal{P}_2)$ and $G[V_1] + G[V_2] = G$. Then, obviously, $G \in \mathcal{P}_1 \circ \mathcal{P}_2$. Now suppose that $G + e \in \mathcal{P}_1 \circ \mathcal{P}_2$ for some $e \in E(\overline{G})$, and let $\{W_1, W_2\}$ be a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $G + e$. Then $\{W_1, W_2\}$ is also a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G and, by our assumption, $G = G[W_1] + G[W_2]$ and moreover $G[W_i] \in \boldsymbol{M}(\mathcal{P}_i)$; $i = 1, 2$. Without loss of generality, we may assume that e has both ends in W_1. But then $(G + e)[W_1] = G[W_1] + e \notin \mathcal{P}_1$, since $G[W_1] \in \boldsymbol{M}(\mathcal{P}_1)$. This contradicts our assumption that $\{W_1, W_2\}$ is a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $G + e$, so that $G + e \notin \mathcal{P}_1 \circ \mathcal{P}_2$, and hence $G \in \boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$. ∎

The following is a straightforward generalization of the previous lemma.

Corollary 1. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be hereditary properties of graphs. A graph G belongs to $\boldsymbol{M}(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$, $n \geq 2$, if and only if for each $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition $\{V_1, \ldots, V_n\}$ of G the following holds: $G[V_i] \in \boldsymbol{M}(\mathcal{P}_i)$ for $i = 1, \ldots, n$ and $G = G[V_1] + \cdots + G[V_n]$.*

According to Lemma 7 all the maximal graphs of a reducible, additive, hereditary property are decomposable. Thus we have

Corollary 2. *If $\mathcal{P}$ is a hereditary property of graphs such that $\boldsymbol{M}(\mathcal{P})$ contains an indecomposable graph, then $\mathcal{P}$ is irreducible.*

We do not know whether the converse of Corollary 2 is true.

Using Lemma 7 to verify that a graph G is $\mathcal{P}_1 \circ \mathcal{P}_2$-maximal can be difficult, since all possible $(\mathcal{P}_1, \mathcal{P}_2)$-partitions need to be checked. In general, the join of a $\mathcal{P}_1$-maximal graph and a $\mathcal{P}_2$-maximal graph need not be $\mathcal{P}_1 \circ \mathcal{P}_2$-maximal, not even if the resulting graph is uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable, as is shown by the following example.

Example 1. Let H_1 be the graph consisting of two disjoint paths, each of length two, and an edge joining the central vertices of the two paths. Put $G_1 = H_1 \cup K_2$. Let G_2 be the 4-cycle, and put $\mathcal{P}_1 = \mathcal{W}_3$ and $\mathcal{P}_2 = \mathcal{I}_1$. Then $G_1 \in \boldsymbol{M}(\mathcal{P}_1)$ and $G_2 \in \boldsymbol{M}(\mathcal{P}_2)$. Now suppose that $\{W_1, W_2\}$ is any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $G = G_1 + G_2$. If both $W_2 \cap V(G_1)$ and $W_2 \cap V(G_2)$ are independent sets, then $|W_2 \cap V(G_1)| \leq 5$ and $|W_2 \cap V(G_2)| \leq 2$. But then $|W_1 \cap V(G_1)| \geq 3$ and $|W_1 \cap V(G_2)| \geq 2$, contradicting our assumption that $G[W_1] \in \mathcal{W}_3$. If $W_2 \cap V(G_2)$ is not an independent set, then $W_2 \cap V(G_1) = \emptyset$ and then, since G_1 is $\mathcal{W}_3$-strict, it follows that $W_1 = V(G_1)$. We can prove in a similar manner that, if $W_2 \cap V(G_1)$ is not an independent set, then $W_1 = V(G_2)$. (Note that $G_1 \in \mathcal{P}_2$ and $G_2 \in \mathcal{P}_1$.) Thus $\{V(G_1), V(G_2)\}$ is the only $\{\mathcal{P}_1, \mathcal{P}_2\}$-partition of $G_1 + G_2$ (up to permutation of the partition sets), and hence $G_1 + G_2$ is uniquely $(\mathcal{P}_1, \mathcal{P}_2)$- partitionable.

However, since G_1 is not $\mathcal{P}_2$-maximal, there is an edge $e \in E(\overline{G_1})$ such that $G_1 + e \in \mathcal{P}_2$, and hence $\{V(G_2), V(G_1 + e)\}$ is a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of $G_1 + G_2 + e$. Thus $G_1 + G_2$ is not $(\mathcal{P}_1, \mathcal{P}_2)$-maximal.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are any properties of graphs, we shall say that a graph G is *strongly uniquely* $(\mathcal{P}_1, \mathcal{P}_2)$*-partitionable* if there exists only one $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G with a permutation of the partition sets not being allowed unless $\mathcal{P}_1 = \mathcal{P}_2$. An easy application of Lemma 7 now yields

Lemma 8. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be hereditary properties of graphs. If $G_1 \in \boldsymbol{M}(\mathcal{P}_1)$ and $G_2 \in \boldsymbol{M}(\mathcal{P}_2)$ and $G_1 + G_2$ is strongly uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable, then $G \in \boldsymbol{M}(\mathcal{P}_1 \circ \mathcal{P}_2)$.*

The next theorem presents well-known results of graph colouring theory.

Theorem 1. *A graph G belongs to $\boldsymbol{M}(\mathcal{O}^n)$ if and only if $G = H_1 + \cdots + H_n$ with $H_i \in \boldsymbol{M}(\mathcal{O}) = \mathcal{O}$ for $i = 1, \ldots, n$, that is, G is a complete n-partite graph.*

The following results provide other sufficient conditions for a graph to be maximal with respect to specific reducible hereditary properties.

Theorem 2. *Let $\mathcal{P}$ be a hereditary property of graphs with $c(\mathcal{P}) = 1$ and let $k \geq 2$ and l be integers. If $G_1 \in \boldsymbol{M}(l, \mathcal{O})$, $G_2 \in \boldsymbol{M}(k, \mathcal{P})$ and*

1. $\chi(G_2) > 2$ *or*
2. $\chi(G_2) = 2$ *but G_2 is not a complete bipartite graph and* $l \geq \max\{|U_1|, |U_2| : \{U_1, U_2\}$ *is any $(\mathcal{O}, \mathcal{O})$-partition of $G_2\}$ or*
3. *G_2 is a complete bipartite graph,*

then $G_1 + G_2 \in \boldsymbol{M}(k + l, \mathcal{O} \circ \mathcal{P})$.

Proof. Let $\{W_1, W_2\}$ be any $(\mathcal{O}, \mathcal{P})$-partition of $G = G_1 + G_2$ (the existence of such a partition follows from the assumption $G_1 \in \mathcal{O}$ and $G_2 \in \mathcal{P}$).

If $W_1 \cap V(G_1) \neq \emptyset$, then $W_1 \cap V(G_2)$ must be empty. Hence $V(G_2) \subseteq W_2$. Since the order of G_2 is at least $2 = c(\mathcal{P}) + 1$, we immediately have by Lemma 6 that $W_2 \cap V(G_1) = \emptyset$. It follows that $W_1 = V(G_1)$ and $W_2 = V(G_2)$.

If $W_1 \cap V(G_1) = \emptyset$, then obviously $V(G_1) \subseteq W_2$. Since $c(\mathcal{P}) = 1$, the complete graph K_3 does not belong to $\mathcal{P}$ and therefore $W_2 \cap V(G_2)$ must be an independent set in G. Now, we shall proceed with respect to the structure of G_2:

1. In this case we arrive at a contradiction since the partition $\{W_1, W_2 \cap V(G_2)\}$ shows that $\chi(G_2) \leq 2$.
2. In this case it is easy to see that the graph $G[W_2]$ is a complete bipartite graph which properly contains $G_2 + e$ for a suitable choice of an edge $e \in \overline{G_2}$. Hence $G[W_2] \notin \mathcal{P}$, a contradiction.
3. In the last case G_2 is the join of two independent sets and therefore G is the join of three independent sets. Hence $G \in \boldsymbol{M}(k + l, \mathcal{O} \circ \mathcal{P})$.

Thus, by Lemma 7, G is $\mathcal{O} \circ \mathcal{P}$-maximal. ∎

Corollary 3. *Let $\mathcal{P}$ be any hereditary property such that $\mathcal{O}^2 \subseteq \mathcal{P} \subseteq \mathcal{I}_1$. If $G_1 \in \boldsymbol{M}(k, \mathcal{O})$, $G_2 \in \boldsymbol{M}(l, \mathcal{P})$, $l \geq 2$, then $G = G_1 + G_2$ is $\mathcal{O}\circ\mathcal{P}$-maximal.*

Proof. Since $\mathcal{O}^2 \subseteq \mathcal{P}$, no bipartite graph is forbidden for $\mathcal{P}$, but $K_3 \notin \mathcal{P}$. Hence, either $\chi(G_2) > 2$ or, by Lemma 5, G_2 is a complete bipartite graph. Thus, by an application of Theorem 2, we have $G \in \boldsymbol{M}(k + l, \mathcal{O}\circ\mathcal{P})$. ■

Corollary 4. *Let $\mathcal{P}$ be any hereditary property such that $\mathcal{P} \subseteq \mathcal{D}_1$. If $G_1 \in \boldsymbol{M}(k, \mathcal{O})$ and $G_2 \in \boldsymbol{M}(l, \mathcal{P})$ with $k \geq l \geq 2$, then $G_1 + G_2$ is $\mathcal{O}\circ\mathcal{P}$-maximal.*

Proof. From the assumptions follows that G_2 is a forest. Hence G_2 is bipartite and it must be either a star or it is not a complete bipartite graph. Since both cases are covered by Theorem 2, $G_1 + G_2$ is $\mathcal{O}\circ\mathcal{P}$-maximal. ■

The next theorem provides a sufficient condition for a graph to be $\mathcal{O}\circ\mathcal{P}$-maximal regardless of the completeness of $\mathcal{P}$, but in terms of the vertex-independence number β.

Theorem 3. *Let $\mathcal{P}$ be any hereditary property of graphs. If $G_1 \in \boldsymbol{M}(l, \mathcal{O})$ and $G_2 \in \boldsymbol{M}(k, \mathcal{P})$ with $l \geq \beta(G_2)$ and $k \geq c(\mathcal{P}) + 2$, then $G_1 + G_2 \in \boldsymbol{M}(k + l, \mathcal{O}\circ\mathcal{P})$.*

Proof. If $\{W_1, W_2\}$ is any $(\mathcal{O}, \mathcal{P})$-partition of $G = G_1 + G_2$, then two cases can occur.

Firstly, if $W_1 \cap V(G_1) \neq \emptyset$, then (similarly as in the previous proofs) $V(G_1) = W_1$ and $V(G_2) = W_2$ and one can see that $G_1 + G_2 + e \notin \mathcal{O}\circ\mathcal{P}$ for every $e \in E(\overline{G})$.

Secondly, if $W_1 \cap V(G_1) = \emptyset$, then $V(G_1) \subseteq W_2$ and $W_1 \subseteq V(G_2)$. Since $|W_1| \leq \beta(G_2) \leq |V(G_1)|$, we have that G_2 is isomorphic to a subgraph of $G[W_2]$. Hence, there is a set $W_3 \subseteq W_2$ such that $|W_3| = |V(G_2)|$ and $G_2 \subseteq G[W_3]$. If G_2 is a proper subgraph of $G[W_3]$, then $G[W_2] \notin \mathcal{P}$ and this partition is not admissible. If $G_2 = G[W_3]$, then $G_2 = G[V(G_1) \cap W_3] + G[V(G_2) \cap W_2]$. Again we see that $G_1 + G_2 + e \notin \mathcal{O}\circ\mathcal{P}$ for every $e \in E(\overline{G})$.

Thus $G = G_1 + G_2 \in \boldsymbol{M}(k + l, \mathcal{O}\circ\mathcal{P})$ by Lemma 7. ■

Untill now we treated reducible hereditary properties which are divisible by $\mathcal{O}$. In the remainder of this section we shall provide sufficient conditions for some other particular reducible hereditary properties.

Theorem 4. *Let $G_1 \in \boldsymbol{M}(k, \mathcal{D}_1)$, $G_2 \in \boldsymbol{M}(l, \mathcal{D}_1)$ and $\min\{k, l\} \geq 3$ or $\{k, l\} \in \{\{1, 1\}, \{1, 2\}, \{2, 2\}, \{2, 3\}\}$. Then $G = G_1 + G_2 \in \boldsymbol{M}(k + l, \mathcal{D}_1\circ\mathcal{D}_1)$.*

Proof. It is easy to check that each of the graphs K_2, K_3, K_4 and $K_5 - e$ is a $\mathcal{D}_1\circ\mathcal{D}_1$-maximal graph.

So, suppose $\min\{k, l\} \geq 3$ and let $\{W_1, W_2\}$ be any $(\mathcal{D}_1, \mathcal{D}_1)$-partition of $V(G)$ distinct from $\{V(G_1), V(G_2)\}$. Since G_1 and G_2 each contains at least two edges and $K_3 \notin \mathcal{D}_1$, the sets $W_1 \cap V(G_1)$, $W_1 \cap V(G_2)$, $W_2 \cap V(G_1)$ and $W_2 \cap V(G_2)$ must be independent in G (if any of these sets is empty, we obtain the partition $\{V(G_1), V(G_2)\}$). Moreover, since $C_4 \notin \mathcal{D}_1$, we have $\min\{|W_1 \cap V(G_1)|, |W_1 \cap V(G_2)|\} \leq 1$ and simultaneously $\min\{|W_2 \cap V(G_1)|, |W_2 \cap V(G_2)|\} \leq 1$. Without loss of generality we may assume that $|W_1 \cap V(G_1)| = 1$. Then $G[W_1]$ and $G[V(G_1)]$ are stars, $|W_2 \cap V(G_1)| \geq 2$, $|W_2 \cap V(G_2)| = 1$ and $G[W_2], G[V(G_2)]$ are also stars. But this means that $G_2 = G[W_1 \cap V(G_2)] + G[W_2 \cap V(G_2)]$, $G_1 = G[W_1 \cap V(G_1)] + G[W_2 \cap V(G_1)]$ so that $G \in \boldsymbol{M}(\mathcal{D}_1\circ\mathcal{D}_1)$. ■

The gaps in the previous theorem are necessary, as is shown by the graphs in the next example.

Example 2. If $k = 1$ and $l = 3$, then $G_1 + G_2 = K_4 - e$ which is not $\mathcal{D}_1\circ\mathcal{D}_1$-maximal. If $k = 1$ and $l > 3$, then G_2 can be chosen to be bipartite but not a complete bipartite graph. Thus we can partition $G_1 + G_2$ into one star and one independent set which immediately yields that $G_1 + G_2$ is not $\mathcal{D}_1\circ\mathcal{D}_1$-maximal.

If $k = 2$ and $l \geq 4$ then, with G_2 as above, $G_1 + G_2$ can be partitioned into two stars and again $G_1 + G_2$ is not $\mathcal{D}_1\circ\mathcal{D}_1$-maximal.

Theorem 5. *Suppose $G_1 \in \boldsymbol{M}(k, \mathcal{D}_1)$, $G_2 \in \boldsymbol{M}(l, \mathcal{I}_1)$ and both l and k are at most two, or $l \geq 2$, $k \geq 3$ and $G_2 \neq K_{1,l-1}$. Then $G = G_1 + G_2$ is $\mathcal{D}_1\circ\mathcal{I}_1$-maximal.*

Proof. The verification that K_2, K_3 and K_4 are $\mathcal{D}_1\circ\mathcal{I}_1$-maximal graphs is trivial. Hence assume that $l \geq 2$, $k \geq 3$ and $G_2 \neq K_{1,l-1}$. Further let $\{W_1, W_2\}$ be an arbitrary $(\mathcal{D}_1, \mathcal{I}_1)$-partition of $V(G)$. We consider four cases.

(1) $|V(G_1) \cap W_1| = 0$: Then $V(G_1) \subseteq W_2$ and, since $G[V(G_1) \cap W_2]$ contains at least one edge, $V(G_2) \cap W_2 = \emptyset$. But then $G_2 = G[W_1]$ is a star, contradicting our assumption.

(2) $|V(G_1) \cap W_1| = 1$: Then $|W_1 \cap V(G_2)|$ is an independent set. If $G[V(G_1) \cap W_2]$ contains an edge, then $W_2 \cap V(G_2)$ must be empty, which is not possible. Hence $G[V(G_1) \cap W_2]$ is an edgeless graph and therefore G_1 is a star. Then $G[V(G_2) \cap W_2]$ must be independent too. But this means

that G_2 is bipartite and, by Lemma 5, even complete bipartite. But then it is clear that, in this case, $G + e \notin \mathcal{D}_1 \circ \mathcal{I}_1$ for every $e \in E(\overline{G})$.

(3) $|V(G_1) \cap W_1| \geq 2$ and $V(G_1) \cap W_1$ is an independent set in G: Then evidently $|V(G_2) \cap W_1| \leq 1$. If $G[V(G_2) \cap W_2]$ would contain an edge, then $W_2 \cap V(G_1)$ must be empty, which is a contradiction. Hence $V(G_2) \cap W_2$ also has to be independent and again it follows that G_2 is a star, a contradiction.

(4) $|V(G_1) \cap W_1| \geq 2$ and $V(G_1) \cap W_1$ is not an independent set in G: It immediately follows that $W_1 \cap V(G_2) = \emptyset$. Hence $V(G_2) \subseteq W_2$ and $V(G_1) \subseteq W_1$ and in this case we have the partition $\{V(G_1), V(G_2)\}$.

The result now follows by an application of Lemma 7. ∎

The next example shows that most of the gaps in the previous theorem again cannot be avoided.

Example 3. Let $G_1 \neq K_{1,m}$ be a tree of order at least 4 and $G_2 = K_{1,n}$. Then we clearly have that $G_1 \in \boldsymbol{M}(\mathcal{D}_1)$ and that $G_2 \in \boldsymbol{M}(\mathcal{D}_1) \cap \boldsymbol{M}(\mathcal{I}_1)$. However, if we take any pair of vertices x and y of G_1 which are at distance at least three, then $G_1 + xy \in \mathcal{I}_1$. Hence we have that $\{V(G_2), V(G_1)\}$ is a $\mathcal{D}_1 \circ \mathcal{I}_1$-partition of $G_1 + G_2 + e$, where $e = xy$, and therefore $G_1 + G_2$ is not $\mathcal{D}_1 \circ \mathcal{I}_1$-maximal.

4. Maximal Graphs with respect to Irreducible Properties

If $\mathcal{P}$ is any hereditary property and G is a $\mathcal{P}$-maximal graph of order less than $c(\mathcal{P}) + 2$, then G is a complete graph and is thus either trivial or decomposable. Our next result ensures the existence of an indecomposable $\mathcal{P}$-maximal graph of order $c(\mathcal{P}) + 2$ for certain properties $\mathcal{P}$.

Proposition 1. *If $\mathcal{P}$ is any additive, hereditary property such that $\boldsymbol{F}(\mathcal{P})$ contains some tree of order $c(\mathcal{P}) + 2$, then there exists an indecomposable $\mathcal{P}$-maximal graph of order $c(\mathcal{P}) + 2$.*

Proof. Put $G = K_{c(\mathcal{P})+1} \cup K_1$. By the definition of $c(\mathcal{P})$ and the additivity of $\mathcal{P}$, the graph G has property $\mathcal{P}$. However, if $e \in E(\overline{G})$, the graph $G + e$ does not have property $\mathcal{P}$, since $G + e$ contains every tree of order $c(\mathcal{P}) + 2$. Consequently, G is $\mathcal{P}$-maximal. ∎

Corollary 5. *If $\mathcal{P}$ is any additive, hereditary property such that $\boldsymbol{F}(\mathcal{P})$ contains some tree of order $c(\mathcal{P}) + 2$, then $\mathcal{P}$ is irreducible.*

Proof. By Proposition 1 and Corollary 2. ∎

Properties that satisfy Proposition 1 are, for example, $\mathcal{O}_k$, $\mathcal{S}_k$ and $\mathcal{W}_k$. However, $\boldsymbol{F}(\mathcal{I}_k)$ contains no trees, and we shall see that there are no nontrivial indecomposable $\mathcal{I}_k$-maximal graphs of order less than $2k+3$.

Clearly, $\boldsymbol{M}(n,\mathcal{O}) = \{\overline{K_n}\}$ for al $n \geq 1$ and $\boldsymbol{M}(\mathcal{O}) = \mathcal{O}$. A complete characterization of the graphs that are maximal with respect to the so-called *hom-properties*, that is, the properties of the form $\rightarrow H$ for some $H \in \mathcal{I}$, is given in [11]. (Some hom-properties are reducible and some are irreducible.) We now present some results on graphs that are maximal with respect to the other properties listed in Section 1.

4.1. $\mathcal{O}_k$-maximal graphs

If $n \geq k+2$, then the $\mathcal{O}_k$-maximal graphs of order n are disjoint unions of complete graphs; more precisely

$$\begin{aligned}\boldsymbol{M}(\mathcal{O}_k) \quad &= \quad \{K_{r_1} \cup \ldots \cup K_{r_s} : s \geq 2,\ r_i \leq k+1 \text{ and } r_i + r_j \geq k+2 \text{ if} \\ &\qquad i \neq j,\ 1 \leq i,j \leq s\}.\end{aligned}$$

Hence every $\mathcal{O}_k$-maximal graph of order at least $k+2$ is disconnected (and thus indecomposable).

4.2. $\mathcal{S}_k$-maximal graphs

It is not difficult to see that $\mathcal{S}_k$-maximal graphs are almost k-regular and their structure can be described as follows

$$\begin{aligned}\boldsymbol{M}(\mathcal{S}_k) \quad &= \quad \{G \in \mathcal{I} : \Delta(G) = k \text{ and every two vertices of degree less than} \\ &\qquad k \text{ are adjacent}\}.\end{aligned}$$

The $\mathcal{S}_k$-maximal graphs of order $k+2$ are characterized by the following result.

Proposition 2. *A graph G of order $k+2$ is an $\mathcal{S}_k$-maximal graph if and only if every component of $\overline{G}$ is a star.*

Proof. Let G be a graph of order $k+2$. Then G is $\mathcal{S}_k$-maximal if and only if $\Delta(G) = k$ and $\Delta(G+e) = k+1$ for every $e \in E(\overline{G})$, that is, if and only if $\delta(\overline{G}) = 1$ and $\delta(\overline{G}-e) = 0$ for every $e \in E(\overline{G})$. Clearly, this is the case if and only if every component of $\overline{G}$ is a star. ∎

The complement of the star of order $k+2$ is the graph $K_{k+1} \cup K_1$. Since a graph is indecomposable if and only if its complement is connected, we have

Corollary 6. *The graph $K_{k+1} \cup K_1$ is the only indecomposable $\mathcal{S}_k$-maximal graph of order $k+2$.*

4.3. $\mathcal{W}_k$-maximal graphs

$\mathcal{W}_k$-maximal trees are characterized in [10]. We know the following about the structure of decomposable $\mathcal{W}_k$-maximal graphs.

Theorem 6. *Suppose G is a $\mathcal{W}_k$-maximal graph of order at least $k+2$ and $G = G_1 + G_2$, where G_1 and G_2 are graphs of order n_1 and n_2 respectively with $0 < n_1 \leq n_2$. Then $n_1 \leq \frac{k}{2}$, $G_1 = K_{n_1}$ and $G_2 \in \mathcal{W}_{k+1-2n_1}$.*

Proof. Note that $n_1 < n_2$, otherwise G has a path of length at least $k+1$ which alternates between G_1 and G_2.

Suppose $n_1 > \frac{k}{2}$. Then there is a path in G that starts in G_2, then alternates between G_1 and G_2 until it has passed through all the vertices of G_1, and ends in G_2. Such a path has length $2n_1 > k$, contradicting our assumption that $G \in \mathcal{W}_k$. This proves that $n_1 \leq \frac{k}{2}$.

Now suppose that G_1 contains two non-adjacent vertices x and y. Then, since G is $\mathcal{W}_k$-maximal, the graph $G + xy$ contains a path of length $k+1$, say $v_1 \ldots v_{k+2}$, with $x = v_r$ and $y = v_{r+1}$; $1 \leq r \leq k+1$. We may assume without loss of generality that G_2 contains an edge in this path that precedes v_r. Let $v_{s-1}v_s$ be the last such edge. Now replace the subpath $v_{s-1}v_s \ldots v_{r+1}$ with the path $v_{s-1}v_{s+1}v_{s+2} \ldots v_r v_s v_{r+1}$. Note that all the edges of the resulting path are indeed edges of G. Thus we have a path of length $k+1$ in G. This contradiction proves that G_1 is a complete graph.

Now suppose G_2 has a path P of length at least $k+2-2n_1$. Then

$$|V(G_2 - P)| = |V(G)| - |V(G_1| - |V(P)| \geq k+2-n_1-(k+2-2n_1) = n_1.$$

Let P^* be the path in G whose first segment is P and then alternates between G_1 and G_2, until all n_1 vertices of G_1 and n_1 of the vertices of $G_2 - P$ have been used. Then P^* is a path of length $k+1$ in G, a contradiction. ■

Corollary 7. *A $\mathcal{W}_k$-maximal graph is indecomposable if and only if it has no universal vertices.*

A graph G of order $k+2$ is a $\mathcal{W}_k$-maximal graph if and only if G has no hamiltonian path, but $G+e$ has a hamiltonian path for every $e \in E(\overline{G})$. The disconnected $\mathcal{W}_k$-maximal graphs of order $k+2$ are easy to characterize.

Proposition 3. *Let G be a disconnected graph of order $k+2$. Then G is $\mathcal{W}_k$-maximal if and only if $G = K_a \cup K_b$ for some pair of positive integers $\{a, b\}$ such that $a + b = k + 2$.*

Proof. Suppose G is $\mathcal{W}_k$-maximal. Let $e \in E(\overline{G})$. Then $G + e$ has a hamiltonian path and is thus a connected graph. It follows that G has exactly two components, say A and B. If $e \in E(\overline{A})$ or $e \in E(\overline{B})$, then $G+e$ does not have a hamiltonian path. This proves that A and B are complete graphs. ∎

The connected $\mathcal{W}_k$-maximal graphs of order $k + 2$ have not yet been characterized. We conjecture that these graphs are block graphs with at least three blocks each. (A *block graph* is a graph whose blocks are all complete graphs.)

4.4. $\mathcal{D}_k$-maximal graphs

$\mathcal{D}_k$-maximal graphs were studied intensively in [3], [8], [12], [15], [16], [18] and [19]. We present two results from these papers.

Theorem 7 [12]**.** *Let G be a graph of order n and let $v \in V$ be a vertex of degree k in G. Then $G \in \boldsymbol{M}(n, \mathcal{D}_k)$ if and only if $G - v \in \boldsymbol{M}(n-1, \mathcal{D}_k)$.*

The structure of maximal k-degenerate graphs is characterized in the next result in an algebraic way.

Theorem 8 [3]**.** *Let k and p be positive integers. Then the sequence $(d_1, \dots, d_p)$ with $d_1 \le \cdots \le d_p$ is the degree sequence of some $\mathcal{D}_k$-maximal graph if and only if it satisfies the following four conditions:*

1. $d_1 = k$;
2. $d_p \le p - 1$;
3. $d_i \le k + i - 1$ *for* $i = 1, \dots, k-1$;
4. $\sum_{i=1}^{p} d_i = 2kp - k(k+1)$.

4.5. $\mathcal{I}_k$-maximal graphs

$\mathcal{I}_k$-maximal graphs are also known as $(k+1)$-*saturated* graphs (see for example [9]). The following two theorems on the structure of $\mathcal{I}_k$-maximal graphs are proved in [9].

Theorem 9 [9]**.** *If G is an $\mathcal{I}_k$-maximal graph of order n, then G contains at least $2k+2-n$ universal vertices.*

Theorem 10 [9]**.** *If G is an $\mathcal{I}_k$-maximal graph without universal vertices, then $\delta(G) \geq 2k$.*

Let

$$\mathcal{A}_k = \{G \in \mathcal{I} : \alpha(G) = k\}$$

where $\alpha(G)$ is the vertex-covering number of G. Note that $\mathcal{A}_k$ is not a hereditary property. We say that a graph G is $\mathcal{A}_k$*- vertex-critical* ($\mathcal{A}_k$*-edge-critical* respectively) if $G \in \mathcal{A}_k$ but $G - x \notin \mathcal{A}_k$ for every vertex (edge respectively) in G. A graph that is $\mathcal{A}_k$-vertex-critical as well as $\mathcal{A}_k$-edge-critical is called $\mathcal{A}_k$*-critical.* Since $\omega(G) = |V(G)| - \alpha(\overline{G})$, we have

Lemma 9. *Let G be a graph of order n. Then G is $\mathcal{I}_k$-maximal if and only if $\overline{G}$ is $\mathcal{A}_{n-k-1}$-edge-critical.*

Since the $\mathcal{A}_k$-critical graphs are exactly the $\mathcal{A}_k$-edge-critical graphs without universal vertices, we have

Lemma 10. *Let G be a graph of order n without universal vertices. Then G is $\mathcal{I}_k$-maximal if and only if $\overline{G}$ is $\mathcal{A}_{n-k-1}$-critical.*

The following result is proved in [7].

Theorem 11 [7]**.** *If G is an $\mathcal{A}_k$-critical graph, then $|V(G)| \leq 2k$, with equality only if $G = kK_2$.*

In the light of Lemma 10, Theorem 11 can be restated as follows:

Theorem 12. *If G is an $\mathcal{I}_k$-maximal graph without universal vertices, then $|V(G)| \geq 2k+2$, with equality only if $G = \overline{(k+1)K_2}$.*

We now characterize the smallest indecomposable $\mathcal{I}_k$-maximal graph.

Theorem 13. *If G is an indecomposable $\mathcal{I}_k$-maximal graph, then $|V(G)| \geq 2k+3$, with equality only if $G = \overline{C_{2k+3}}$.*

Proof. Since an indecomposable graph contains no universal vertices, and the graph $\overline{(k+1)K_2}$ is decomposable (since its complement is disconnected), it follows that $|V(G)| \geq 2k+3$. Now suppose that $|V(G)| = 2k+3$. Then, by Lemma 10, $\alpha(\overline{G}) = k+2$. But by Theorem 10, $\delta(G) \geq 2k$, and hence $\Delta(\overline{G}) \leq 2$. Since the only connected graph of order $2k+3$ with minimum degree 2 and vertex connectivity $k+2$ is the cycle C_{2k+3}, it follows that $G = \overline{C_{2k+3}}$. ∎

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