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A SURVEY OF HEREDITARY PROPERTIES OF GRAPHS

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Abstract

In this paper we survey results and open problems on the structure of additive and hereditary properties of graphs. The important role of vertex partition problems, in particular the existence of uniquely partitionable graphs and reducible properties of graphs in this structure, is emphasized. Many related topics, including questions on the complexity of related problems, are investigated.

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1. Introduction and Notation

All graphs considered are finite, simple, i.e., undirected, loopless and without multiple edges. For undefined concepts we refer the reader to [8], [17] and [29].

A *graph property* is any isomorphism-closed subclass of graphs. Since we have, in general, no reason to distinguish between isomorphic copies of a graph, we use the notation $\mathcal{I}$ to denote the set of all unlabelled graphs. Therefore, by saying that H is a subgraph of G, we mean that H is isomorphic to a subgraph of G.

If $\mathcal{P}$ is a subset of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$. We shall use the terms *set of graphs* and *property of graphs* interchangeably. The symbol $\mathcal{E}$ will stand for the empty property, i.e., the subsets of $\mathcal{I}$ containing no graphs. Properties $\mathcal{I}$ and $\mathcal{E}$ are called *trivial*.

By the *join* $G_1 + G_2 + \cdots + G_n$ of n graphs $G_1, G_2, \ldots, G_n$ we mean the graph consisting of disjoint copies of $G_1, G_2, \ldots, G_n$ and all the edges between $V(G_i)$ and $V(G_j)$ for all $i \neq j$. Every graph which is the join of at least two graphs is called *decomposable*. A graph that is not decomposable is called *indecomposable*. It is easy to see that a graph G is decomposable if and only if its complement $\overline{G}$ is not connected. Then G is a join of the complements of the components of $\overline{G}$. Thus every decomposable graph G can be expressed in a unique way as the join of indecomposable graphs.

A *homomorphism* of a graph G to a graph H is a mapping f of the vertex set $V(G)$ into $V(H)$ which preserves the edges, i.e., if $e = \{u, v\} \in E(G)$ then $f(e) = \{f(u), f(v)\} \in E(H)$.

If a homomorphism of G to H exists, we say that G is *homomorphic* to H and write $G \to H$. It is easily seen that in such case for the chromatic number, the following inequality $\chi(G) \leq \chi(H)$ holds.

A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have the property $\mathcal{P}$ it follows that $G \in \mathcal{P}$ too. Throughout the paper we let $\preceq$ be a partial order on the set $\mathcal{I}$. A property $\mathcal{P}$ is said to be $\preceq$-*hereditary* if, whenever $G \in \mathcal{P}$ and $H \preceq G$, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is *induced hereditary* if it is $\leq$-hereditary with respect to the relation $\leq$ *to be an induced subgraph* and $\mathcal{P}$ is *hereditary* if it is $\subseteq$-hereditary with respect to the relation $\subseteq$ *to be a subgraph* (some authors prefer the term *monotone* instead of hereditary, see [7], [101]). In the first three sections of this survey we shall concentrate on additive hereditary properties; properties that are

hereditary with respect to other partial orderings are the subject of study in further sections.
Hereditary properties have been studied extensively (see e.g., [7], [17], [23], [25], [26], [48], [64], [83] and [101]).

Example 1. We list some important hereditary properties, using partially the notation of [17].

$$
\begin{array}{rcl}
\mathcal{O} & = & \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\}, \\
\mathcal{O}_k & = & \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\}, \\
\mathcal{S}_k & = & \{G \in \mathcal{I} : \text{ the maximum degree } \Delta(G) \leq k\}, \\
\mathcal{W}_k & = & \{G \in \mathcal{I} : \text{ the length of the longest path in } G \text{ is at most } k\}, \\
\mathcal{D}_k & = & \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate,} \\
 & & \text{ i.e., the minimum degree } \delta(H) \leq k \text{ for each } H \subseteq G\}, \\
\mathcal{T}_k & = & \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or} \\
 & & K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\
\mathcal{I}_k & = & \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}, \\
\to H & = & \{G \in \mathcal{I} : G \text{ is homomorphic to a given graph } H\}.
\end{array}
$$

It is easy to see that for $k = 1$ the properties $\mathcal{O}_1$, $\mathcal{S}_1$ and $\mathcal{W}_1$ are equal to each other.

Let $\mathcal{P}$ be a nontrivial hereditary property. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$ – it is called the *completeness* of $\mathcal{P}$. Obviously

$$c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{W}_k) = c(\mathcal{D}_k) = c(\mathcal{T}_k) = c(\mathcal{I}_k) = k$$

and for additive properties $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$.

By a *partition of a set* S into n parts we mean an unordered family $\{S_1, S_2, \ldots, S_n\}$ of pairwise disjoint subsets of S with $\bigcup_{i=1}^{n} S_i = S$. Note that some of the S_i's may be empty. In the case that an *ordered partition* is required we shall denote the ordered partition by $[S_1, S_2, \ldots, S_n]$.

In Section 2 we investigate the lattice of additive and hereditary properties of graphs. It is shown how minimal forbidden subgraphs and maximal graphs are used to characterize properties.

In Section 3 we discuss vertex partitions of graphs. Reducibility of properties plays a major role in this section and the relationship between this concept and the existence of uniquely partitionable graphs is explained. The concept of minimal reducible bounds is shown to provide a wealth of difficult problems.

In Section 4 lattices arising from other partial orderings on $\mathcal{I}$ are studied. These lattices can be used to explain many seemingly unrelated concepts and results from Graph Theory.

Section 5 is devoted to show how many known invariants of graphs can be related to chains of such properties of graphs.

In Section 6 we list some important results concerning the complexity of partition problems related to properties of graphs (see also [98]).

2. The Lattice of Additive and Hereditary Properties of Graphs

In this section we first recall some known results on the lattices of additive and hereditary properties of graphs given in [17], [70], [77], [102]. More general lattices of this nature will be discussed in Section 4.

If $\preceq$ is a given partial order on $\mathcal{I}$, then the set of all $\preceq$-hereditary properties will be denoted by $\mathbb{K}_{\preceq}$. As special cases the set $\mathbb{K}_{\subseteq}$ of all hereditary properties will be denoted by $\mathbb{L}$ and the set of all additive hereditary properties will be denoted by $\mathbb{L}^a$. The first results in this section describe the structure of these sets, ordered by inclusion.

Theorem 1 ([17]). *The partially ordered sets* $(\mathbb{L}, \subseteq)$ *and* $(\mathbb{L}^a, \subseteq)$ *are distributive lattices.*

Note that $\mathbb{L}$ is closed under (arbitrary) intersections and unions while $\mathbb{L}^a$ is closed under (arbitrary) intersections. Hence the properties in $\mathbb{L}$ and in $\mathbb{L}^a$ form *closure systems* (see [3]). From this it follows that the lattices in Theorem 1 are complete. Hence we have

Theorem 2 ([17]). *The partially ordered sets* $(\mathbb{L}, \subseteq)$ *and* $(\mathbb{L}^a, \subseteq)$ *are complete distributive lattices with the least element* $\mathcal{E}$ *and the greatest element* $\mathcal{I}$.

It should be noted that $(\mathbb{L}^a, \subseteq)$ is not a sublattice of $(\mathbb{L}, \subseteq)$ since the join of properties in the former is not the union of these properties. Therefore the distributivity of $(\mathbb{L}^a, \subseteq)$ is not completely trivial. In [17], [70] an isomorphism between $(\mathbb{L}^a, \subseteq)$ and a suitable lattice of sets of connected graphs is used to prove it.

For a graph $H \in \mathcal{I}$, $\to H$ denotes the class of all graphs that admit homomorphisms to H. Such classes of graphs are called *hom-properties* or *colour classes* and were investigated e.g., in [55], [61], [62], [77] and [102]. It was pointed out that they play an interesting role in the lattice $(\mathbb{L}^a, \subseteq)$.

Theorem 3 ([61]). *The hom-properties form a sublattice of* $\mathbb{L}^a$.

We will present more information about the structure of the sublattice of hom-properties in Section 3.

For a given nonnegative integer k, let

$$\mathbb{L}_k^a = \{\mathcal{P} \in \mathbb{L}^a : c(\mathcal{P}) = k\}.$$

Since $\mathbb{L}_k^a$ is also a closure system, we have

Theorem 4 ([17]). *The ordered set* $(\mathbb{L}_k^a, \subseteq)$ *is a complete distributive lattice and it is a sublattice of* $(\mathbb{L}^a, \subseteq)$. $\mathcal{O}_k$ *is the least element and* $\mathcal{I}_k$ *is the greatest element of* $(\mathbb{L}_k^a, \subseteq)$.

Since the set $\mathbb{L}$ forms a lattice with a set union and a set intersection as operations, it is natural to ask, for any two hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$, what the completeness of $\mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2$ is in terms of the completenesses of $\mathcal{P}_1$ and $\mathcal{P}_2$. Our next result provides an answer to this question.

Theorem 5 ([17]). *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be any nontrivial hereditary properties of graphs. Then* $c(\mathcal{P}_1 \cup \mathcal{P}_2) = \max\{c(\mathcal{P}_1), c(\mathcal{P}_2)\}$ *and* $c(\mathcal{P}_1 \cap \mathcal{P}_2) = \min\{c(\mathcal{P}_1), c(\mathcal{P}_2)\}$.

The algebraicity of these lattices will be discussed in Section 4.

2.1. Minimal Forbidden Subgraphs

If $\mathcal{P}$ is a nontrivial hereditary property, we define (see [26], [48]) the set of *minimal forbidden subgraphs* of $\mathcal{P}$ as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

Note that $\boldsymbol{F}(\mathcal{P})$ may be finite or infinite.

Some direct consequences of this definition are contained in the next lemma. It shows, amongst others, how a hereditary property can uniquely be determined by its set of minimal forbidden subgraphs.

Lemma 6. *Let* $\mathcal{P}$ *be a hereditary property. Then*
(a) $G \in \mathcal{P}$ *if and only if no subgraph of* G *is in* $\boldsymbol{F}(\mathcal{P})$.
(b) *If* $\mathcal{P}$ *is additive, then* $\boldsymbol{F}(\mathcal{P})$ *contains only connected graphs.*

The set of minimal forbidden subgraphs of each of the properties in Example 1, except for $\mathcal{D}_k$ and hom-properties, is easy to describe. A list of them is given in the following example.

Example 2. Let S_n and P_n denote the star and the path on n vertices, respectively. Then

$$
\begin{array}{lcl}
\boldsymbol{F}(\mathcal{O}) & = & \{K_2\}, \\
\boldsymbol{F}(\mathcal{O}_k) & = & \{H \in \mathcal{I} : H \text{ is a tree on } k+2 \text{ vertices}\}, \\
\boldsymbol{F}(\mathcal{S}_k) & = & \{S_{k+2}\}, \\
\boldsymbol{F}(\mathcal{W}_k) & = & \{P_{k+2}\}, \\
\boldsymbol{F}(\mathcal{T}_k) & = & \{H \in \mathcal{I} : H \text{ is homeomorphic to } K_{k+2} \text{ or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\
\boldsymbol{F}(\mathcal{I}_k) & = & \{K_{k+2}\}.
\end{array}
$$

To characterize the set $\boldsymbol{F}(\mathcal{D}_k)$ we need some more notation. For a nonnegative integer k and a graph G, we denote the set of all vertices of G of degree $k+1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of G and $G_1, \ldots, G_s$, $s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by H_i, $i = 1, \ldots, s$.

Theorem 7 ([67]). *A graph G belongs to $\boldsymbol{F}(\mathcal{D}_k)$ if and only if G is connected, $\delta(G) \geq k+1$, $V(G) - M(G)$ is an independent set of vertices of G and for each cutset $S \subset V(G) - M(G)$ we have that $\delta(H_i) \leq k$ for each $i = 1, \ldots, s$.*

The characterization of the the forbidden graphs for a hom-property $\rightarrow H$ seems to be very complicated and it is known just for very particular choices of the graph H (see [62]). Such choices of the graph H yield to the properties which were already discussed in Example 2.

In the next theorem the following notation is useful. If $\mathcal{S}$ is a set of graphs, we write $\min_{\preceq}[\mathcal{S}]$ for the set of graphs in $\mathcal{S}$ that are minimal with respect to the partial order $\preceq$ on $\mathcal{I}$. Note that, in this notation, $\boldsymbol{F}(\mathcal{P}) = \min_{\subseteq}[\mathcal{I} - \mathcal{P}]$ for any nontrivial hereditary property $\mathcal{P}$.

Theorem 8 ([17]). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any nontrivial hereditary properties of graphs. Then*

1. $\boldsymbol{F}(\mathcal{P}_1 \cap \mathcal{P}_2) = \min_{\subseteq}[\boldsymbol{F}(\mathcal{P}_1) \cup \boldsymbol{F}(\mathcal{P}_2)]$,
2. $\boldsymbol{F}(\mathcal{P}_1 \cup \mathcal{P}_2) = \min_{\subseteq}[\{H \in \mathcal{I} :$ *there exists a pair of graphs* $G_1 \in \boldsymbol{F}(\mathcal{P}_1)$ *and* $G_2 \in \boldsymbol{F}(\mathcal{P}_2)$ *such that* $G_1 \subseteq H$ *and* $G_2 \subseteq H\}]$.

The completeness of a property is now described in terms of its forbidden subgraphs.

Theorem 9 ([17]). *Let $\mathcal{P}$ be any nontrivial hereditary property of graphs. Then $c(\mathcal{P}) = \min\{|V(H)| - 2 : H \in \boldsymbol{F}(\mathcal{P})\}$.*

Inclusion between properties, and hence equality between properties, can also be described in terms of forbidden subgraphs.

Theorem 10 ([17]). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any nontrivial hereditary properties. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if for every $H \in \boldsymbol{F}(\mathcal{P}_2)$ there exists a graph $H' \in \boldsymbol{F}(\mathcal{P}_1)$ such that $H' \subseteq H$.*

2.2. $\mathcal{P}$-maximal Graphs

A nontrivial hereditary property $\mathcal{P}$ can also be characterized by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs. More precisely, let us define the set of $\mathcal{P}$-*maximal graphs* by

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{I} : G \in \mathcal{P} \text{ and } G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}$$

and the set of $\mathcal{P}$-*maximal graphs of order* n by

$$\boldsymbol{M}(n, \mathcal{P}) = \{G \in \mathcal{I} : G \in \mathcal{P},\ G+e \notin \mathcal{P} \text{ for each } e \in E(\overline{G}) \text{ and } |V(G)| = n\}.$$

Our next result follows immediately from these definitions.

Lemma 11. *Let $\mathcal{P}$ be a nontrivial hereditary property. Then*

1. *A graph G belongs to $\mathcal{P}$ if and only if there exists a graph $H \in \boldsymbol{M}(\mathcal{P})$ such that $G \subseteq H$.*
2. *A graph G of order n belongs to $\mathcal{P}$ if and only if there exists a graph $H \in \boldsymbol{M}(n, \mathcal{P})$ such that $G \subseteq H$.*
3. $\boldsymbol{M}(\mathcal{P}) = \bigcup_n \boldsymbol{M}(n, \mathcal{P})$.
4. *If a graph G belongs to $\boldsymbol{M}(\mathcal{P})$, then every component of G belongs to $\boldsymbol{M}(\mathcal{P})$.*

From the definition of $\mathcal{P}$-maximal graphs and the completeness of a property we immediately deduce the following result.

Lemma 12 ([21]). *Let $\mathcal{P}$ be a nontrivial hereditary property. Then*

1. $\boldsymbol{M}(n,\mathcal{P}) = \{K_n\}$ *for each n with* $1 \leq n \leq c(\mathcal{P})+1$.
2. *If $G \in \boldsymbol{M}(n,\mathcal{P})$, $H \in \boldsymbol{M}(n,\mathcal{P})$ and G is a subgraph of H or H is a subgraph of G, then $G = H$.*

It is natural, that Statement 2 of the previous lemma inspires us to introduce a more general concept—the *generator of a hereditary* (*an additive hereditary*) *property.* To be more exact, consider an arbitrary set X, a subset of $\mathcal{I}$. It is quite easy to see that the property

$$[X] = \{G \in \mathcal{I} : G \text{ is a subgraph of some graph } H \in X\}$$

is hereditary, the property

$$[X]^a = \{G \in \mathcal{I} : \text{ each component of } G \text{ is a subgraph of some } H \in X\}$$

is in addition additive and both are generated by the set X, called *generator.*

The previous lemma yields that the set $\boldsymbol{M}(\mathcal{P})$ generates the (additive) hereditary property $\mathcal{P}$. On the other hand, it is not difficult to see that the set $\boldsymbol{M}(\mathcal{P})$ is neither the unique nor the minimal (with respect to the set inclusion) generator of $\mathcal{P}$. In fact, for any positive integer k the sets $\boldsymbol{M}_k = \bigcup_{n\geq k} \boldsymbol{M}(n,\mathcal{P})$ and $\boldsymbol{M}^*_k = \bigcup_{n\geq k, n\equiv 0 (mod 2)} \boldsymbol{M}(n,\mathcal{P})$ are generators of $\mathcal{P}$, too. The concept of generator will be discussed, in more detail and in general, with respect to a partial order, in Section 4.

From the definitions it follows that there is a very strong dependence between the structure of minimal graphs and the structure of $\mathcal{P}$-maximal graphs for a given hereditary property $\mathcal{P}$. One can expect that it is quite easy to deduce the structure of forbidden graphs when all $\mathcal{P}$-maximal graphs are known and vice versa. Unfortunately, this problem seems to be very difficult even for such familiar hereditary properties as $\mathcal{O}^2$ (to be a bipartite graph) and $\mathcal{D}_1$ (to be a forest). Indeed, although the structure of the sets $\boldsymbol{F}(\mathcal{O}^2)$, $\boldsymbol{F}(\mathcal{D}_1)$, $\boldsymbol{M}(\mathcal{O}^2)$ and $\boldsymbol{M}(\mathcal{D}_1)$ is well described, there are known no general rules, which can be used for a transformation between $\boldsymbol{F}(\mathcal{O}^2)$ and $\boldsymbol{M}(\mathcal{O}^2)$, and $\boldsymbol{F}(\mathcal{D}_1)$ and $\boldsymbol{M}(\mathcal{D}_1)$.

However, the dependence can be expressed by means of a graph theoretical invariant. In order to present it, we need the following notation. If τ is any graph theoretical invariant, then the symbol $\tau(\mathcal{P})$ stands for

$$\tau(\mathcal{P}) = \min\{\tau(F) : F \in \boldsymbol{F}(\mathcal{P})\}.$$

Theorem 13 ([72]). *Let $\tau(G)$ be a graph theoretical invariant satisfying:*

1. *for each subgraph H of G the value $\tau(H)$ is at most $\tau(G)$,*
2. *for any graph $G \in \mathcal{I}$ and for an arbitrary edge e from the complement of G holds $\tau(G+e) \leq \tau(G)+1$.*

Then for any graph $G \in \boldsymbol{M}(n,\mathcal{P})$ with $n \geq c(\mathcal{P})+2$ holds the following: $\tau(G) \geq \tau(\mathcal{P})-1$.

The next lemma shows that inclusion between properties can also be described in terms of the corresponding sets of maximal graphs.

Lemma 14 ([21]). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any nontrivial hereditary properties. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if for every $G \in \boldsymbol{M}(\mathcal{P}_1)$ there is a graph $H \in \boldsymbol{M}(\mathcal{P}_2)$ such that $G \subseteq H$.*

The next lemma provides a criterion for the verification of $\mathcal{P}$-maximal graphs in terms of the features of comparable hereditary properties.

Lemma 15 ([84]). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any hereditary properties of graphs. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, $G \in \boldsymbol{M}(n,\mathcal{P}_2)$ and $G \in \mathcal{P}_1$, then G belongs to $\boldsymbol{M}(n,\mathcal{P}_1)$.*

We now discuss the maximal graphs of order n for some of the properties listed in Example 1. In view of Lemma 12, we can restrict our attention to a property $\mathcal{P}$ and an integer n with $n \geq c(\mathcal{P})+2$. This is then our choice in each case of the following example.

Example 3. Let n be a positive integer, $n \geq c(\mathcal{P})+2$. Then

$$\begin{array}{rcl}
\boldsymbol{M}(n,\mathcal{O}) & = & \{\overline{K_n}\}, \\
\boldsymbol{M}(n,\mathcal{O}_k) & = & \{K_{r_1} \cup \ldots \cup K_{r_s} : \sum_{i=1}^{s} r_i = n \text{ and } r_i + r_j \geq k+2, \\
 & & r_i \leq k+1 \text{ for each } 1 \leq i,j \leq s, i \neq j\}, \\
\boldsymbol{M}(n,\mathcal{S}_k) & = & \{G \in \mathcal{I} : \Delta(G) \leq k,\ G \text{ is of order } n \text{ and for every pair} \\
 & & \text{of nonadjacent vertices of } G \text{ at least one has degree } k\}, \\
\boldsymbol{M}(n,\mathcal{D}_1) & = & \{G \in \mathcal{I} : G \text{ is a tree of order } n\}, \\
\boldsymbol{M}(n,\mathcal{T}_2) & = & \{G \in \mathcal{I} : G \text{ is a triangulation of order } n \text{ of the disc with} \\
 & & \text{no inner vertices}\}, \\
\boldsymbol{M}(n,\mathcal{T}_3) & = & \{G \in \mathcal{I} : G \text{ is a triangulation of order } n \text{ of the plane}\}, \\
\boldsymbol{M}(n,\mathcal{I}_1) & = & \{G \in \mathcal{I} : G \text{ is a triangle-free graph of order } n \text{ and every} \\
 & & \text{pair of non-adjacent vertices of } G \text{ are at distance } 2\}.
\end{array}$$

The structure of maximal k-degenerate graphs has been studied extensively in [37], [64] and [75] and one description is given in the following theorem.

Theorem 16 ([64]). *Let $G = (V, E)$ be a graph of order n, $n \geq k+1$, and $v \in V$ be a vertex of degree k. Then $G \in \boldsymbol{M}(n, \mathcal{D}_k)$ if and only if $G - v \in \boldsymbol{M}(n-1, \mathcal{D}_k)$.*

The degree sequences of maximal k-degenerate graphs can also be characterized (see [15]). The characterization of graphs maximal with respect to hom-properties was obtained in [62] and will be presented in Section 3.

We now introduce the notation $\max_{\preceq}[\mathcal{S}]$, for a set $\mathcal{S}$ of graphs, to denote the set of graphs in $\mathcal{S}$ that are maximal with respect to the partial order $\preceq$ on $\mathcal{I}$. This concept is useful to describe $\boldsymbol{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2)$ and $\boldsymbol{M}(n, \mathcal{P}_1 \cap \mathcal{P}_2)$ in terms of $\boldsymbol{M}(n, \mathcal{P}_1)$ and $\boldsymbol{M}(n, \mathcal{P}_2)$.

Theorem 17 ([84]). *Let n be a positive integer and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be hereditary properties of graphs. Then*

1. $\boldsymbol{M}(n, \mathcal{P}_1 \cup \mathcal{P}_2) = \max_{\subseteq}[\boldsymbol{M}(n, \mathcal{P}_1) \cup \boldsymbol{M}(n, \mathcal{P}_2)]$,
2. $\boldsymbol{M}(n, \mathcal{P}_1 \cap \mathcal{P}_2) = \max_{\subseteq}[\{G \in \mathcal{I}:$ *there exists a pair of graphs* $H_1 \in \boldsymbol{M}(n, \mathcal{P}_1)$ and $H_2 \in \boldsymbol{M}(n, \mathcal{P}_2)$ *such that* $G \subseteq H_1$ *and* $G \subseteq H_2, \}]$.

2.3. Extremal Graph Problems

One type of extremal graph problem could be formulated in the following way: for a graph of given order a certain type of subgraphs is prohibited, and one is to determine the maximum and minimum possible number of edges in such a graph.

A general extremal problem, in our terminology, can be formulated as follows. Given a hereditary property $\mathcal{P}$ with a family $\boldsymbol{F}(\mathcal{P})$ of forbidden subgraphs, find the numbers

$$\begin{aligned} \mathrm{ex}(n, \mathcal{P}) &= \max\Big\{|E(G)| : G \in \boldsymbol{M}(n, \mathcal{P})\Big\}, \\ \mathrm{sat}(n, \mathcal{P}) &= \min\Big\{|E(G)| : G \in \boldsymbol{M}(n, \mathcal{P})\Big\}. \end{aligned}$$

The set of all $\mathcal{P}$-maximal graphs of order n with exactly $\mathrm{ex}(n, \mathcal{P})$ edges is denoted by $\mathrm{Ex}(n, \mathcal{P})$. The members of $\mathrm{Ex}(n, \mathcal{P})$ are called $\mathcal{P}$-*extremal* graphs. By the symbol $\mathrm{Sat}(n, \mathcal{P})$ we shall denote the set of all $\mathcal{P}$-maximal graphs on n vertices with $\mathrm{sat}(n, \mathcal{P})$ edges. These graphs are called $\mathcal{P}$-*saturated*.

A problem of this type was first formulated by Turán (see [95], [96]) and his original problem asked for the maximum number of edges in any graph of order n which does not contain the complete graph K_p (i.e. he was interested in the number $\mathrm{ex}(n, \mathcal{I}_{p-2})$).

The following two theorems belong to the fundamental results of extremal graph theory.

Theorem 18 ([88])**.** *If $\mathcal{P}$ is a hereditary property with chromatic number $\chi(\mathcal{P})$, then*

$$\mathrm{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem 19 ([58])**.** *If $\mathcal{P}$ is a given hereditary property and*

$$\begin{aligned} u = u(\mathcal{P}) &= \min\Big\{|V(F)| - \alpha(F) - 1 : F \in \boldsymbol{F}(\mathcal{P})\Big\} \\ d = d(\mathcal{P}) &= \min\Big\{|E(F')| \,\Big|\, F' \subseteq F \in \boldsymbol{F}(\mathcal{P}) \textit{ is induced by a set } S \cup \{x\}, \\ &\quad S \subseteq V(F) \textit{ is independent and } |S| = |V(F)| - u - 1, \\ &\quad x \in V(F) - S\Big\}, \end{aligned}$$

then

$$\mathrm{sat}(n, \mathcal{P}) \le un + \frac{1}{2}(d-1)(n-u) - \binom{u+1}{2},$$

if n is large enough.

As a consequence of the previous two theorems we immediately have

Corollary 20. *If $\mathcal{P}$ is a hereditary property of graphs and* $\mathrm{sat}(n, \mathcal{P}) = \mathrm{ex}(n, \mathcal{P})$ *for every positive n, then $\chi(\mathcal{P}) = 2$.*

The hereditary properties with $\chi(\mathcal{P}) = 2$ are called *degenerate*. It will turn out in the next sections that these properties play an important role in the lattice $\mathbb{L}^a$. We shall also deal with degenerate hereditary properties, which have some tree among forbidden subgraphs. As their position in the investigated lattice is also interesting, we shall call them *very degenerate* (see [89]).

The numbers $\mathrm{ex}(n, \mathcal{P})$ and $\mathrm{sat}(n, \mathcal{P})$ have been studied very extensively e.g., in [34], [35], [58], [80], [86], [87], [88], [97] and [84], but there are not many hereditary properties, for which the exact values of $\mathrm{sat}(n, \mathcal{P})$ and $\mathrm{ex}(n, \mathcal{P})$ are known. It seems that the determination of $\mathrm{sat}(n, \mathcal{P})$ is much more complicated than the estimation of the number $\mathrm{ex}(n, \mathcal{P})$. There are two important causes for this fact. Firstly, unlike the number $\mathrm{ex}(n,\mathcal{P})$, the behaviour of $\mathrm{sat}(n,\mathcal{P})$ is not monotone in general (see [58]). Secondly, the determination of $\mathrm{ex}(n,\mathcal{P})$ requires max-max optimization, while for the evaluation of $\mathrm{sat}(n,\mathcal{P})$ the min-max type of optimization is necessary.

It was pointed out that the language of the lattice of hereditary properties enables us to investigate the numbers $\mathrm{ex}(n,\mathcal{P})$ and $\mathrm{sat}(n,\mathcal{P})$ from another point of view and utilize the structure of the lattice instead of the concrete knowledge about $\mathcal{P}$-maximal graphs (for more details see [84]).

The following assertions present some results of utilization of this method.

Theorem 21 ([84]). *If $\mathcal{P}$ is a hereditary property, such that $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$ or $\mathcal{D}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$, $n \geq k+1$, then* $\mathrm{sat}(n,\mathcal{P}) \leq kn - \binom{k+1}{2}$.

Theorem 22 ([84]). *Let $\mathcal{P}$ be a hereditary property and let $\mathcal{D}_1 \subseteq \mathcal{P} \subseteq \mathcal{I}_1$. If the vertex-connectivity $\kappa(\mathcal{P}) \geq 1$, then* $\mathrm{sat}(n,\mathcal{P}) = n-1$.

3. Vertex Partitions and Reducible Properties

In this section we shall consider the reducible properties and their role in the lattice $\mathbb{L}^a$. Reducible properties of graphs are those which are defined by vertex-partitions of graphs; they have been investigated mainly in connection with generalized colourings.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be arbitrary hereditary properties of graphs. A *vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition* of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$ (for convenience, the empty set $\emptyset$ will be regarded as the set inducing the subgraph with any property $\mathcal{P}$).

A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is additive and hereditary, too. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$.

Thus, e.g., $\mathcal{O}^k$, $k \geq 2$ denotes the class of all k-colourable graphs, and $\mathcal{D}_1^k$ — the class of graphs with vertex-arboricity at most k.

An additive hereditary property $\mathcal{R}$ is said to be *reducible* in $\mathbb{L}^a$ if there exist nontrivial additive hereditary properties $\mathcal{P}, \mathcal{Q}$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and *irreducible* in $\mathbb{L}^a$, otherwise.

If $\mathcal{R} = \mathcal{P}\circ\mathcal{Q}$ we say $\mathcal{P}$ and $\mathcal{Q}$ divide $\mathcal{R}$. The next lemma, which determines the completeness of reducible hereditary properties, follows straightforwardly from the definitions.

Lemma 23 ([17], [70])**.** *For any reducible property* $\mathcal{R} = \mathcal{P}_1\circ\mathcal{P}_2$

$$c(\mathcal{R}) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 1.$$

3.1. Minimal Reducible Bounds

Using the concepts and notation introduced in previous sections, we are able to express many known results on generalized colourings in terms of partial ordering in $\mathbb{L}^a$ in such a way that we shall prove that a reducible property is an upper bound for the given class of graphs $\mathcal{P}$ in $\mathbb{L}^a$ (the concept of reducible bound will be precisely defined later in this section).

It is not hard to see ([9, 57]) that for all $p, q \geq 0$,

$$\mathcal{O}_{p+q+1} \subseteq \mathcal{O}_p\circ\mathcal{O}_q \text{ and } \mathcal{D}_{p+q+1} \subseteq \mathcal{D}_p\circ\mathcal{D}_q.$$

The well-known Theorem of Lovász states.

Theorem 24 ([65])**.** *For* $p, q \geq 0$*, holds the following*

$$\mathcal{S}_{p+q+1} \subseteq \mathcal{S}_p\circ\mathcal{S}_q.$$

Let us remark that, in spite of its very similar nature, the analogous question for $\mathcal{W}_k$ still remains open.

Problem 1. Is it true that $\mathcal{W}_{p+q+1} \subseteq \mathcal{W}_p\circ\mathcal{W}_q$ for all $p, q \geq 0$?

Bollobás and Manvel proved the following refinement of Brooks' Theorem.

Theorem 25 ([4])**.** *Let* $p, q \geq 1$ *satisfy* $pq > 1$*, then*

$$\mathcal{S}_{p+q} \cap \mathcal{I}_{p+q-1} \subseteq (\mathcal{D}_{p-1} \cap \mathcal{S}_p)\circ(\mathcal{D}_{q-1} \cap \mathcal{S}_q).$$

In connection with the Four Colour Problem, different types of partitions of the vertices of graphs have been investigated. A short survey of reducible bounds for the class $\mathcal{T}_3$ of planar graphs is given in [11]. Let us recall some of them.

The Four Colour Theorem ([1]) $\mathcal{T}_3 \subset \mathcal{O}^4$ implies

$$\mathcal{T}_3 \subseteq (\mathcal{O}^2 \cap \mathcal{T}_3)^2 \text{ and } \mathcal{T}_3 \subseteq \mathcal{O} \circ (\mathcal{O}^3 \cap \mathcal{T}_3).$$

Improving the bound $\mathcal{T}_3 \subseteq \mathcal{O} \circ \mathcal{D}_1^2$ ([10, 51, 90]) Thomassen [93] proved the conjecture of Borodin [9]:

$$\mathcal{T}_3 \subseteq \mathcal{D}_1 \circ (\mathcal{D}_2 \cap \mathcal{T}_3) \text{ and that } \quad \mathcal{T}_3 \subseteq \mathcal{C}_3 \circ \mathcal{C}_3,$$

where $\mathcal{C}_3 = \{G \in \mathcal{I} : \text{ each cycle in G has length } 3\}$.

Another type of bound was proved by Poh [82] and independently by Goddard [45]:

$$\mathcal{T}_3 \subseteq (\mathcal{D}_1 \cap \mathcal{S}_2) \circ (\mathcal{T}_3 \cap (\mathcal{D}_1 \cap \mathcal{S}_2)^2).$$

In order to compare these results, the notion of a minimal reducible bound has been introduced (see [57, 71]).

For a given irreducible property $\mathcal{P}$, a reducible property $\mathcal{R}$ is called *minimal reducible bound* for $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{R}$ and there is no reducible property $\mathcal{R}' \subset \mathcal{R}$ satisfying $\mathcal{P} \subseteq \mathcal{R}'$. In other words, $\mathcal{R}$ is a minimal reducible bound for $\mathcal{P}$ in $\mathbb{L}^a$ if in the interval $(\mathcal{P}, \mathcal{R}\ )$ of the lattice $\mathbb{L}^a$ there are only irreducible properties. This means that the corresponding "theorem" is sharp and in some sense cannot be improved. The set of all minimal reducible bounds for $\mathcal{P}$ will be denoted by $\boldsymbol{B}(\mathcal{P})$. It is worth to pointing out that $\boldsymbol{B}(\mathcal{P})$ might also be empty.

The problem of finding all minimal reducible bounds for the class of planar graphs, formulated by Mihók and Toft in 1993 (see Problem 17.9 in [57]), seems to be very difficult. Some partial results are presented in [11]. For example, in [71] it has been proved that the class of all outerplanar graphs $\mathcal{T}_2$ has exactly two minimal reducible bounds. On the other hand, the set of all minimal reducible bounds of the set of all 1-non-outerplanar graphs is infinite (see [11], [14]).

The are many possible difficulties in proving the minimality of reducible bound for a property $\mathcal{P}$ in $\mathbb{L}^a$, let us present some of them. The detailed proofs of the corresponding theorems can be found in [11, 66].

The set $\boldsymbol{B}(\mathcal{P})$ of all minimal reducible properties for $\mathcal{P}$ in $\mathbb{L}^a$ can be determined for many properties of small completeness using the knowledge of the structure of reducible properties. Let us start with some easy examples.

The property $\mathcal{O}^2$, the smallest reducible property in $\mathbb{L}^a$ and the only reducible property of completeness 1, is obviously the unique minimal reducible bound for $\mathcal{P}$ if and only if $\mathcal{P} \subset \mathcal{O}^2$. Hence,

$$\boldsymbol{B}(\mathcal{O}) = \boldsymbol{B}(\mathcal{O}_1) = \boldsymbol{B}(\mathcal{S}_1) = \boldsymbol{B}(\mathcal{W}_1) = \boldsymbol{B}(\mathcal{D}_1) = \{\mathcal{O}^2\}.$$

The structure of the reducible properties with completeness 2 follows from the next more general result.

Theorem 26 ([73])**.** *Let $\mathcal{P}$ be an additive degenerate hereditary property. Let $\mathcal{P}_1, \mathcal{P}_2$ be any additive hereditary properties. Then $\mathcal{P}\circ\mathcal{P}_1 \subseteq \mathcal{P}\circ\mathcal{P}_2$ if and only if $\mathcal{P}_1 \subseteq \mathcal{P}_2$.*

As a corollary, using Lemma 23, we can describe the structure of $\mathbb{R}_2 = \{\mathcal{R} : \mathcal{R}$ is a reducible property with $c(\mathcal{R}) = 2\}$.

Corollary 27. *The set $\mathbb{R}_2 = \{\mathcal{R} : \mathcal{R} = \mathcal{O}\circ\mathcal{P},\ c(\mathcal{P}) = 1\}$ of reducible properties of the completeness 2 partially ordered by set-inclusion forms a lattice isomorphic to $\mathbb{L}_1^a$.*

The following result states that one implication of Theorem 26 holds in general.

Theorem 28 ([73])**.** *Let $\mathcal{P}$ be a hereditary property of graphs. If $\mathcal{P}_1, \mathcal{P}_2$ are any hereditary properties such that $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\mathcal{P}\circ\mathcal{P}_1 \subseteq \mathcal{P}\circ\mathcal{P}_2$.*

Combining the facts mentioned above, we obtain the following minimal reducible bounds:

$$\boldsymbol{B}(\mathcal{O}_2) = \boldsymbol{B}(\mathcal{S}_2) = \boldsymbol{B}(\mathcal{W}_2) = \{\mathcal{O}\circ\mathcal{O}_1\}.$$

The fact that the class of 2-degenerate graphs $\mathcal{D}_2$ has exactly one minimal reducible bound $\mathcal{R} = \mathcal{O}\circ\mathcal{D}_1$ follows from the construction of Broere (see [11]) which implies

Theorem 29. *Let T be any tree and $T \in \boldsymbol{F}(\mathcal{P})$. Then there exists a planar 2-degenerate graph G which has no vertex $(\mathcal{P}, \mathcal{P})$-partition.*

This theorem implies that if a class $\mathcal{P}$ contains the class of all 2-degenerate planar graphs and $\mathcal{R} = \mathcal{P}_1\circ\mathcal{P}_2$ is a reducible bound for $\mathcal{P}$, then at least one of the factors $\mathcal{P}_i$, $(i \in \{1, 2\})$, contains the class $\mathcal{D}_1$ of all forests.

Theorem 29 also generalizes the result of Cowen, Cowen and Woodall [31] which states that there is no positive integer k such that the class of all planar graphs has the bound $\mathcal{S}_k^2$.

Let us remark that the construction used in the proof of Theorem 29 gives in general non-outerplanar graphs. Thus for the class of outerplanar graphs we may have more reducible bounds. In [71] it has been proved that $\boldsymbol{B}(\mathcal{T}_2) = \{\mathcal{O}\circ\mathcal{D}_1, (\mathcal{D}_1 \cap \mathcal{S}_2)^2\}$.

Surprisingly, the minimal reducible bounds for the properties $\mathcal{I}_k$, $k = 1, 2, \ldots$ are trivial and they follow from a result of Nešetřil and Rödl.

Theorem 30 ([78]). *Let $\boldsymbol{F}(\mathcal{P})$ be a finite set of 2-connected graphs. Then for every graph G of property $\mathcal{P}$ there exists a graph H of property $\mathcal{P}$ such that for any partition $\{V_1, V_2\}$of $V(H)$ there is an i, $i = 1$ or $i = 2$, for which the subgraph $H[V_i]$ induced by V_i in H contains G.*

Corollary 31 ([66]). *Let $\boldsymbol{F}(\mathcal{P})$ be a finite set of 2-connected graphs, then the property $\mathcal{P}$ has exactly one minimal reducible bound $\mathcal{R} = \mathcal{O}\circ\mathcal{P}$.*

By Corollary 31 we have, that for any $k \geq 1$, the property $\mathcal{I}_k$ has the only minimal reducible bound $\mathcal{O}\circ\mathcal{I}_k$ i.e., $\boldsymbol{B}(\mathcal{I}_k) = \{\mathcal{O}\circ\mathcal{I}_k\}$.

Since the structure of reducible properties of completeness $c(\mathcal{R}) \geq 3$ is very complicated (see [74]) there are only some partial results on rather simple properties with completeness 3 (see [11, 66]). The proofs of these results are based on the following lemma.

Lemma 32 ([11]). *Let $\mathcal{P}_1$,$\mathcal{P}_2$ be additive degenerate hereditary properties and $\mathcal{P}_3\circ\mathcal{P}_4 \subseteq \mathcal{P}_1\circ\mathcal{P}_2$ for $\mathcal{P}_i \in \mathbb{L}^a, i = 1, 2, 3, 4$. Then $\mathcal{P}_3 \subseteq \mathcal{P}_1$ and $\mathcal{P}_4 \subseteq \mathcal{P}_2$ or $\mathcal{P}_3 \subseteq \mathcal{P}_2$ and $\mathcal{P}_4 \subseteq \mathcal{P}_1$.*

Using Lemma 32 and Lemma 23 we obtain:

Theorem 33 ([66]). *For any positive integer k,*

$$\boldsymbol{B}(\mathcal{O}_k) = \{\mathcal{O}_p\circ\mathcal{O}_q : p + q + 1 = k\}.$$

For the class of k-degenerate graphs $\mathcal{D}_k$ we can prove

Theorem 34 ([66]). *For any positive integer k,*

$$\boldsymbol{B}(\mathcal{D}_k) = \{\mathcal{D}_p\circ\mathcal{D}_q : p + q + 1 = k\}.$$

In general, in order to prove that $\boldsymbol{B}(\mathcal{P}) = \{\mathcal{R}_i : i \in I\}$, it is sufficient:

(i) to verify that $\mathcal{R}_i$ is a reducible bound for $\mathcal{P}$, $i \in I$,

(ii) to verify that the set of reducible properties $\{\mathcal{R}_i : i \in I\}$ is an antichain in $\mathbb{L}^a$,

(iii) to verify that there is no reducible property in the interval $(\mathcal{P}, \mathcal{R}_i)$ for each $i \in I$,

(iv) to prove that, if $\mathcal{P} \subseteq \mathcal{R}$ for some reducible property $\mathcal{R}$, then there exists an $i \in I$ such that $\mathcal{R}_i \subseteq \mathcal{R}$.

Let us remark, that step (ii) is a straightforward consequence of step (iii) provided that the reducible properties one considers are pairwise distinct.

An effort to verify the steps may be met with resistance of different kinds (see [11, 66]). In order to get over these difficulties, more information on the structure of reducible properties in $\mathbb{L}^a$ is necessary. We therefore present some results on the structure of reducible properties in the sequel.

3.2. Which Properties are Reducible?

The notion of reducible properties was introduced while studying the existence of uniquely partitionable graphs (see [69]), where it has been proved that if the property $\mathcal{R}$ is reducible, then there are no uniquely $(\mathcal{R}, \mathcal{R})$-partitionable graphs. (More results on uniquely partitionable graphs are presented in Section 3.3.)

The structure of minimal forbidden subgraphs for any reducible property $\mathcal{R} \neq \mathcal{O}^2$ is very complicated. There are many partial results to support the conjecture that, for any reducible property $\mathcal{R} \in \mathbb{L}^a$, the set $\boldsymbol{F}(\mathcal{R})$ of minimal forbidden subgraphs for $\mathcal{R}$ is infinite (see [68, 69]). Some other results on the structure of $\boldsymbol{F}(\mathcal{R})$, for example, the generalization of the famous Gallai's theorem, are presented in [17] (see also Section 5).

The relationship between the structure of minimal forbidden subgraphs and maximal graphs for a reducible property, by Theorem 18, leads to the following interesting result.

Theorem 35 ([72])**.** *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are arbitrary hereditary properties of graphs then*

$$\chi(\mathcal{P}_1 \circ \mathcal{P}_2) = \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) - 1.$$

The graphs maximal with respect to reducible properties have the following structure.

Theorem 36 ([21, 24])**.** *A graph G is $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$-maximal if and only if for every vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ it holds*

$$G = G[V_1] + G[V_2] + \ldots + G[V_n]$$

and the graphs $G[V_i]$, $i = 1, 2, \ldots, n$, are $\mathcal{P}_i$-maximal.

On the other hand, the join of any $\mathcal{P}_i$-maximal graphs need not to be $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$-maximal (see [21, 62]).

From Theorem 36 we immediately have

Theorem 37. *If $\mathcal{P} \in \mathbb{L}^a$ and there exists an indecomposable $\mathcal{P}$-maximal graph, then the property $\mathcal{P}$ is irreducible.*

Example 4. Since there exist indecomposable $\mathcal{I}_k$-maximal graphs for any $k \geq 0$, the property $\mathcal{I}_k$ is irreducible for each $k \geq 0$.

It is not difficult to find indecomposable $\mathcal{P}$-maximal graphs for any degenerate property $\mathcal{P} \in \mathbb{L}^a$, so that all degenerate properties are irreducible. This fact also follows from Theorem 35.

In the next sections we will present more results supporting the following conjecture.

Conjecture 38. *An additive hereditary property $\mathcal{P}$ is irreducible if and only if there are indecomposable $\mathcal{P}$-maximal graphs.*

3.3. Uniquely Partitionable Graphs

In this section we give a survey of some general results on uniquely partitionable graphs.

A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is said to be *uniquely* $(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$-*partitionable* if G has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. The set of all uniquely $(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$-partitionable graphs will be denoted by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n)$.

Thus, e.g., $\boldsymbol{U}(\mathcal{O}^n)$ denotes the set of all uniquely n-colourable graphs (see [5, 27, 52]); $\boldsymbol{U}(\mathcal{S}_k^n)$ denotes the set of so-called uniquely $(m, k)^{\Delta}$-colourable graphs (see [38, 39, 103]); $\boldsymbol{U}(\mathcal{W}_k^n)$ has been studied in [2, 38]; $\boldsymbol{U}(\mathcal{D}_k^n)$ in [6, 85] and $\boldsymbol{U}(\mathcal{I}_k^n)$ in [19, 38], respectively.

Another generalization of uniquely colourable graphs has been introduced by Zhu in [104].

The basic properties of uniquely $\mathcal{P}^n$-partitionable graphs have been investigated e.g., in [6, 38, 69]. Let us recall some necessary conditions for the existence of uniquely $(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$-partitionable graphs.

Theorem 39 ([24])**.** *If one of the following holds:*

1. $\mathcal{P}$ *divides* $\mathcal{Q}$,
2. $\mathcal{Q}$ *divides* $\mathcal{P}$,
3. *there exists* $\mathcal{S}$ *such that* $\mathcal{S}$ *divides both* $\mathcal{P}$ *and* $\mathcal{Q}$,

then $U(\mathcal{P}\circ\mathcal{Q}) = \emptyset$.

Let $G \in \mathcal{P}$. We say that G is $\mathcal{P}$-*strict* if $G + K_1 \notin \mathcal{P}$.

Theorem 40 ([20, 24])**.** *Let* G *be a uniquely* $(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n)$*-partitionable graph and let* $\{V_1, V_2, \ldots, V_n\}$ *be the unique* $(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n)$*-partition of* $V(G)$, $n \geq 2$. *Then*

1. $G \notin \mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_{j-1}\circ\mathcal{P}_{j+1}\circ\cdots\circ\mathcal{P}_n$, *for every* $j = 1, 2, \ldots, n$,
2. *the subgraphs* $G[V_i]$ *are* $\mathcal{P}_i$*-strict,* $i = 1, 2, \ldots, n$,
3. *for* $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ *the set* $V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k}$ *induces a uniquely* $(\mathcal{P}_{i_1}\circ\mathcal{P}_{i_2}\circ\cdots\circ\mathcal{P}_{i_k})$*-partitionable subgraph of* G,
4. $\delta(G) \geq \max_j \sum_{i=1, i\neq j}^{n} \delta(\mathcal{P}_i)$,
5. $|V(G)| \geq \sum_{i=1}^{n} (c(\mathcal{P}_i) + 2) - 1$,
6. *the graph* $G = G[V_1] + G[V_2] + \cdots + G[V_n]$ *is uniquely* $(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n)$*-partitionable.*

Theorem 41 ([69])**.** *If* $H \in \mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n$ *and* $\boldsymbol{U}(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n) \neq \emptyset$*, then* H *is an induced subgraph of some uniquely* $(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n)$*-partitionable graph* G.

Many partial results concerning the existence of uniquely $\mathcal{P}^n$-partitionable graphs for different types of properties $\mathcal{P}$ can be generalized. For example, the next theorem implies that the bound in (5) of Theorem 40 is sharp for many properties including $\mathcal{O}_k$, $\mathcal{S}_k$ and $\mathcal{W}_k$, $k \geq 1$.

Theorem 42 ([20])**.** *Suppose* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *are additive hereditary properties such that* $\boldsymbol{F}(\mathcal{P}_i)$ *contains some tree* T_i *of order* $c(\mathcal{P}_i) + 2$. *Then there exists a uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable graph* G *with*

$$|V(G)| = \sum_{i=1}^{n} (c(\mathcal{P}_i) + 2) - 1.$$

On the other hand, for the properties $\mathcal{I}_k$ we have

Theorem 43 ([20]). *If G is a uniquely $(\mathcal{I}_{k_1}, \mathcal{I}_{k_2} \ldots, \mathcal{I}_{k_n})$-partitionable graph and $k_n \geq k_i$ for $i = 1, \ldots, n$, then*

$$|V(G)| \geq \sum_{i=1}^{n-1}(2k_i + 3) + k_n + 1,$$

with equality only if

$$G = \overline{C_{2k_1+3}} + \cdots + \overline{C_{2k_{n-1}+3}} + K_{k_n+1}.$$

In [69] the fact that degenerate properties are irreducible is proved by a construction of a uniquely $(\mathcal{P}, \mathcal{P})$-partitionable graph for an arbitrary degenerate property $\mathcal{P} \in \mathbb{L}^a$. This result can be generalized to

Theorem 44 ([24]). *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be any degenerate additive and hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph.*

Trying to prove that the necessary conditions for the existence of uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graphs given by Theorem 39 are also sufficient, we succeeded in proving it for degenerate properties. The divisibility condition contained in the following definition leads to Theorem 46.

Let $\mathcal{P}, \mathcal{Q}$ be hereditary properties of graphs and let $G \in \mathcal{P}$. If S is a subset of the vertex set $V(G)$ such that $G[S] \in \mathcal{Q}$ and for every graph $T \in \mathcal{Q}$ the graph $T + (G - S) \in \mathcal{P}$, then S is said to be a $(\mathcal{Q}, \mathcal{P})$-*extendable set* of G.

Recall that $\mathcal{Q}$ divides $\mathcal{P}$ in $\mathbb{L}$ if there exists a property $\mathcal{P}^* \in \mathbb{L}$ such that $\mathcal{P} = \mathcal{Q} \circ \mathcal{P}^*$.

Theorem 45 ([24]). *Let $\mathcal{P}, \mathcal{Q}$ be additive hereditary properties of graphs. Then $\mathcal{Q}$ divides $\mathcal{P}$ in $\mathbb{L}$ if and only if every $\mathcal{P}$-maximal graph contains a $(\mathcal{Q}, \mathcal{P})$-extendable set.*

Theorem 46 ([24]). *Let $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a$ and let $\mathcal{Q}$ be a degenerate property. Then $U(\mathcal{Q} \circ \mathcal{P}) \neq \emptyset$ if and only if $\mathcal{Q}$ does not divide $\mathcal{P}$ in $\mathbb{L}$.*

The following conjecture (if true) would play an important role in characterizing reducible properties.

Conjecture 47. *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ be a factorization of a reducible property $\mathcal{R} \in \mathbb{L}^a$ into irreducible factors. Then $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) \neq \emptyset$. In particular, $\boldsymbol{U}(\mathcal{P} \circ \mathcal{P} \circ \cdots \circ \mathcal{P}) \neq \emptyset$ if and only if $\mathcal{P}$ is irreducible.*

In the next section, we will show that this conjecture is true for hom-properties.

3.4. Structure of Reducible Hom-Properties

A characterization of reducible hom-properties was given by Kratochvíl Mihók and Semanišin in [61, 62]. We present here the main results of these papers. In this field of study, the notion of a core of a graph G plays an important role. A *core of a graph* G is any subgraph G' of G such that $G \to G'$ while G fails to be homomorphic to any proper subgraph of G'. It is known that every graph G has a unique core up to isomorphism; it is denoted by $C(G)$ (see [77]). A graph G is called a *core* if G is its own core, i.e., $G \cong C(G)$.

Hom-properties can be given in various ways, for example, the property $\to C_6$ is the same as the property $\to C_{38}$. A standard way is to describe a hom-property by a core.

Proposition 48. *For any graph H, its core $C(H)$ generates $\to H$.*

Thus, writing $\to H$ we assume H be a core.

We have mentioned in Section 2 that it is very difficult to characterize the set of forbidden graphs for hom-properties. Fortunately, this is not the case for the set of maximal graphs with respect to hom-properties. In order to characterize their structure we need to introduce the following concepts and notation.

For any graph $G \in \mathcal{I}$ with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define a *multiplication* $G^{::}$ of G in the following way:

1. $V(G^{::}) = W_1 \cup W_2 \cup \ldots \cup W_n$,
2. for each $1 \leq i \leq n : |W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \leq i \leq j \leq n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{::})$ if and only if $\{v_i, v_j\} \in E(G)$.

The sets $W_1, W_2, \ldots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \ldots, v_n$, respectively. It is not difficult to see that $G^{::}$ is homomorphic to G. In order to emphasize the structure of $G^{::}$ we also use the notation $G^{::}(W_1, W_2, \ldots, W_n)$.

Theorem 49 ([62])**.** *A graph G is $(\to H)$-maximal if and only if G is a multiplication of a graph $\tilde{H} \subseteq H$ such that*

1. *$\tilde{H}$ is a core,*
2. *$\tilde{H}$ is $(\to H)$-maximal, and*

3. $|W_i| = 1$ *for every vertex* $v_i \in V(\tilde{H})$ *for which there exists a homomorphism* $\varphi : \tilde{H} \longrightarrow H$ *and a vertex* $y \in V(H) - V(\varphi(\tilde{H}))$ *such that the closed* $\varphi(\tilde{H})$*-neighborhood of* $\varphi(v_i)$ *is contained in the* H*-neighborhood of* y.

Corollary 50 ([62]). *Let* $\rightarrow H$ *be a hom-property. Then any multiplication* $H^{::}$ *of the core* H *is a* $(\rightarrow H)$*-maximal graph.*

Theorem 49 and Corollary 50 implies that the set of all multiplication of a core H generates the (additive) hereditary property $\rightarrow H$.

It follows from Theorem 49 that neither join of maximal graphs with respect to hom-properties has to be a maximal graph with respect to the composition of these properties. The next result provides one type of sufficient condition.

Corollary 51 ([62]). *If* $G^{::}$ *is a multiplication of a core* G *and* $H^{::}$ *is a multiplication of a core* H*, then* $G^{::} + H^{::}$ *belongs to* $\boldsymbol{M}((\rightarrow G)\circ(\rightarrow H))$.

The reducibility of hom-property $\rightarrow H$ is given by the structure of H.

Theorem 52 ([61]). *Let a graph* H *be a core. A hom-property* $\rightarrow H$ *is irreducible if and only if* H *is indecomposable.*

It is easy to see that any multiplication of an indecomposable graph is indecomposable, too. Hence, by Theorem 49 we obtained for hom-properties an affirmative answer to Conjecture 38.

Theorem 53. *A hom-property* $\rightarrow H$ *is irreducible if and only if every* $(\rightarrow H)$*-maximal graph is an induced subgraph of an indecomposable* $(\rightarrow H)$*-maximal graph.*

Since the decomposition of a decomposable graph into the join of indecomposable graphs is unique, the factorization of reducible hom-properties into irreducible hom-properties is also unique. It was proved that this factorization is unique also in $\mathbb{L}^a$:

Theorem 54 ([61]). *Let a core* $H = H_1 + H_2 + \cdots + H_n$ *be the join of indecomposable graphs* H_i, $i = 1, 2, \ldots, n$. *Then* $\rightarrow H = (\rightarrow H_1)\circ(\rightarrow H_2)\circ\cdots\circ(\rightarrow H_n)$ *is the unique factorization of* $\rightarrow H$ *into irreducible factors in* $\mathbb{L}^a$*, apart from the order of the factors.*

Let us remark that any multiplication of the core $H = H_1 + H_2 + \cdots + H_n$ with H_i indecomposable for $i = 1, 2, \ldots, n$ is uniquely $(\to H_1)\circ(\to H_2)\circ\ldots\circ(\to H_n)$-partitionable which gives for hom-properties the affirmative answer to Conjecture 47.

The characterization of decomposable cores follows from Theorem 54:

Theorem 55 ([61]). *Let the graph $H = H_1 + H_2 + \cdots + H_n$ be the join of indecomposable graphs H_i, $i = 1, 2, \ldots, n$. Then the graph H is a core if and only if each H_i is a core for $i = 1, 2, \ldots, n$.*

3.5. Factorization into Irreducible Properties

It is natural to ask whether the factorization of any reducible property into irreducible factors is unique. This problem was formulated in the book of Jensen and Toft [57] as Problem 17.9. According to the results presented above we can afford to conjecture.

Conjecture 56. *Let $\mathcal{R} \in \mathbb{L}^a$ be a reducible property of graphs and $\mathcal{R} = \mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n$, $n \geq 2$, be a factorization of $\mathcal{R}$ into irreducible factors. Then the factorization is unique* (*apart from the order of factors*).

By Theorem 54 the answer is affirmative for hom-properties. In [74] the unique factorization of all additive and hereditary properties of completeness at most three has been proved. It turns out that Theorem 35 provides an important contribution to the solution of the mentioned general problem. Indeed, it yields the unique factorization of the product of two degenerate additive hereditary properties.

Some related questions on cancellation in $\mathbb{L}^a$ have been investigated in [73].

4. Lattices with Respect to Other Orderings

In this section we consider the posets of the form $(\mathcal{I}, \preceq)$ where $\preceq$ is any partial order on $\mathcal{I}$.

The set $\mathbb{K}_{\preceq}$ of all $\preceq$-hereditary properties has been defined in Section 2. We now add $\mathbb{K}^a_{\preceq}$ to our notation for the set of all additive $\preceq$-hereditary properties.

If we deal with the partitions of the vertices, many partial orders $\preceq$ considered preserve the order $\leq$ "to be an induced subgraph" i.e., they have the following property:

O1: If $G \leq H$ then $G \preceq H$.

Let us mention some of them in the following example.

Example 5.

1. $H \subseteq G$ (*to be a subgraph*), i.e., a graph H is a subgraph of G,
2. $H \leq G$ (*to be an induced subgraph*), i.e., H is an induced subgraph of G,
3. $H \preceq_S G$ (*to be topologically contained*), which means that a subdivision of H is a subgraph of G [59],
4. $H \preceq_M G$ (*to be a minor*), i.e., H can be obtained from a subgraph of G by the contractions of some edges of G [18].

Let us remark that $\preceq_M$-hereditary properties are also called *minor hereditary* (see [18]) and are related to the well-known Hadwiger conjecture; on the other hand, the $\preceq_S$-hereditary properties are involved in Hajós' conjecture [94]. Obviously, both minor hereditary and $\preceq_S$-hereditary properties are hereditary.

Clearly, if $\preceq$ satisfies O1 and if $\mathcal{P}$ is $\preceq$-hereditary, then $\mathcal{P}$ is induced hereditary. Hence in such a case we have that $\mathbb{K}_{\preceq} \subseteq \mathbb{M}$ and that $\mathbb{K}^a_{\preceq} \subseteq \mathbb{M}^a$.

Our first result describes the structure of these systems of sets with respect to any partial order $\preceq$ on $\mathcal{I}$ as ordered systems with respect to inclusion. The proofs are easy and similar results are discussed in [17].

Lemma 57. *Let $\preceq$ be any partial order on $\mathcal{I}$. Then*

1. *The intersection of any subset of $\mathbb{K}_{\preceq}$ is a member of $\mathbb{K}_{\preceq}$.*
2. *The intersection of any subset of $\mathbb{K}^a_{\preceq}$ is a member of $\mathbb{K}^a_{\preceq}$.*
3. *The union of any subset of $\mathbb{K}_{\preceq}$ is a member of $\mathbb{K}_{\preceq}$.*
4. *The union of any directed subset of $\mathbb{K}^a_{\preceq}$ is a member of $\mathbb{K}^a_{\preceq}$.*

By the above lemma, $\mathbb{K}_{\preceq}$ is a closure system and $\mathbb{K}^a_{\preceq}$ is an algebraic closure system (see [47]). We can also define (using parts 1 and 2 of this lemma) for any set $\mathcal{G}$ of graphs, the property

$$[\mathcal{G}]_{\preceq} = \bigcap\{\mathcal{P} : \mathcal{P} \in \mathbb{K}_{\preceq}, \mathcal{G} \subseteq \mathcal{P}\}$$

and call it the $\preceq$-*hereditary property generated by* $\mathcal{G}$ and

$$[\mathcal{G}]^a_{\preceq} = \bigcap\{\mathcal{P} : \mathcal{P} \in \mathbb{K}^a_{\preceq}, \mathcal{G} \subseteq \mathcal{P}\}$$

and call it the *additive* $\preceq$-*hereditary property generated by* $\mathcal{G}$. In both these cases a property will be called *finitely generated* if it is generated by a finite set of graphs.

Since $[\mathcal{G}]_{\preceq}$ is the least $\preceq$-hereditary property containing $\mathcal{G}$ in $\mathcal{I}$, we can describe it using only graph theoretical constructions:

$$[\mathcal{G}]_{\preceq} = \{G : G \in \mathcal{I},\ G \preceq H \text{ for some } H \in \mathcal{G}\}.$$

In order to describe $[\mathcal{G}]^a_{\preceq}$ in a similar way using only graph theoretical constructions, we have to assume that $\preceq$ has the property that, for arbitrary graphs G_i and H_i with $G_i \preceq H_i$, $i = 1, 2$, we have that $G_1 \cup G_2 \preceq H_1 \cup H_2$. For such a partial order we can prove that

$$[\mathcal{G}]^a_{\preceq} = \{G : G \in \mathcal{I},\ G \preceq H \text{ for some finite union } H \text{ of graphs in } \mathcal{G}\}.$$

The partially ordered sets $(\mathbb{K}_{\preceq}, \subseteq)$ and $(\mathbb{K}^a_{\preceq}, \subseteq)$ are lattices. Indeed, by Lemma 57, the meet and join operations of these lattices can be described as follows: If $\mathbb{P} \subseteq \mathbb{K}_{\preceq}$ and $\mathbb{R} \subseteq \mathbb{K}^a_{\preceq}$, then

$$\bigwedge \mathbb{P} = \bigcap \mathbb{P} \text{ and } \bigvee \mathbb{P} = \bigcup \mathbb{P} \text{ in } (\mathbb{K}_{\preceq}, \subseteq) \text{ and}$$

$$\bigwedge \mathbb{R} = \bigcap \mathbb{R} \text{ and } \bigvee \mathbb{R} = \left[\bigcup \mathbb{R}\right]^a_{\preceq} \text{ in } (\mathbb{K}^a_{\preceq}, \subseteq).$$

Also, $(\mathbb{K}_{\preceq}, \subseteq)$ and $(\mathbb{K}^a_{\preceq}, \subseteq)$ are complete algebraic lattices — a fact which is trivial for the former. The compact elements of these algebraic lattices are, like for any closure system, the finitely generated elements. We describe this in the following result.

Lemma 58. *Let* $\mathcal{P}$ *be an element of* $(\mathbb{K}_{\preceq}, \subseteq)$ *or of* $(\mathbb{K}^a_{\preceq}, \subseteq)$. *Then the following are equivalent*

1. $\mathcal{P}$ *is a compact element of this lattice.*
2. $\mathcal{P}$ *is finitely generated in this lattice.*
3. $\mathcal{P}$ *contains only finitely many pairwise non-isomorphic connected graphs.*

The fact that these lattices are algebraic can also be expressed by the statements in the following lemma.

Lemma 59. *Let $\mathcal{P}$ be any element of $(\mathbb{K}_{\preceq}, \subseteq)$ or of $(\mathbb{K}^a_{\preceq}, \subseteq)$. Then $\mathcal{P}$ is the join of all the compact elements that are smaller than $\mathcal{P}$. In fact:*

1. *If $\mathcal{P} \in \mathbb{K}_{\preceq}$, then $\mathcal{P} = \bigcup_{G \in \mathcal{P}} [G]_{\preceq}$.*
2. *If $\mathcal{P} \in \mathbb{K}^a_{\preceq}$, then $\mathcal{P} = \bigvee_{G \in \mathcal{P}} [G]^a_{\preceq}$.*

Clearly, $\mathcal{E}$ is the least element of both lattices $(\mathbb{K}_{\preceq}, \subseteq)$ and $(\mathbb{K}^a_{\preceq}, \subseteq)$ while $\mathcal{I}$ is the greatest element of them both. Again, as in the case for the lattices of $\subseteq$-hereditary sets, $(\mathbb{K}^a_{\preceq}, \subseteq)$ is not a sublattice of $(\mathbb{K}_{\preceq}, \subseteq)$ since the join of properties in the latter is the union of these properties and it may be a proper subset of the join of these properties in the former. It is also trivial to see that the lattice $(\mathbb{K}_{\preceq}, \subseteq)$ is a distributive lattice.

For lattices of the form $(\mathbb{K}^a_{\preceq}, \subseteq)$ we need an extra condition to enable us to imitate the proof of Theorem 5 of [17] which will show that the lattices are distributive. For this we say that a partial order $\preceq$ in $\mathcal{I}$ is *union compatible* if every $\mathcal{P} \in \mathbb{K}_{\preceq}$ has the property that for every $H \in \mathcal{I}$, if $H \preceq G$ where G is a union of members of $\mathcal{P}$, then H is equal to a union of members of $\mathcal{P}$. Note that the two partial orders $\subseteq$ and $\leq$ both have this property. Now we can formulate

Theorem 60. *If $\preceq$ is a union compatible partial order, then $(\mathbb{K}^a_{\preceq}, \subseteq)$ is a distributive lattice.*

It is not so difficult to see that also the lattices $\mathbb{L}^a_S$ of subdivision hereditary additive properties and $\mathbb{L}^a_M$ of minor hereditary additive properties are distributive.

In general the completeness of an (additive) hereditary property $\mathcal{P} \in \mathbb{K}_{\preceq}$ need not be defined. However, we can still define, for a given nonnegative integer k,

$$\mathbb{K}_{\preceq,k} = \{\mathcal{P} \in \mathbb{K}_{\preceq} : c(\mathcal{P}) = k\} \text{ and}$$
$$\mathbb{K}^a_{\preceq,k} = \{\mathcal{P} \in \mathbb{K}^a_{\preceq} : c(\mathcal{P}) = k\}.$$

Then we have

Theorem 61. *For any nonnegative integer k, the lattice $(\mathbb{K}_{\preceq,k}, \subseteq)$ is a sublattice of $(\mathbb{K}_{\preceq}, \subseteq)$ and the lattice $(\mathbb{K}^a_{\preceq,k}, \subseteq)$ is a sublattice of $(\mathbb{K}^a_{\preceq}, \subseteq)$.*

According to the result of Greenwell, Hemminger and Klerlein [48] for any $\preceq$, the $\preceq$-hereditary properties can be characterized in terms of forbidden substructures. For example, for the order $\leq$ "to be an induced subgraph" the set of *minimal forbidden subgraphs* of $\mathcal{P}$ we can define as follows: $\boldsymbol{C}(\mathcal{P}) = \{H \notin \mathcal{P} :$ for every $v \in V(G)$, $H - v \in \mathcal{P}\}$. However it is not always possible to describe properties in terms of minimal forbidden subgraphs in this general context. Let us remark that such a characterization exists if the partial order $\preceq$ on $\mathcal{I}$ is so called *well-founded* i.e., every strictly descending chain in $(\mathcal{I}, \preceq)$ is finite (see [59]). One can consider also maximal graphs. Let us define, for a $\preceq$-hereditary property $\mathcal{P} \subseteq \mathcal{I}$, the set of $\preceq$-*maximal graphs* by

$$\begin{aligned} \boldsymbol{M}_{\preceq}(\mathcal{P}) \quad = \quad & \{G \in \mathcal{I} : G \in \mathcal{P}, \text{ and every graph } H \in \mathcal{I} \text{ satisfying} \\ & G \preceq H,\ G \neq H \text{ does not belong to } \mathcal{P}\}. \end{aligned}$$

It is, however, more natural to weaken the condition for membership of $\boldsymbol{M}_{\subseteq}(\mathcal{P})$. Recall the resulting set of maximal graphs $\boldsymbol{M}(\mathcal{P})$ (see Section 2).

$$\boldsymbol{M}(\mathcal{P}) = \{G \in \mathcal{I} : G \in \mathcal{P},\ G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}.$$

Related problems have been investigated also in [59] and [83].

5. Invariants Related to Hereditary Properties

In this section we are going to present a concept of (integer) monotone graph invariants and to show how they can be used to define some chains in the lattice $\mathbb{K}$ (see Section 2), and compare known results corresponding to the invariants. On the other hand, such a comparison might lead to some new results and it would indicate open problems for specific invariants.

The concept of integer valued invariants can be extended to a more general concept of rational (or real) valued invariants.

Let $\mathbb{K}_{\preceq}$ be a given lattice and let $\mathcal{P} \subseteq \mathcal{I}$. A non-negative integer valued function $f: \mathcal{P} \to \mathbb{N}$, such that $f(G) = f(H)$ for any two isomorphic graphs $G, H \in \mathcal{P}$, is called the *graph invariant* (*invariant*, for short).

An invariant f on $\mathcal{P}$ in $\mathbb{K}$ is called *monotone* if $H \preceq G$ implies $f(H) \leq f(G)$, for any $H, G \in \mathcal{P}$. If $f(G \cup H) = \max\{f(G), f(H)\}$, for any disjoint G and H, then f is called *additive*.

The first attempt to define an additive monotone graph invariant by similar conditions has been made by Frick [38].
Suppose that we have an invariant $f: \mathcal{P} \to \mathbb{N}$. Let $k_0 = \min\{f(G): G \in \mathcal{P}\}$ and $\mathcal{P}_0 = \{G \in \mathcal{P}: f(G) = k_0\}$. Then in the lattice $\mathbb{K}$ we have an interval $[\mathcal{P}_0, \mathcal{P}]$ and a chain

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k \subset \cdots, \quad k \in \Omega \subseteq \mathbb{N},$$

of properties in $\mathbb{K}$, defined in the following way

$$\mathcal{P}_k = \{G \in \mathcal{P}: f(G) \leq k_0 + k\}, \text{ for each } k \in \mathbb{N}.$$

Note that in the definition of $\mathcal{P}_0$ the inequality $f(\mathcal{P}) \leq k_0$ is equivalent to the equality $f(\mathcal{P}) = k_0$. It is not difficult to observe that for some invariants $k_0 = 0$ and it leads us to the already investigated properties $\mathcal{O}, \mathcal{W}_0, \mathcal{S}_0$ and so on. Examples of all mentioned cases will be presented later on.

Conversely, let in the lattice $\mathbb{K}$ a finite or countable chain

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k \subset \cdots, \quad k \geq 0,$$

be given and let

$$\mathcal{P} = \bigcup_{k \geq 0} \mathcal{P}_k.$$

Then an invariant $f: \mathcal{P} \to \mathbb{N}$ by this chain is defined as follows:

$$\begin{aligned} f(G) &= 0, \text{ if } G \in \mathcal{P}_0 \quad \text{and} \\ f(G) &= k, \text{ if } G \in \mathcal{P}_k - \mathcal{P}_{k-1}, \text{ for } k \geq 1. \end{aligned}$$

From the above it follows that if $H \preceq G$, then $f(H) \leq f(G)$ and, if $\mathcal{P}_k$ $(k \geq 0)$ are in $\mathbb{K}^a$, then $f(G \cup H) = \max\{f(G), f(H)\}$, for any disjoint G and H.

Let an invariant f be defined by a chain as above and let $\mathcal{Q} \subseteq \mathcal{P}$. Then we call an invariant g the *restriction* f to $\mathcal{Q}$ if g is defined by the chain

$$(*) \qquad \mathcal{P}_0 \cap \mathcal{Q} \subseteq \mathcal{P}_1 \cap \mathcal{Q} \subseteq \mathcal{P}_2 \cap \mathcal{Q} \subseteq \cdots,$$

which we call the *restricted chain*. In each case, for the chain $(*)$ we can ask if the properties $\mathcal{P}_k \cap \mathcal{Q}$, $k \geq 0$, are non-empty or the equality between properties holds. In general, the answer to this question is not trivial.

Let the chain $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n \subset \cdots$ in $\mathbb{K}$ be defined by an invariant f. A graph $G \in \mathcal{P}_k$ is called $(\mathcal{P}_k, \preceq)$-f *critical* if $f(H) < f(G)$ for any $H \preceq G$ and $H \neq G$. It is clear that in the lattice $\mathbb{M}$ of all induced properties, a graph G is $(\mathcal{P}_k, \leq)$-f critical if $f(G-v) < f(G)$, for any vertex v of G. We call such graphs (k, f)-*critical*, for short.

Similarly, in $\mathbb{L}$ a graph G is $(\mathcal{P}_k, \subseteq)$-$f$ critical if $f(G - e) < f(G)$, for any edge e of G. We call such graphs, as usually, (k, f)-*minimal*.

5.1. Examples of Well-Known Graph Invariants

We will restrict our attention mainly to the lattices $\mathbb{M}$ and $\mathbb{L}$, and assume $\mathcal{P} = \mathcal{I}$.

5.1.1. The Largest Order of Components of a Graph G: $o(G)$

This invariant defines the chain

$$\mathcal{O} \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots.$$

Let $\mathbb{L}_k^a = \{\mathcal{P} \in \mathbb{L}^a \colon c(\mathcal{P}) = k\},\ k \geq 0$. $(\mathbb{L}_k^a, \subseteq)$ is a sublattice of $\mathbb{L}^a$ with $\mathcal{O}_k$ as its least element.

5.1.2. The Clique Number: $\omega(G)$

The clique number $\omega(G)$ of a graph G leads us to the chain

$$\mathcal{O} = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots,$$

where $\mathcal{I}_k = \{G \in \mathcal{I} \colon \omega(G) \leq k+1\}$.

The property $\mathcal{I}_k$ is the greatest element in $\mathbb{L}_k^a$.

5.1.3. The Chromatic Number: $\chi(G)$

The chromatic number of a graph G defines the chain

$$\mathcal{O} \subset \mathcal{O}^2 \subset \mathcal{O}^3 \subset \cdots,$$

where $\mathcal{O}^k = \{G \in \mathcal{I} \colon \chi(G) \leq k\}$, i.e., the class of all k-colourable graphs. Thus, $k_0 = 1$ and $\mathcal{P} = \mathcal{I}$.

Restricted chains obtained from this one have been intensively investigated. For example,

$$\mathcal{O} \subset \mathcal{O}^2 \cap \mathcal{I}_1 \subset \mathcal{O}^3 \cap \mathcal{I}_1 \subset \cdots$$

or more generally,

$$\mathcal{O} \subset \mathcal{O}^2 \cap \mathcal{I}_k \subset \mathcal{O}^3 \cap \mathcal{I}_k \subset \cdots,$$

for $k \geq 1$. Non-emptiness of these properties follows from the well-known theorems of Zykov [105] or Mycielski [76].

If $\mathcal{P} = \mathcal{T}_n$, it is known, see [26], that in the chain

$$\mathcal{O} \subset \mathcal{O}^2 \cap \mathcal{T}_n \subset \mathcal{O}^3 \cap \mathcal{T}_n \subset \cdots$$

the $n+1$ consecutive properties are non-empty (in the chain only proper subsets are considered), for $n = 1, 2, 3$.

5.1.4. The Degeneracy: $\rho(G)$

The *degeneracy number* $\rho(G)$ of a graph G is defined as follows: $\rho(G) = \max\{\delta(H) \colon H \leq G\}$. Let $\mathcal{D}_k = \{G \in \mathcal{I} \colon \rho(G) \leq k\}$. Thus, we have the following chain

$$\mathcal{O} = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots .$$

5.1.5. The Maximum Degree: $\Delta(G)$

Let $\mathcal{S}_k = \{G \in \mathcal{I} \colon \Delta(G) \leq k\}$. We have the chain

$$\mathcal{O} = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots .$$

All the above described properties satisfy in $\mathbb{L}^a$, for $k \geq 1$, the following relations

$$\mathcal{O}_k \subseteq \mathcal{S}_k \subset \mathcal{D}_k \subset \mathcal{O}^{k+1} \subset \mathcal{I}_k,$$

each of them has completeness which equals to k. Only for $k = 1$ we have $\mathcal{O}_1 = \mathcal{S}_1$.

5.1.6. The Path Number: $l(G)$

For the invariant $l(G)$, the length of the longest path, we get the following chain of hereditary properties:

$$\mathcal{O} = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots .$$

5.1.7. The Size of a Graph G: $e(G)$

Let $e(G) = |E(G)|$. If we denote by $\mathcal{E}_k = \{G \in \mathcal{I} \colon |E(G)| \leq k\}$, then we have immediately the chain

$$\mathcal{O} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots .$$

5.2. Generalized Chromatic Number

Let $\mathcal{P}$ be a property of graphs. A *$\mathcal{P}$-partition* (*colouring*) of a graph G is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that the subgraph $G[V_i]$ induced by the set V_i has the property $\mathcal{P}$ for each $i = 1, \ldots, n$. If $(V_1, \ldots, V_n)$ is a $\mathcal{P}$-partition of a graph G, then the corresponding vertex colouring c is defined by $c(v) = i$ whenever $v \in V_i$, for $i = 1, \ldots, n$. The smallest integer n for which G has $\mathcal{P}$-partition is called the *$\mathcal{P}$-chromatic* (or *$\mathcal{P}$-vertex-partition*) *number* of G and is denoted by $\chi_{\mathcal{P}}(G)$. The $\mathcal{O}$-chromatic number is the ordinary chromatic number (see [17] for a survey and more details).

The $\mathcal{P}$-chromatic number defines the chain

$$\mathcal{P} \subset \mathcal{P}^2 \subset \mathcal{P}^3 \subset \cdots,$$

where $\mathcal{P}^k = \{G \in \mathcal{I} \colon \chi_{\mathcal{P}}(G) \leq k\}$.

5.3. The Choice Number

Let G be a graph and let $L(v)$ be a list of colours (say, positive integers) prescribed for the vertex v, and $\mathcal{P} \in \mathbb{M}$. A $(\mathcal{P}, L)$-*colouring* is a graph $\mathcal{P}$-colouring $c(v)$ with the additional requirement that for all $v \in V(G), c(v) \in L(v)$. If G admits a $(\mathcal{P}, L)$-colouring, then G is said to be $(\mathcal{P}, L)$-*colourable.* The graph G is $(\mathcal{P}, k)$-*choosable* if it is $(\mathcal{P}, L)$-colourable for every list L of G satisfying $|\, L(v) \,| = k$ for every $v \in V(G)$. The *$\mathcal{P}$-choice number* $\mathrm{ch}_{\mathcal{P}}(G)$ is the smallest natural number k such that G is $(\mathcal{P}, k)$-choosable. For more details see [13], where these concepts were introduced.

Vizing [100] and Erdős, Rubin and Taylor [36] independently introduce the idea of considering $(\mathcal{O}, L)$-colouring and $(\mathcal{O}, k)$-choosability (k-choosability, for short). In both the papers, the choosability version of Brooks' theorem [22] has been proved but the choosability version of Gallai's theorem [41] has been proved independently, by Thomassen [92] and by Kostochka et al. [60].

In [13] some extensions of these two basic theorems to $(\mathcal{P}, k)$-choosability have been proved. If $L(v)$ is the same for all vertices of G, these results generalize also the corresponding theorems of [17]. In [13] is proved that $\mathrm{ch}_{\mathcal{P}}(G) - \chi_{\mathcal{P}}(G)$ can be arbitrarily large in the following sense.

Theorem 62. *Let $\mathcal{P} \in \mathbb{M}^a$ and $1 \leq c(\mathcal{P}) < \infty$. Then for any nonnegative integer s there exists a graph G_s such that $\mathrm{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > s$.*

From the definition of $(\mathcal{P}_k, \preceq)$-f critical graphs it follows that for a nontrivial property $\mathcal{P} \in \mathbb{M}$, a graph G is $(\mathcal{P}, k)$-choice critical if $\mathrm{ch}_{\mathcal{P}}(G) = k \geq 2$ but $\mathrm{ch}_{\mathcal{P}}(G - v) < k$ for all vertices v of G. Since in any $(\mathcal{P}, k)$-choice critical graph G (see [13]) $\deg_G(v) \geq \delta(\mathcal{P})(k-1)$ for a degree of any vertex v of G, let us denote (the set of low vertices) by $S(G) = \{v : v \in V(G), \deg_G(v) = \delta(\mathcal{P})(k-1)\}$.

Now we present a generalization of Gallai's and Brooks' theorems.

Theorem 63 ([13])**.** *Let $\mathcal{P} \in \mathbb{M}$ and G be a $(k, \mathcal{P})$-choice critical graph. Then any block B of $G[S(G)]$ is one of the following types:*

(i) *B is a complete graph,*
(ii) *B is a $\delta(\mathcal{P})$-regular graph belonging to $\boldsymbol{C}(\mathcal{P})$,*
(iii) *$B \in \mathcal{P}$ and $\Delta(B) \leq \delta(\mathcal{P})$,*
(iv) *B is an odd cycle.*

Theorem 64 ([13])**.** *Let $\mathcal{P} \in \mathbb{M}^a$ and G be a connected graph other than*
(i) *a complete graph of order $n\delta(\mathcal{P}) + 1, n \geq 0$,*
(ii) *a $\delta(\mathcal{P})$-regular graph belonging to $\boldsymbol{C}(\mathcal{P})$,*
(iii) *an odd cycle if $\mathcal{P} = \mathcal{O}$.*
Then
$$\mathrm{ch}_{\mathcal{P}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(\mathcal{P})} \right\rceil .$$

Let $\mathcal{CH}^k = \{G \in \mathcal{I} : \mathrm{ch}_{\mathcal{O}}(G) \leq k\}$, i.e., it is the class of all k-choosable graphs. The completeness $c(\mathcal{CH}^k) = k - 1$. Since any k-choosable graph is k-colourable, then by the result of [12], for $\mathcal{P} = \mathcal{O}$, we have $\mathcal{D}_{k-1} \subseteq \mathcal{CH}^k \subseteq \mathcal{O}^k$.

In [17] similar results for generalized chromatic numbers $\chi_{\mathcal{P}}$ are presented.

5.4. Invariants from Gallai Type Theorems

In 1959 Gallai presented his, now classical, theorem, involving the vertex covering number τ_0, the vertex independence number α_0, the edge covering number α_1 and the edge independence number β_1.

Theorem 65 ([40])**.** *For every nontrivial connected graph G with p vertices, we have*
$$\alpha_0 + \tau_0 = p \text{ and } \alpha_1 + \beta_1 = p.$$

A large number of similar results and generalizations of this theorem have been obtained in subsequent years; they are called *Gallai-Type Equalities.* The typical Gallai-Type Equality has the form

$$f + g = p,$$

where f and g are integer (real) valued functions of some type defined on the class of (connected) graphs and p denotes the number of vertices in a graph.

For all parameters created from this type of theorem, the main problem is to find non-trivial relations with some other parameters and to characterize the maximal graphs with respect to ones. In this subsection examples of two such parameters and their generalizations will be presented.

5.4.1. The Vertex Covering Number

A set S of vertices of G is a *vertex cover* of G if each edge of G has at least one end vertex in the set S. The cardinality of any smallest vertex cover is denoted by $\tau_0(G)$ and is called the *vertex covering number* of G. Let $\mathcal{B}_k = \{G \in \mathcal{I}\colon \tau_0(G) \leq k\}$. By Gallai's Theorem, $\mathcal{B}_k \in \mathbb{M}$.

5.4.2. Generalized Vertex Covering Number

Let $\mathcal{P} \in \mathbb{M}$ and G be a graph. A set $S \subseteq V(G)$ is $\mathcal{P}$-*independent* in G if $G[S] \in \mathcal{P}$. The maximum of the cardinalities of the maximal $\mathcal{P}$-independent sets in G is called the $\mathcal{P}$-*independence number* of G and it is denoted by $\alpha_{\mathcal{P}}(G)$. A subset of $V(G)$ is called a *vertex* $\mathcal{P}$-*cover* if it meets every non $\mathcal{P}$-independent set of G. The minimum cardinality of a vertex $\mathcal{P}$-cover is called the *vertex* $\mathcal{P}$-*covering number* and is denoted by $\tau_{\mathcal{P}}(G)$.

The following generalization of Gallai's Theorem has been proved in [53].

Theorem 66. *For any $\mathcal{P} \in \mathbb{M}$ and a graph G of order p,*

$$\alpha_{\mathcal{P}}(G) + \tau_{\mathcal{P}}(G) = p.$$

Similarly, we can define a chain of the following properties:

$$\mathcal{B}_{k,\mathcal{P}} = \{G \in \mathcal{I}\colon \tau_{\mathcal{P}}(G) \leq k\}.$$

5.4.3. Nieminen's Number or the Leafage Number

A nonempty subset D of the vertex set V of a graph G is a *dominating* set if every vertex in $V - D$ is adjacent to a member of D. If $u \in D$ and $v \in V - D$, and $\{u, v\} \in E$, we say that u *dominates* v and v is *dominated* by u.

The minimum of the cardinalities of the minimal dominating sets in G is called the *domination number* of G and it is denoted by $\gamma(G)$.

The study of domination in graphs has been initiated by Ore [81], for a survey see the special volume of *Discrete Mathematics* **86** (1990).

Theorem 67 ([79]). *Let $\gamma(G)$ be the domination number and $\epsilon(G)$ be the maximum number of pendant edges in a spanning forest of a graph G with p vertices. Then $\gamma(G) + \epsilon(G) = p$.*

We call $\epsilon(G)$ the *leafage number* of G.

5.4.4. Generalized Leafage Number

Let $\mathcal{P} \in \mathbb{M}$ and $G = (V(G), E(G))$ be a graph. Two vertices u and v of G are called $\mathcal{P}$-*adjacent* if there is a subgraph H' of G isomorphic to $H \in \boldsymbol{C}(\mathcal{P})$ containing u and v. For a vertex $v \in V$ by $N_{\mathcal{P}}(v)$ we denote the $\mathcal{P}$-*neighbourhood* of v, i.e., $N_{\mathcal{P}}(v) = \{u \in V : u \text{ is } \mathcal{P}\text{-adjacent to } v\}$. For a set $X \subseteq V$, let $N_{\mathcal{P}}(X) = \bigcup_{v \in X} N_{\mathcal{P}}(v)$. Especially, $N(v) = N_{\mathcal{O}}(v)$.

Next, for a vertex $v \in V(G)$ we denote the set of all forbidden subgraphs containing v by $\boldsymbol{C}_{G,\mathcal{P}}(v) = \{H' \leq G : v \in V(H'), H' \simeq H \in \boldsymbol{C}(\mathcal{P})\}$, where $\simeq$ is an isomorphism relation.

The number $| \boldsymbol{C}_{G,\mathcal{P}}(v) |$ is called a $\mathcal{P}$-*degree* of v in G and it is denoted $\deg_{G,\mathcal{P}}(v)$. If $\deg_{G,\mathcal{P}}(v) = 1$, then v is said to be $\mathcal{P}$-*pendant.* If $\deg_{G,\mathcal{P}}(v) = 0$, then v is said to be $\mathcal{P}$-*isolated.*

For a property $\mathcal{P}$, let $\Delta(\mathcal{P}) = \min\{\Delta(H) : H \in \boldsymbol{C}(\mathcal{P})\}$.

A set $D \subseteq V(G)$ is said to be $\mathcal{P}$-*dominating* in G if $N_{\mathcal{P}}(v) \cap D \neq \emptyset$ for any $v \in V(G) - D$.

A set $D \subseteq V(G)$ is said to be *strongly* $\mathcal{P}$-*dominating* in G if for every $v \in V(G) - D$ there is $H' \leq G$ containing v such that $H' \simeq H \in \boldsymbol{C}(\mathcal{P})$ and $V(H') - \{v\} \subseteq D$.

The minimum of the cardinalities of the (strongly) $\mathcal{P}$-dominating sets of G is called the (*strong*) $\mathcal{P}$-*domination number* of G and is denoted by $\gamma_{\mathcal{P}}(G)$ $(\gamma'_{\mathcal{P}}(G))$, respectively.

Notice, that if $\mathcal{P} = \mathcal{O}$, then $\mathcal{P}$-dominating and strongly $\mathcal{P}$-dominating sets in G are dominating sets in the ordinary sense.

Next, if $\mathcal{P} = \mathcal{I}_{n-2}$, then the $\mathcal{I}_{n-2}$-dominating set in G is the K_n-dominating set in G (see [56]).

Let $\mathcal{P} \in \mathbb{M}$ and G be a graph. Let S be a spanning subgraph of G. A family $X_{\mathcal{P}}(S) = \{G_1, G_2, ..., G_k\}$ of induced subgraphs of S such that

(1) $G_i \simeq H \in \boldsymbol{C}(\mathcal{P})$ and

(2) for any G_i there is a vertex $v_i \in V(G_i)$ such that $v_i \notin V(G_j),\ j \neq i$, $1 \leq i, j \leq k$

is called a *family of $\mathcal{P}$-pendant subgraphs of S.*

A vertex $v_i \in V(G_i)$ satisfying (2) is called a *$\mathcal{P}$-pendant vertex in the family $X_{\mathcal{P}}(S)$*. The *generalized leafage number* $\epsilon_{\mathcal{P}}(G)$ of a graph G is the maximum number of $\mathcal{P}$-pendant subgraphs in a spanning subgraph of the graph G.

Notice, that if $\mathcal{P} = \mathcal{O}$, then $\epsilon_{\mathcal{P}}(G) = \epsilon(G)$.

Theorem 68 ([16])**.** *Let $\mathcal{P} \in \mathbb{M}$. For every graph G of order p, we have*

$$\gamma'_{\mathcal{P}}(G) + \epsilon_{\mathcal{P}}(G) = p.$$

Hedetniemi and Laskar [54] proved a similar equality as in Nieminen's Theorem involving connectivity.

Generalization of the above result to an invariant involving connectivity and a property $\mathcal{P} \in \mathbb{M}$ in [16] is obtained. A survey of Gallai Type Equalities is presented in [30].

5.5. Edge Partitions Invariants

A large class of invariants from an edge partition is derived. We will mention here only two well-known ones. In fact, the Ramsey Theory (see [46]) includes the most important results of this type.

In order to present the next results, we need the following notation

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \{G: E(G) = E_1 \cup E_2,\ G[E_i] \in \mathcal{P}_i,\ i = 1, 2\}.$$

5.5.1. The Chromatic Index: χ'

The chromatic index χ' defines the following chain of properties

$$\mathcal{O} \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_k \subset \cdots,$$

where $\mathcal{J}_k = \{G \in \mathcal{I}: \chi'(G) \leq k\}$, i.e., $\mathcal{J}_k = \oplus_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i = \mathcal{O}_1,\ i = 1, ..., k$.

A well-known theorem of Vizing [99] states that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.

It easy to see that $\mathcal{J}_1 = \mathcal{S}_1$. But $\mathcal{S}_k \subset \mathcal{J}_{k+1} \subset \mathcal{S}_{k+1}$, for $k \geq 2$. On the other hand, the relation $\mathcal{S}_k \cap \mathcal{O}^2 \subseteq \mathcal{J}_k$ seems to be very interesting.

The decision problem:

$$\text{Is a graph } G \text{ in } \mathcal{S}_k \cap \mathcal{J}_k,\ k \geq 3,$$

is NP-complete (see[43]).

5.5.2. Arboricity: Υ

Let $\mathcal{A}_k = \oplus_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i = \mathcal{D}_1$. The minimum k for which a graph G is in $\mathcal{A}_k$ is called the *arboricity* of G and is denoted by $\Upsilon(G)$. The well-known result of Nash-Williams states that $\Upsilon(G) = \max\{\lceil |E(H)|/(|V(H)|-1)\rceil\}$, where the maximum is taken over all induced subgraphs H of G with at least two vertices.

An example of a fine relation is (see [50]):

$$\mathcal{T}_3 \cap \mathcal{I}_1 \subset \mathcal{A}_2,$$

i.e., every planar triangle-free graph is the union of two forests.

6. Complexity Results

Let $\mathcal{P}$ and $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$ be given (induced) hereditary properties. The $(\mathcal{P} : \mathcal{R})$-*partition* (*colouring*) *problem* is stated as follows.

$(\mathcal{P} : \mathcal{R})$**-PARTITION** ($(\mathcal{P} : \mathcal{R})$**-PART**, for short).
INSTANCE: A graph $G \in \mathcal{P}$.
QUESTION: Does there exist a $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-partition of G ?

The question can be of course reformulated in the following way: Is the graph G $\mathcal{R}$-partitionable?

For some hereditary properties $\mathcal{R}$, the problem $(\mathcal{I} : \mathcal{R})$-PART is not even decidable. For some such examples see [23]. However, for many properties $\mathcal{P}$ and $\mathcal{R}$, the problem $(\mathcal{P} : \mathcal{R})$-PART is well defined.

Let $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$. Note, that if $(\mathcal{I} : \mathcal{P}_i)$-PART is in NP for some i (it means it is NP-hard to decide whether a graph belongs to $\mathcal{P}_i$), then $(\mathcal{I} : \mathcal{R})$-PART $\in$ NP. Let $\overline{\mathcal{R}}$ be a hereditary property defined by $\boldsymbol{C}(\overline{\mathcal{R}}) = \{\overline{H} : H \in \boldsymbol{C}(\mathcal{R})\}$. Then the complexity of $(\mathcal{I} : \mathcal{R})$-PART and $(\mathcal{I} : \overline{\mathcal{R}})$-PART are the same, as the graph G is $\mathcal{R}$-partitionable if and only if $\overline{G}$ is $\overline{\mathcal{R}}$-partitionable.

For the complexity terminology see [43].

6.1. $\mathcal{O}^k$-Partition

Theorem 69 ([43])**.** (a) *For* $k \leq 2$ *the problem* $(\mathcal{I} : \mathcal{O}^k)$-PART *is in* P.
(b) *For* $k \geq 3$ *the problem* $(\mathcal{I} : \mathcal{O}^k)$-PART *is* NP*-complete.*

Brooks' Theorem implies that 3-colourability of a graph with maximum vertex degree 3 can be determined in polynomial time (we should only verify that the graph does not contain K_4). Thus, the problem $(\mathcal{S}_3 : \mathcal{O}^3)$-PART is in P.

Since Brooks' Theorem applies to all graphs, then it might be expected that with the additional restriction of planarity, a stronger result could be obtained. Unfortunately, it is not so.

Theorem 70 ([91], [43])**.** *The problem* $(\mathcal{T}_3 \cap \mathcal{S}_4 : \mathcal{O}^3)$-PART *is* NP*-complete.*

6.2. List Colouring

The LIST COLOURING problem includes graph k-colourability as a particular case by putting $L(v) = \{1, \ldots, k\}$ for all $v \in V(G)$. Hence, the LIST COLOURING problem is NP-complete even if all lists have length three. Let us denote by $\mathbf{LC}[\mathcal{P} : (k, s)]$ and $\mathbf{LC}[\mathcal{P} : (= k, s)]$ the following two subproblems.

$\mathbf{LC}[\mathcal{P} : (k, s)]$
INSTANCE: A graph $G \in \mathcal{P}$ and a collection of lists $\{L(v) :\mid L(v) \mid\geq k, v \in V(G)\}$, and each colour occurs in at most s lists.
QUESTION: Does there exist a list colouring of G ?

$\mathbf{LC}[\mathcal{P} : (= k, s)]$
INSTANCE: A graph $G \in \mathcal{P}$ and a collection of lists $\{L(v) :\mid L(v) \mid= k, v \in V(G)\}$, and each colour occurs in at most s lists.
QUESTION: Does there exist a list colouring of G ?

It was observed in [36] and [100] that $\mathbf{LC}[\mathcal{I} : (2, \infty)]$ is in P while $\mathbf{LC}[\mathcal{I} : (3, \infty)]$ is NP-complete.

Theorem 71 ([63])**.** *Let* $k \geq 3$ *be an arbitrary fixed integer. Then*
(a) $\mathbf{LC}[\mathcal{I} : (= k, k)]$ *is in* P*,*
(b) $\mathbf{LC}[\mathcal{I} : (= k, k + 1)]$ *is* NP*-complete.*

For $k = 3$ the last problem remains NP-complete even for a class of planar graphs.

Theorem 72 ([63]). $\mathbf{LC}[\mathcal{T}_3 \cap \mathcal{S}_3 : (3,3)]$ *is* NP-*complete.*

Theorem 73. *The following cases of the* LIST COLOURING *problem are in* P.

(a) [63]: $\mathbf{LC}[\mathcal{I} : (2,\infty)]$,

(b) [63]: $\mathbf{LC}[\mathcal{I} : (\infty,2)]$,

(c) [63]: $\mathbf{LC}[\mathcal{S}_2 : (k,s)]$,

(d) [33]: $\mathbf{LC}[\mathcal{I} : (\geq \deg(v),*)]$, *where* $(\geq \deg(v),*)$ *denotes a collection of lists with* $\mid L(v) \mid \geq \deg(v)$ *for all* $v \in V(G)$ *and* $\mid \bigcup_{v\in V(G)} L(v) \mid \leq \Delta(G)$.

6.3. $(\mathcal{I} : \mathrm{Hom}(H))$-Part

The complexity of $(\mathcal{I} : \mathrm{Hom}(H))$-PART for undirected graphs was determined by Hell and Nešetřil in [55].

Theorem 74 ([55]). (a) *If H is bipartite, then* $(\mathcal{I} : \mathrm{Hom}(H))$-PART *is in* P.
(b) *If H is not bipartite, then* $(\mathcal{I} : \mathrm{Hom}(H))$-PART *is* NP-*complete.*

6.4. $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-Partition

Let $\mathcal{P}$ and $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_k$ be given (induced) hereditary properties. It is easy to see that for some properties $\mathcal{P}$ and $\mathcal{R}$, the $(\mathcal{P} : \mathcal{R})$-PART is polynomial. For example, if $\mathcal{P} = \mathcal{O}_{km}$, then $\mathcal{P} \subseteq \mathcal{R} = (\mathcal{O}_m)^k, k \geq 1, m \geq 2$. By Ramsey's theorem there is a polynomial time algorithm for an $(\mathcal{I} : \mathcal{R})$-PART, when $\mathcal{R} = \mathcal{Q}^k$, $\boldsymbol{F}(\mathcal{Q}) = \{K_m, \overline{K}_l\}$. But we do not know it explicitly, as the determination of the precise value of the Ramsey numbers is a very difficult problem.

In [23] Brown has stated the following conjecture.

Conjecture 75. *Let* $\mathcal{R} = \mathcal{Q}^k$, $\boldsymbol{C}(\mathcal{Q}) = \{H\}, |V(H)| \geq 3\}$. *Then* $(\mathcal{I} : \mathcal{R})$-PART *is* NP-*complete for any* $k \geq 2$.

Conjecture 75 has been verified in [23] for:

(a) $H = H_1 + H_2$ (the join of two nonempty graphs) and $k \geq 3$,

(b) H is 2-connected and $k \geq 2$.

In particular, Conjecture 75 is true for:

(1) $H = K_m$, $m \geq 3, k \geq 2$,
(2) $H = C_m, m \geq 3, k \geq 2$,
(3) $H = P_m, m = 3, 4, k \geq 3$ and for $m \geq 5, k \geq 2$.

One of the most intriguing problems is the following

Problem. *Is Conjecture* 75 *true for* $H = P_3$ *and* $k = 2$?

The following theorem provides some particular result.

Theorem 76 ([51])**.** *Let* $\mathcal{R} = \mathcal{O}^{k-r} \circ \mathcal{D}_1^r$*. Then* $(\mathcal{I} : \mathcal{R})$-PART *is* NP-*complete for all* $k \geq 3$ *and* $0 \leq r \leq k$.

For $k = 2$, in [51] the following results are proved.

Theorem 77 ([51])**.** (a) $(\mathcal{S}_6 : \mathcal{O} \circ \mathcal{D}_1)$-PART *is* NP-*complete,*
(b) $(\mathcal{I} : \mathcal{D}_1^2)$-PART *is* NP-*complete.*

Theorem 78 ([32])**.** $(\mathcal{I} : \mathcal{S}_d^k)$-PART *is* NP-*complete for any* $k \geq 3$ *and* $d \geq 0$.

Question [32]**.** *What is the complexity of* $(\mathcal{S}_{k(d+1)} : \mathcal{S}_d^k)$-PART ?

6.5. $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-Partition of Planar Graphs

It is well-known [28] that every planar graph has vertex arboricity at most 3, so $(\mathcal{T}_3 : \mathcal{D}_1^3)$-PART is always true. Stein [90] (see also [49] and [10]) strengthened this result by proving that every planar graph can be partitioned into two forests and an edgeless subgraph, so $(\mathcal{T}_3 : \mathcal{O} \circ \mathcal{D}_1^2)$-PART is always true. On the other hand, Stein [90] proved that a maximal planar graph G has vertex arboricity 2 if and only if the dual G^*(which is planar, cubic and 3-connected) is hamiltonian.

But in [42] it is proved that HAMILTONIAN CYCLE problem is NP-complete even when restricted to planar, cubic and 3-connected graphs. From this it follows that $(\mathcal{T}_3 : \mathcal{D}_1^2)$-PART is NP- complete.

Theorem 79. *The following problems are* NP-*complete*
(a) [44]: $(\mathcal{T}_3 \cap \mathcal{S}_4 : \mathcal{O}^3)$-PART,
(b) [51]: $(\mathcal{T}_3 \cap \mathcal{S}_4 : \mathcal{O} \circ \mathcal{D}_1)$-PART,

(c) [32]: $(\mathcal{T}_3 \cap \mathcal{S}_4 : \mathcal{S}_1^2)$-PART,
(d) [51]: $(\mathcal{T}_3 : \mathcal{O}^2 \circ \mathcal{D}_1)$-PART *for graphs in which each face has size* 3 *or* 4,
(e) [32]: $(\mathcal{T}_3 : \mathcal{S}_d^2)$-PART *for* $d \geq 1$,
(f) [32]: $(\mathcal{T}_3 : \mathcal{S}_1^3)$-PART.

Conjecture 80 ([51])**.** $(\mathcal{T}_3 : \mathcal{O}^2 \circ \mathcal{D}_1)$-PART *is* NP-*complete for maximal planar graphs.*

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