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PROBLEMS ON FULLY IRREGULAR DIGRAPHS

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A simple graph with more than one vertex is well-known to have two vertices of the same degree. This amounts to saying that no simple nontrivial graph can be fully irregular. Recall that directing each edge of a simple graph results in an *oriented graph* (which is a digraph without 2-cycles $\vec{C}_2$).

A digraph D is called *fully irregular* if distinct vertices of D have distinct degree pairs. The *degree pair* of a vertex is the outdegree followed by the indegree of the vertex. The notion of fully irregular digraphs—introduced by the present author in 1995—is investigated in [1, 2, 3, 5]. Some results on fully irregular digraphs were presented at international conferences held in Poland at Lubiatów '96, Gronów '97, '98, and at Kazimierz Dolny '97.

Theorem 1. *Let D be a digraph of order n. There exists an injection $D \mapsto D'$ which associates with D a fully irregular digraph D' of order $n + 2\lceil\sqrt{n}\,\rceil$ such that D is an induced subdigraph of D' and such that deleting all arcs of D from D' results in an oriented graph.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of D. Let $t = \lceil\sqrt{n}\,\rceil - 1$. Consider two disjoint linearly ordered sets U and W which comprise altogether $2(t + 1)$ new vertices respectively u_i and w_i, which are ordered by increasing subscripts i, $i = 0, 1, \ldots, t$. Let B be the bipartite digraph whose vertex set is $U \cup W$ and all arcs are of the form (w_j, u_i) for each $i \in \{0, 1, \ldots, t - j\}$ where $j = 0, 1, \ldots, t$. Let D' be a digraph of order $n+2t+2$ which includes disjoint digraphs D and B, all arcs both from V to u_0 and from w_0 to V, and possibly arcs (v, u_i) and/or (w_i, v) where $v \in V$ and, moreover, the neighbours of any such v both in U and W make up precisely initial segments of U and W, respectively. Hence the outdegrees and indegrees of vertices from U and W, respectively, are all zero. Therefore the two obligatory arcs (v, u_0) and (w_0, v) for each vertex v of D enable us to identify D as the subdigraph of D' induced by all vertices whose outdegrees and indegrees are all positive.

For any vertex v of D the optional arcs (possibly none) from v to a segment of U can be chosen in $t+1$ ways. The same is the number of choices for remaining optional arcs from a segment of W to v. Thus the degree pair in D' of each vertex v can be any of some $(t+1)^2$ $(\geq n)$ points in the plane integral lattice. Therefore distinct degree pairs in D' for all n vertices of D can be designed and realized. The construction associates mutually distinct degree pairs with all remaining vertices, too. Therefore a required injection exists. ■

Corollary 2. *There are at least as many fully irregular digraphs* (*oriented graphs*) *of order* $n+2\lceil\sqrt{n}\,\rceil$ *as there are digraphs* (*oriented graphs*) *of order* n. ■

It seems likely that fully irregular digraphs can contribute to finding a constructive proof (which is lacking) of the fact (cf. [4]) that almost all digraphs have trivial automorphism group. Given a digraph D on n vertices, let $f(D)$ (and $f'(D)$) be the smallest integer t such that a fully irregular digraph $\tilde{D}$ on $n+t$ vertices includes D as an induced subdigraph (and such that deleting the arc set $A(D)$ of D from $\tilde{D}$ results in an oriented graph). Name $f(D)$ and $f'(D)$ respectively the *irregularity deficit* and *irregularity o-deficit* of D. Clearly, $f(D) \leq f'(D)$. Let $f(n)$ (and $f'(n)$) be the *largest irregularity deficit* (resp. *largest irregularity o-deficit*) among n-vertex digraphs.

Corollary 3. *The irregularity o-deficit among n-vertex digraphs is bounded by* $2\lceil\sqrt{n}\,\rceil$. *Thus*

$$f(n) \leq f'(n) \leq 2\lceil\sqrt{n}\,\rceil.$$ ■

Problem 1 (**Problem 1'**)**.** Characterize n-vertex digraphs D with the largest possible irregularity deficit $f(D)$ (o-deficit $f'(D)$), i.e., with $f(D) = f(n)$ (resp. $f'(D) = f'(n)$).

Given a nonnegative integer r, a digraph D is called *r-diregular* if degree pairs in D are all (r, r).

Theorem 4 (Górska et al. [2])**.** *If D is an r-diregular oriented graph on n vertices then* $f'(D) = \lfloor\sqrt{2n} - \frac{1}{2}\rfloor$ *for* $n \geq 1$ *unless* $n = 3$, $r = 1$, *and then* $f'(D) = 2$. ■

Theorem 5 (Górska et al. [3]). *If D is an r-diregular digraph on n vertices then* $f(D) = \lfloor\sqrt{n-1}\rfloor (= \lceil\sqrt{n}\,\rceil - 1)$ *for* $n \geq 1$ *unless* $n = 4$, $r \in \{1, 2\}$, *and then* $f(D) = 2$. ■

References

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