

## A NOTE ON KERNELS AND SOLUTIONS IN DIGRAPHS

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### Abstract

For given nonnegative integers  $k, s$  an upper bound on the minimum number of vertices of a strongly connected digraph with exactly  $k$  kernels and  $s$  solutions is presented.

**Keywords:** kernel of digraph, solution of digraph.

**1991 Mathematics Subject Classification:** 05C20.

Kernels (solutions) are vertex subsets of digraphs that are studied in [2, 3, 5, 8]. The decision problem of the existence of a kernel in a digraph is known to be NP-complete (see e.g. the book [4]). The number of kernels (solutions) was investigated in the papers [1, 7, 9]. In [6] the first author has shown that for given nonnegative integers  $k, s$  there are infinitely many pairwise nonisomorphic strongly connected digraphs with no pair of opposite arcs that have exactly  $k$  kernels and  $s$  solutions. An upper bound on the minimum number of vertices of those digraphs is also presented there. In the following a better upper bound is established.

## 1. PRELIMINARIES

An ordered pair  $D = (V, A)$  is said to be a *digraph* whenever  $V$  is a non-empty set (vertices of  $D$ ) and  $A$  (arcs of  $D$ ) is a subset of the set of ordered pairs of elements  $V$  such that  $\overrightarrow{vv} \notin A$  for each  $v \in V$ , and if  $u, v \in V$  then  $\overrightarrow{uv} \in A$  implies  $\overrightarrow{vu} \notin A$ .

A set of vertices  $W \subseteq V$  is called *independent* if for every pair of vertices  $u, v \in W$  neither  $\overrightarrow{uv}$  nor  $\overrightarrow{vu}$  is present in the digraph.  $W \subseteq V$  is *absorbent* if for each  $u \in V - W$  there exists  $\overrightarrow{uw} \in A$  with  $w \in W$  and *dominant* if for each  $v \in V - W$  there exists  $\overrightarrow{uv} \in A$  with  $u \in W$ . A set  $W \subseteq V$  is a *kernel* of  $D$  if  $W$  is independent and absorbent and  $W$  is a *solution* of  $D$  if  $W$  is independent and dominant.

As usual, a digraph is *strongly connected*, if for every  $u, v \in V$  there exists a sequence  $u\overrightarrow{a_1}, \overrightarrow{a_1a_2}, \overrightarrow{a_2a_3}, \dots, \overrightarrow{a_k v}$  in  $A$ . Let  $\mathcal{G}$  denote the class of all finite strongly connected digraphs.

## 2. RESULTS

Let  $\mathcal{G}_{(k,s)}$  denote the set of all strongly connected digraphs with  $k$  kernels and  $s$  solutions. It is known (see [6], Theorem 2.6) that the set  $\mathcal{G}_{(k,s)}$  is infinite whenever  $k$  and  $s$  are nonnegative integers. A digraph belonging to  $\mathcal{G}_{(k,s)}$  with the minimum number of vertices is called a *minimum digraph* of  $\mathcal{G}_{(k,s)}$ . The number of vertices of a minimum digraph of  $\mathcal{G}_{(k,s)}$  will be denoted by  $k \star s$ . The following assertions were proved:

**Proposition 1** ([6] 1.1, 1.2, 2.7). *Let  $k, s$  be nonnegative integers. Then*

- (i)  $k \star s = s \star k$ ,
- (ii)  $0 \star 0 = 3$ ,  $0 \star 1 = 1 \star 0 = 5$ ,  $0 \star 2 = 2 \star 0 = 6$ ,
- (iii)  $1 \star 1 = 2 \star 2 = 4$ ,  $1 \star 2 = 2 \star 1 = 5$ ,
- (iv) if  $k > 1$  then  $k \star 0 \leq 4k$  and  $k \star 1 \leq 4k + 1$ , and
- (v)  $k \star s \leq 4(k + s) - 7$  whenever  $k > 1$  and  $s > 1$ .

Let  $k, s$  be positive integers. By part (i) of the previous proposition  $k \leq s$  can be supposed without loss of generality. Define a digraph  $D_{(k,s)}$  as follows. Denote by  $T$  and  $U$  two disjoint copies of an acyclic tournament with  $s$  vertices such that  $t_1, t_2, \dots, t_s$  are the vertices of  $T$ ,  $u_1, u_2, \dots, u_s$  are the vertices of  $U$ ,  $\overrightarrow{t_i t_j}$  (resp.  $\overrightarrow{u_i u_j}$ ) are the arcs of  $T$  (of  $U$ ) for  $i, j \in \{1, 2, \dots, s\}$  whenever  $i < j$ . Take  $T, U$ , two new vertices  $v, w$  and add the following arcs:  $\overrightarrow{t_i u_j}$  and  $\overrightarrow{u_i t_j}$  for  $i, j \in \{1, 2, \dots, s\}$  whenever  $i > j$ ,

$\overrightarrow{t_i v}, \overrightarrow{v u_i}, \overrightarrow{u_i w}$  for every  $i \in \{1, 2, \dots, s\}$ ,  
 $\overrightarrow{w t_i}$  for  $i \leq k$ ,  $\overrightarrow{t_i w}$  for  $k < i \leq s$  and  $\overrightarrow{w v}$ .

**Proposition 2.** *The digraph  $D_{(k,s)}$  belongs to  $\mathcal{G}_{(k,s)}$  whenever  $k, s$  are positive integers.*

**Proof.**  $D_{(k,s)}$  has a hamiltonian cycle (for instance  $t_1, t_2, \dots, t_{s-1}, t_s, v, u_1, u_2, \dots, u_{s-1}, u_s, w, t_1$ ), thus it is strongly connected. Since no vertex of the digraph  $D_{(k,s)}$  creates absorbent (dominant) set then every kernel (solution) of  $D_{(k,s)}$  contains at least two vertices. On the other hand no triple of vertices of  $D_{(k,s)}$  is independent. Thus any kernel (solution) must contain exactly two vertices. But if  $\{x, y\}$  is an independent subset of the vertex set of  $D_{(k,s)}$  then there exists  $i \in \{1, 2, \dots, s\}$  such that  $x = t_i, y = u_i$  or  $x = u_i, y = t_i$ . It is easy to check that  $S$  is a solution of  $D_{(k,s)}$  if and only if  $S = \{t_i, u_i\}$  for  $i \in \{1, 2, \dots, s\}$  and  $K$  is a kernel of  $D_{(k,s)}$  if and only if  $K = \{t_i, u_i\}$  for  $i \in \{1, 2, \dots, k\}$ . ■

**Corollary.** *Let  $k, s$  be positive integers. Then  $k \star s \leq 2 \cdot \max\{k, s\} + 2$ .*

**Proof.** By the previous proposition it suffices to take the digraph  $D_{(k,s)}$  having  $\max\{k, s\} + 2$  vertices. Therefore the number of the vertices of a minimum digraph of  $\mathcal{G}_{(k,s)}$  is at most  $2 \cdot \max\{k, s\} + 2$ . ■

**Remark.** The upper bound of  $k \star s$  above is sharp in the case  $k = s = 1$  and also if  $k = 0, s = 2$ . On the contrary it is not attained for  $k = 0$  and  $s \in \{0, 1\}$ . The new bound improved the bound from (v) in Proposition 1 in all cases where  $k > 1, s > 1$  or  $k = 1, s > 2$  or  $k > 2, s = 1$ .

#### REFERENCES

- [1] M. Behzad and F. Harary, *Which directed graphs have a solution?*, Math. Slovaca **27** (1977) 37–42.
- [2] V.V. Belov, E.M. Vorobjov and V.E. Shatalov, *Graph Theory*, Vyshshaja Shkola, Moskva, 1976. (Russian)
- [3] C. Berge, *Graphs and Hypergraphs* (Dunod, Paris, 1970). (French)
- [4] M.R. Garey and D.S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness* (Freeman, San Francisco, 1979).
- [5] F. Harary, R.Z. Norman and D. Cartwright, *Structural Models* (John Wiley & Sons, Inc., New York – London – Sydney, 1965).

- [6] M. Harminc, *Kernel and solution numbers of digraphs*, Acta Univ. M. Belii **6** (1998) 15–20.
- [7] M. Harminc and T. Olejnikova, *Binary operations on digraphs and solutions*, Slovak, Zb. ved. prac, VŠT, Košice (1984) 29–42.
- [8] L. Lovasz, *Combinatorial Problems and Exercises* (Akademiai Kiado, Budapest, 1979).
- [9] R.G. Nigmatullin, *The largest number of kernels in graphs with  $n$  vertices*, Kazan. Gos. Univ. Učen. Zap. **130** (1970) kn.3, 75–82. (Russian)

Received 2 February 1999

Revised 29 October 1999